Note di Matematica Vol. IX - n. 2, 221-227(1989)

FLOW-INVARIANT SETS

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INTRODUCTION

Let S be a subset of some normed space E and let $\chi : U \to E$ be a continuous function defined on an open subset U containing S. We consider functions $x : [0,T] \to U$ which satisfy the differential equation

$$(1) \qquad \dot{x}(t) = \chi(x(t))$$

for all $t \in [0,T]$, where T is a positive number, depending on x. Such functions will be called *solutions of* (1). We say that S is *invariant with respect to* (1), if every solution of (1) with $x(0) \in S$ remains in S, that is $x([0,T]) \subseteq S$.

Early results concerning the characterization of invariant sets in the finite dimensional case $E = \mathbb{R}^n$ were obtained by Nagumo [Na42]. Further research was independently done by Bony [Bo69], Brezis [Br70], Crandall [Cr72], Hartman [Ha72] and Yorke [Yo67], [Yo70]. Different conditions imposed on S and χ by Bony and Brezis respectively were related with one another by Redheffer [Re72]. He elucidated as well as generalized their results. There was even some progress in the case of infinite dimensional spaces. Redheffer and Walter [ReWa75] were the first to investigate this field. Their results were extended by Volkmann [Vo73], [Vo75]. Finally, Martin [Ma73] and Volkmann [Vo76] came to far-reaching generalizations in case E is a Banach-space.

This note offers an alternative approach based on an elementary lemma from the theory of functions of one real variable. As a corollary we obtain a slightly more general version of Brezis' Invariance Theorem. In this fashion it becomes evident that the latter is a higher dimensional generalization of the lemma from one-dimensional calculus.

Lemma. *Let* $f : [a,b] \to \mathbb{R}$ *be a continuous function with* $f(a) = 0$. *Assume there exists a number* $c \geq 0$ *such that the differential inequality*

$$(2) \qquad \liminf_{h \to 0^+} \frac{f(x+h) - f(x)}{h} \leq c f(x)$$

holds for all $x \in [a,b)$ *with the possible exception of some countable subset* $D \subseteq (a,b)$. *Then* $f \leq 0$.

Proof. Let $\{r_n\}_{n \in \mathbb{N}}$ be some enumeration of D and let $\varepsilon > 0$ be an arbitrary positive number.

For each $y \in [a,b]$ we define the following subset of $[a,b]$

$$M(y) := \left\{ x \in [a,y] : f(x) \leq \varepsilon e^{cx}(x-a) \left(1 + \sum_{r_n < x} \frac{1}{2^n} \right) \right\}. \tag{3}$$

This set depends also on ε. Now, $M(y)$ is not the empty set because of $a \in M(y)$. We set $m := \sup\ M(y)$. Since $\sum_{r_n<x} \frac{1}{2^n}$ is a left-continuous function with respect to x, every increasing sequence in $M(y)$ converges to a point in $M(y)$. This yields $m \in M(y)$. We claim that $m = y$ and prove it by deriving a contradiction from the assumption $m < y$.

Suppose $m < y$. We distinguish two cases.

(i) $m \notin \{r_n\}_{n\in\mathrm{N}}$

By (2) there exists a positive number $d > 0$ satisfying $m + d < y$ and

$$\frac{f(m+d) - f(m)}{d} \leq cf(m) + (1+cd)\varepsilon e^{cm} \left(1 + \sum_{r_n<m} \frac{1}{2^n} \right).$$

A simple calculation yields

$$f(m+d) \leq (1+cd) f(m) + (1+cd)\varepsilon e^{cm} d \left(1 + \sum_{r_n<m} \frac{1}{2^n} \right).$$

On account of $m \in M(y)$ and $1 + cd > 0$ we get

$$\begin{aligned} f(m+d) &\leq (1+cd) \left(\varepsilon e^{cm}(m-a) \left(1 + \sum_{r_n<m} \frac{1}{2^n} \right) \right) \\ &\quad + (1+cd)\varepsilon e^{cm} d \left(1 + \sum_{r_n<m} \frac{1}{2^n} \right) \\ &= (1+cd)\varepsilon e^{cm}(m+d-a) \left(1 + \sum_{r_n<m} \frac{1}{2^n} \right). \end{aligned}$$

Since $0 < 1 + ch < e^{ch}$ for $h > 0$, it finally follows

$$\begin{aligned} f(m+d) &\leq e^{cd}\varepsilon e^{cm}(m+d-a) \left(1 + \sum_{r_n<m+d} \frac{1}{2^n} \right) \\ &= \varepsilon e^{c(m+d)}(m+d-a) \left(1 + \sum_{r_n<m+d} \frac{1}{2^n} \right). \end{aligned}$$

This implies $m+d \in M(y)$ because of (3), contradicting $m = \sup M(y)$.

