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FUNCTIONAL DIFFERENTIAL INEQUALITIES OF PARABOLIC TYPE *

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INTRODUCTION

We prove a theorem wich generalizes a result of J. Szarski (see [4]; Theor. 2) concerning weak inequalities for a diagonal system of second order differential functional inequalities of the type

$$u_t^i(t,x) \leq f^i(t,x,u(t,x),u_x^i(t,x),u_{x,x}^i(t,x),u(t,\cdot)) \qquad i=1,\ldots,m,$$

assuming that f^i is parabolic with respect to u for any $i=1,\ldots,m$.

After introducing the definition of left parabolic (or right parabolic) function with respect to another one, we obtain the over mentioned generalization (Theor. 2.2) as a consequence of a theorem about strong inequalities (Theor. 2.1) which is a generalization of Theor. 1 of [4] in the case of left parabolic (or right parabolic) functions.

These generalizations have been suggested by the following example in wich we have the assertion of Theor. 2 of [4] even if hypotheses of the theorem are not all verified. Consider the function f defined as [(1)]

$$f(t,x,u,q,r,z) = (x_1^2 + x_2^2 - 1)\, sgn\, r_{11}$$

for $(t,x) \in D =]0,T[\times\{x=(x_1,x_2)\in \mathbf{R}^2 : x_1^2+x_2^2<1\}, u \in \mathbf{R}, q \in \mathbf{R}^2, r = (r_{ij})_{1\leq i,j\leq 2}$ belonging to the set of real and symmetried 2×2 matrices and z continuous function in $\overline{D}$, with continuous in D partial second derivatives with respect to x as well as functions u and v defined assuming

$$u(t,x) = t\cdot(x_1^2+x_2^2-1) \text{ and } v(t,x) = 0$$

for every $(t,x) \in D$.

1. DEFINITIONS AND ASSUMPTIONS

If $(\bar{t},\bar{x}) \in \mathbf{R}^{1+n}$ and $r>0$ we set

$$(1.1) \qquad U_r^-(\bar{t},\bar{x}) = \left\{(t,x)\in\mathbf{R}^{1+n} : (t<\bar{t}) \wedge (t-\bar{t})^2 + \sum_{j=1}^{n}(x_j-\bar{x}_j)^2 < r^2\right\}$$

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(1) Let $x \in \mathbf{R}$. Then $sgn x = 1$ if $0 < x$, $sgn x = 0$ if $x = 0$ and $sgn x = -1$ if $x < 0$.

the *lower − half neighborood* with center $(\bar{t},\bar{x})$ and radius $r > 0$.

For all set $D \in \mathbf{R}^{1+n}$ with $\mathring{D} \neq \emptyset$, we denote by D_p the subset

$$(1.2) \qquad D_p = \{(\bar{t},\bar{x}) \in \overline{D} : (\exists r > 0)(U_r^-(\bar{t},\bar{x}) \subset \mathring{D})\}.$$

We say *parabolic boundary of D* the set

$$(1.3) \qquad \Gamma_p = \overline{D} \backslash D_p.$$

If $m \in \mathbf{N}$ and D_j is a subset of $\mathbf{R}^{1+n}$ containing D_p for $j = 1,\dots,m$, we set

$$Z = \{\varphi = (\varphi^j)_{1\leq j\leq m} : (\forall j = 1,\dots,m)(\varphi_j : D_j \to \mathbf{R})\}.$$

Furthermore, for $\varphi = (\varphi^j)_{1\leq j\leq m}, \psi = (\psi^j)_{1\leq j\leq m} \in Z$, we denote

$$(1.5) \qquad \varphi < \psi \quad on \quad \Gamma_p^+ \qquad (or \quad \varphi \leq \psi \quad on \quad \Gamma_p^+)$$

if it results

$$(1.6) \qquad \limsup_k (\psi^j(t^k,x^k) - \varphi^j(t^k,x^k)) > 0$$

(or

$$\limsup_k (\psi^j(t^k,x^k) - \varphi^j(t^k,x^k)) \geq 0),$$

for $j = 1,\dots,m$, for any sequence $(t^k,x^k) \in D_p$ such that t^k is decreasing sequence and $\lim_k (t^k,x^k) \in \Gamma_p$ and denote

$$(1.7) \qquad \varphi < \psi \quad on \quad \Gamma_\infty \qquad (or \quad \varphi \leq \psi \quad on \quad \Gamma_\infty)$$

if the (1.6) is true for $j = 1,\dots,m$ for any sequence $(t^k,x^k) \in D_p$ such that t^k is decreasing to $-\infty$ or $||x^k|| \to +\infty$.

A function $\varphi = (\varphi^j)_{1\leq j\leq m} \in Z$ is called *regular in D* if φ_t^j, φ_x^j and φ_{xx}^j are continuous in D_p for $j = 1,\dots,m$ where $x = (x_1,\dots,x_n) \in \mathbf{R}^n, \varphi_x^j = (\varphi_{x_\ell}^j)_{1\leq \ell\leq n}$ and $\varphi_{xx}^j = (\varphi_{x_\ell x_k}^j)_{1\leq \ell,k\leq m}$.

Finally we denote Z_0 the subspace of Z containing regular functions.

Assumption K: *Let* $f^j(t,x,z,q,r,w)(j = 1,\dots,m)$ *be a real function defined for* $(t,x) \in D_p, (z,q) \in G_j \subset \mathbf{R}^m \times \mathbf{R}^n, r \in M_j \subset \mathbf{R}^{n\times n}$ *and* $Z_j \subset Z_0$.

For any $j = 1,\dots,m$ we set $D(f^j) = D_p \times G_j \times M_j \times Z_j, f = (f^j)_{1\le j\le m}$ and

$$Z_0(f) = \{\varphi = (\varphi^j)_{1\le j\le m} \in Z_0 : (\forall j = 1,\dots,m) \\ ((t,x,\varphi(t,x),\varphi^j_x(t,x),\varphi^j_{xx}(t,x),\varphi) \in D(f^j))\}. \tag{1.8}$$

The defect $P_\varphi(t,x)$ *of a function* $\varphi = (\varphi^j)_{1\le j\le m} \in Z_0(f)$ *at the point* $(t,x) \in D_p$ is the vector in $\mathbf{R}^m$ whose components $P^j_\varphi(t,x)$ are defined by

$$P^j_\varphi(t,x) = \varphi^j_t(t,x) - f^j(t,x,\varphi(t,x),\varphi^j_x(t,x),\varphi^j_{xx}(t,x),\varphi). \tag{1.9}$$

The function $\varphi = (\varphi^j)_{1\le j\le m} \in Z_0(f)$ is said to be *left parabolic* (or *right parabolic*) *with respect to* f and we write $\varphi \in \mathscr{P}^-_0(f)$ (or $\varphi \in \mathscr{P}^+_0(f)$) if we have

$$\varphi^j_{xx}(t,x) - s \in M_j \tag{1.10$_s$}$$

$$f^j(t,x,\varphi(t,x),\varphi^j_x(t,x),\varphi^j_{xx}(t,x),\varphi)- \\ - f^j(t,x,\varphi(t,x),\varphi^j_x(t,x),\varphi^j_{xx}(t,x) - s,\varphi) \ge 0 \tag{1.11$_s$}$$

(or

$$\varphi^j_{xx}(t,x) + s \in M_j \tag{1.10$_d$}$$

$$f^j(t,x,\varphi(t,x),\varphi^j_x(t,x),\varphi^j_{xx}(t,x) + s,\varphi)- \\ - f^j(t,x,\varphi(t,x),\varphi^j_x(t,x),\varphi^j_{xx}(t,x),\varphi) \ge 0)cr \tag{1.11$_d$}$$

for any real symmetric matrix $s \in \mathbf{R}^{n\times n}, s \ge 0$, for any $j = 1,\dots,m$ and $(t,x) \in D_p$.