(ii) $m = r_k$ for some $k \in \mathbf{N}$.

Since f is continuous and $m \neq a$, there exists a positive number $d > 0$ such that $m+d < y$ and

$$f(m+d) \leq f(m) + \varepsilon e^{cm}(m-a)\frac{1}{2^k}.$$

By (3) we may thus infer

$$\begin{aligned} f(m+d) &\leq \varepsilon e^{cm}(m-a)\left(1+\sum_{r_n<m}\frac{1}{2^n}\right) + \varepsilon e^{cm}(m-a)\frac{1}{2^k} \\ &= \varepsilon e^{cm}(m-a)\left(1+\left(\sum_{r_n<m}\frac{1}{2^n}\right)+\frac{1}{2^k}\right) \\ &\leq \varepsilon e^{c(m+d)}(m+d-a)\left(1+\sum_{r_n<m+d}\frac{1}{2^n}\right). \end{aligned}$$

Hence $m+d \in M(y)$, a contradiction.

We thus conclude $m = y$, yielding $y \in M(y)$. Since $y \in M(y) \subseteq M(b)$ for each $y \in [a,b]$, we have $[a,b] \subseteq M(b) \subseteq [a,b]$. That gives $M(b) = [a,b]$. Consequently

$$f(x) \leq \varepsilon e^{cx}(x-a)\left(1+\sum_{r_n<x}\frac{1}{2^n}\right) \leq 2\varepsilon e^{cb}(b-a)$$

holds for each $x \in [a,b]$. Since $\varepsilon > 0$ was arbitrary, $f \leq 0$ follows. ∎

If we set $S := (-\infty, 0]$ and $\chi(x) := cx$ for all $x \in \mathbf{R}$ then the Lemma simply says that S is invariant with respect to (1). We have thus treated a very special case. But soon this case will show itself to be the core of the Invariance Theorem of Brezis.

Definition. *Let S be a closed subset of a normed space L, and suppose $x, y \in L$. We say that y is a subtangent vector to S at x if there exist a subset $D \subseteq \mathbf{R}_0^+ \setminus \{0\}$ having 0 as a limit point and a function $r : D \to S$ satisfying*

$$\lim_{h \to 0^+} \frac{r(h) - x}{h} = y. \tag{4}$$

The set of all subtangent vectors of S at x is denoted by $L_x(S)$.

Theorem. *Let S be a closed subset of $\mathbf{R}^n$, and let $U \subseteq \mathbf{R}^n$ be an open subset containing S. Suppose that $\chi : U \to \mathbf{R}^n$ is a locally Lipschitz-continuous function satisfying $\chi(y) \in L_y(S)$ for each $y \in S$. Then S is invariant with respect to (1).*

Proof. Let $x : [0,T] \to U$ be a solution of the differential equation

$$\dot{x}(t) = \chi(x(t))$$

for all $t \in [0,T]$. Assume $x(0) \in S$.

Since S is closed, there exists at least one element in S attaining the minimal distance $\inf_{s \in S} \|x(t) - s\|$ between $x(t)$ and S. Let $z(t)$ be such an element for each $t \in [0,T]$. This defines a function $z : [0,T] \to S$ satisfying

$$\|x(t) - z(t)\| = \inf_{s \in S} \|x(t) - s\|. \tag{5}$$

We give our proof by deriving a contradiction from $x([0,T]) \not\subseteq S$. Let us therefore assume $x([0,T]) \not\subseteq S$.

Since x is continuous and S is closed, the set $\{t \in [0,T] : x(t) \notin S\}$ is open with respect to $[0,T]$. Hence there exists an interval $[a,b] \subseteq [0,T]$ satisfying $x(a) \in S$ and $x((a,b]) \cap S = \emptyset$. Now, we choose a strictly positive number r, such that χ satisfies a Lipschitz-condition with constant c on $B_{2r}(x(a))$, and a number $b' \in (a,b]$ with

$$x([a,b']) \subseteq B_r(x(a)). \tag{6}$$

From $x(a) \in S$ we infer $\|x(t) - x(a)\| \geq \inf_{s \in S} \|x(t) - s\| = \|x(t) - z(t)\|$. Hence $\|z(t) - x(a)\| \leq \|z(t) - x(t)\| + \|x(t) - x(a)\| \leq \|x(a) - x(t)\| + \|x(t) - x(a)\| < r + r = 2r$ for each $t \in [a,b']$. We conclude