Finally, f is *left increasing* (or *right increasing*) *with respect to* φ if for any $z = (z^j)_{1\le j\le m} \in Z_0, (\bar t,\bar x) \in D_p, j = 1,\dots,m$, and $r \in M_j$ we have that

$$(z^j \ge 0 \ in\ E_j(\bar t,\bar x)) \wedge (\varphi - z \in Z_j) \Rightarrow \\ \Rightarrow f^j(\bar t,\bar x,\varphi(\bar t,\bar x),\varphi^j_x(\bar t,\bar x),\varphi^j_{xx}(\bar t,\bar x),\varphi - z) \le \\ \le f^j(\bar t,\bar x,\varphi(\bar t,\bar x),\varphi^j_x(\bar t,\bar x),\varphi^j_{xx}(\bar t,\bar x),\varphi) \tag{1.12$_s$}$$

(or

$$(1.12)_d \qquad \begin{aligned} &(z^j \geq 0 \; in \, E_j(\bar{t},\bar{x})) \wedge (\varphi + z \in Z_j) \Rightarrow \\ &\Rightarrow f^j(\bar{t},\bar{x},\varphi(\bar{t},\bar{x}),\varphi^j_x(\bar{t},\bar{x}),\varphi^j_{xx}(\bar{t},\bar{x}),\varphi) \leq \\ &\leq f^j(\bar{t},\bar{x},\varphi(\bar{t},\bar{x}),\varphi^j_x(\bar{t},\bar{x}),\varphi^j_{xx}(\bar{t},\bar{x}),\varphi + z)) \end{aligned}$$

where

$$(1.13) \qquad E_j(\bar{t},\bar{x}) = D_j \cap \{(t,x) \in \mathbf{R}^{1+n} : t \leq \bar{t}\}.$$

We denote a left increasing (or right increasing) function f with respect to φ with the symbol $f \uparrow_- \varphi$ (or $f \uparrow_+ \varphi$).

2. FUNCTIONAL INEQUALITIES

Theorem 2.1. (Strong inequalities). *Assume that*

1) *Assumption K,*
2) *For any $j = 1,\dots,m$, $f^j(t,x,u,q,r,w)$ is increasing with respect to u^i, $i = 1,\dots,m$, $i \neq j$,*
3) $u, v \in Z_0(f)$,
4) $Pu < Pv$ *in* D_p
5) $u < v$ *on* Γ^+_p, $u < v$ *on* Γ_∞ *and for any* $j = 1,\dots,m$ $u^j < v^j$ *in* $D_j \setminus D_p$,
6) $v \in \mathscr{P}^-_0(f)$ (or $u \in Pr^+_0(f)$),
7) $f \uparrow_+ u$ (or $f \uparrow_- v$).

Under these assumptions we have for any $j = 1,\dots,m$

$$u^j < v^j \quad in \quad D_j,$$

Proof. Let be $v \in \mathscr{P}^-_0(f)$ and $f \uparrow_+ u$.

Let be

$$(2.1) \qquad S_t = \{x \in \mathbf{R}^n : (t,x) \in D_p\}$$

for any $t \in \mathbf{R}$ and

$$(2.2) \qquad B = \{t \in \mathbf{R} : S_t \neq \emptyset\}, \quad l_1 = inf(B) \quad and \quad l_2 = sup(B).$$

At first we will prove that

$$(I) \qquad (\forall t^* \in [l_1, l_2[\setminus B)(\exists \bar{t} \in]t^*, l_2[)((\forall (t,x) \in D_p \cap (]t^*, \bar{t}[\times \mathbb{R}^n), \forall j = 1, \dots, m)(u^j(t,x) < v^j(t,x)))$$

and

$$(II) \qquad \begin{aligned} &(t^* \in B, t^* < l_2) \wedge ((\forall x \in S_{t^*}, \forall j = 1, \dots, m)(u^j(t^*, x) < v^j(t^*, x))) \Rightarrow \\ &\Rightarrow (\exists \bar{t} \in]t^*, l_2[)((\forall (t,x) \in D_p \cap ([t^*, \bar{t}] \times \mathbb{R}^n), \forall j = 1, \dots, m) \\ &(u^j(t,x) < v^j(t,x))). \end{aligned}$$

Suppose I is not true. Then, $t^* \in]l_1, l_2[\setminus B$, a sequence $(t^k, x^k) \in D_p$ and an index j exist such that t^k is decreasing to t^*, $\lim_k x^k = x$ or $\lim_k \|x^k\| = +\infty$ and $u^j(t^k, x^k) \geq v^j(t^k, x^k)$. This conclusion contradicts assumption 5). Similarly, we deduce the II.

Now, we let

$$(2.3) \qquad A = \{t \in B : (\exists x \in S_t, \exists j = 1, \dots, m)(u^j(t,x) \geq v^j(t,x)\}$$

and suppose that $A \neq \emptyset$. Put $\bar{l}_1 = inf(A)$ by I we have $l_1 < \bar{l}_1$. Furthermore,

$$(III) \qquad (\forall (t,x) \in D_p \cap (]l_1, \bar{l}_1[\times \mathbb{R}^n), \forall j = 1, \dots, m)(u^j(t,x) < v^j(t,x))$$

and

$$(IV) \qquad (\forall (t,x) \in D_p \cap (]l_1, \bar{l}_1] \times \mathbb{R}^n), \forall j = 1, \dots, m)(u^j(t,x) \leq v^j(t,x))$$

Now, if $\bar{l}_1 = l_2$, then $A = \{l_2\}$ and by IV we have

$$(2.4) \qquad (\exists \bar{x} \in S_{\bar{l}_1}, \exists j = 1, \dots, m)(u^j(\bar{l}_1, \bar{x}) = v^j(\bar{l}_1, \bar{x}))$$

If $\bar{l}_1 < l_2$, from I it follows that $S_{\bar{l}_1} \neq \emptyset$ and from IV and II we have that (2.4) is true.

Hence there is a point $\bar{x} \in S_{\bar{l}_1}$ and an index j so that

$$(2.5) \qquad u^j_x(\bar{l}_1, \bar{x}) = v^j_x(\bar{l}_1, \bar{x}) \quad and \quad u^j_{xx}(\bar{l}_1, \bar{x}) \leq v^j_{xx}(\bar{l}_1, \bar{x}).$$

Then by 4, 7, 6 and 2 we get successively

$$\begin{aligned}0 &< P_v^j(\bar{l}_1,\bar{x}) - P_u^j(\bar{l}_1,\bar{x}) = \\ &= v_t^j(\bar{l}_1,\bar{x}) - f^j(\bar{l}_1,\bar{x},v(\bar{l}_1,\bar{x}),v_x^j(\bar{l}_1,\bar{x}),v_{xx}^j(\bar{l}_1,\bar{x}),v) - \\ &- u_t^j(\bar{l}_1,\bar{x}) + f^j(\bar{l}_1,\bar{x},u(\bar{l}_1,\bar{x}),u_x^j(\bar{l}_1,\bar{x}),u_{xx}^j(\bar{l}_1,\bar{x}),u) \le \\ &\le v_t^j(\bar{l}_1,\bar{x}) - u_t^j(\bar{l}_1,\bar{x}) - f^j(\bar{l}_1,\bar{x},v(\bar{l}_1,\bar{x}),v_x^j(\bar{l}_1,\bar{x}),v_{xx}^j(\bar{l}_1,\bar{x})_1 v) + \\ &+ f^j(\bar{l}_1,\bar{x},u(\bar{l}_1,\bar{x}),u_x^j(\bar{l}_1,\bar{x}),u_{xx}^j(\bar{l}_1,\bar{x}),v) \le \\ &\le v_t^j(\bar{l}_1,\bar{x}) - u_t^j(\bar{l}_1,\bar{x})\end{aligned}$$