$$z([a,b']) \subseteq B_{2r}(x(a)). \tag{7}$$

Now, let us take $t \in [a,b']$. By assumption $\chi(z(t)) \in L_{z(t)}(S)$. According to (4) this yields a set $D_t \subseteq \mathbf{R}_0^+ \backslash \{0\}$ having 0 as a limit point and a function $r_t : D_t \to S$ satisfying

$$\lim_{h \to 0^+} \frac{r_t(h) - z(t)}{h} = \chi(z(t)).$$

We thus have a function $s_t : D_t \to \mathbf{R}^n$ with

$$r_t(h) = z(t) + h\chi(z(t)) + hs_t(h) \quad \text{and} \quad \lim_{h \to 0^+} s_t(h) = 0. \tag{8}$$

Since $x(t) = \chi(x(t))$, there exists a function $u_t : D_t \to \mathbf{R}^n$ satisfying

$$(9) \qquad x(t+h) = x(t) + h\chi(x(t)) + hu_t(h) \quad \text{and} \quad \lim_{h\to 0^+} u_t(h) = 0.$$

Now, we have $r_t(h) \in S$ for all $h \in D_t$. This yields

$$\|x(t+h) - z(t+h)\| = \inf_{s\in S} \|x(t+h) - s\| \le \|x(t+h) - r_t(h)\|.$$

Combining this inequality with (8) and (9), we infer

$$\begin{aligned}\|x(t+h) - z(t+h)\| &\le \|x(t) + h\chi(x(t)) + hu_t(h) - z(t) - h\chi(z(t)) - hs_t(h)\| \\ &\le \|x(t) - z(t)\| + h\|\chi(x(t)) - \chi(z(t))\| + h\|u_t(h) - s_t(h)\|.\end{aligned}$$

Since χ is Lipschitz-continuous on $B_{2r}(x(a))$ with constant c, and $x(t), z(t) \in B_{2r}(x(a))$ by (6) and (7) respectively, we get

$$(10) \quad \|x(t+h) - z(t+h)\| \le \|x(t) - z(t)\| + hc\|x(t) - z(t)\| + h(\|u_t(h)\| + \|s_t(h)\|).$$

Setting $f(t) := \|x(t) - z(t)\|$ for $t \in [a, b']$, we conclude

$$\frac{f(t+h) - f(t)}{h} \le cf(t) + \|u_t(h)\| + \|s_t(h)\| \; \forall h \in D_t.$$

Now, $\lim_{h\to 0^+} u_t(h) = 0$ and $\lim_{h\to 0^+} s_t(h) = 0$ yield

$$(11) \qquad \liminf_{h\to 0^+} \frac{f(t+h) - f(t)}{h} \le cf(t).$$

Since $d(y,S) := \inf_{s\in S} \|y - s\|$ is continuous with respect to y, continuity of f follows. In addition, we have $f(0) = 0$ and (11). We may thus invoke the Lemma to conclude $f \le 0$. But f assumes only positive values, hence $f = 0$. This means $x([a, b']) \subseteq S$, contradicting $x((a, b']) \cap S = \emptyset$. ∎

Note. In fact the Lemma shows that $f(x) := kx$ for $k > 0$ is a *restricted uniqueness function* in the sense of Redheffer [Re72].

Our restriction to finite dimensional spaces is necessitated by the existence of a function $z : [0, T] \to S$ satisfying (5), which is a consequence of the fact that S contains at least one element z' of minimal distance between S and $x(t)$. This of course is true for closed subsets of $\mathbb{R}^n$. We may therefore easily generalize the Invariance-Theorem to infinite dimensional normed spaces in case S is a distance set, that is for each $x \in L$ we have a $z' \in S$ satisfying $\inf_{s \in S} ||x - s|| = ||x - z'||$.

Yet, there is a more satisfactory generalization, proved independently by Martin [Ma73] and Volkmann [Vo76]. They showed that the Invariance-Theorem of Brezis remains valid for arbitrary closed subsets of Banach-spaces, provided that χ satisfies a more stringent condition than being locally Lipschitz-continuous.

Further generalizations were obtained by Hörmander [Hö81] and Lawson [La87]. They investigated closed subsets of finite dimensional euclidean spaces which are invariant with respect to certain vector-field-distributions.

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Received December 16, 1988.
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