and, hence,

$$u_t^j(\bar{l}_1,\bar{x}) < v_t^j(\bar{l}_1,\bar{x}). \tag{2.6}$$

On the other hand, by IV and (2.4) the function $u^j(\cdot,\bar{x}) - v^j(\cdot,\bar{x})$, defined for t in the interval $]l_1,\bar{l}_2)$ attains its maximum at $t = \bar{l}_1$. Hence we have $u_t^j(\bar{l}_1,\bar{x}) \ge v_t^j(\bar{l}_1,\bar{x})$ in contradiction with (2.6). This completes the proof.

In the same manner we proof the theorem if $u \in \mathscr{P}_0^+(f)$ and $f \uparrow -v$.

Remark 1. Using the notations of [4], if $\sum_i^* = \emptyset$ then Theorem 1 of [4] follows from Theorem 2.1.

Theorem 2.2. (weak inequalities). *Under assumptions 1,2 and 3 of Theorem 2.1 suppose that*

8) $Pu \le Pv$ *in* D_p,

9) $u \le v$ *on* Γ_p^+, $u \le v$ *on* Γ_∞ *and, for any* $j = 1,\dots,m$ $u^j \le v^j$ in $D_j \backslash D_p$.

10) *let* $I = (l_1,l_2)$ *(see* (2.2)*). There exists a sequence* y_ν *of real functions defined in* I *and a function* $K : I \times \mathbf{R} \to \mathbf{R}$ *such that the function*

$$z_\nu(t) = \int_{l_1}^{\tau} K(\tau,y_\nu(\tau))\,d\tau + \frac{1}{\nu}\cdot t \quad e \quad \bar{z}_\nu = \left(z_\nu + \frac{1}{\nu},\dots,z_\nu + \frac{1}{\nu}\right),$$

where $t \in I$ *and* $\nu \in \mathbf{N}$, *satisfies:*

$$0 \le z_\nu(t) < \infty,\ \lim_{\nu\to\infty} z_\nu(t) = 0,\ v + \bar{z}_\nu \in Z_0(f), \tag{I}$$

$$(\forall(t,x) \in D_p, \forall j = 1,\dots,m)$$

$$\begin{aligned}&(f^j(t,x,(v+\bar{z}_\nu)(t,x),v_x^j(t,x),v_{xx}^j(t,x),v+\bar{z}_\nu) - \\ &- f^j(t,x,v(t,x),v_x^j(t,x),v_{xx}^j(t,x),v) < K(t,y_\nu(t)),\end{aligned} \tag{II}$$

(III) $$v + \overline{z}_\nu \in \mathscr{P}_0^-(f) \quad (or\ v \in \mathscr{P}_0^+(f)),$$

(IV) $$f_+^{\uparrow} u \quad (or\ f_-^{\uparrow}(v + \overline{z}_\nu)).$$

Under these assumptions we have for any $j = 1, \ldots, m$ $u^j \leq v^j$ *in* D_j.

Proof. By observing that, for any $\nu \in \mathbf{N}$, the functions u and $v + \overline{z}_\nu$ satisfy all assumptions of Theorem 2.1, we have that

$$u^j < v^j + z_\nu + \frac{1}{\nu} \tag{2.6}$$

for $\nu \in \mathbf{N}$ and $j = 1, \ldots, m$.

Hence Theorem 2.2, follows from (2.6) and I.

Remark 2. By using notations of [4], if $\sum_i^* = \emptyset$ then Theorem 2 of [4] follows from Theorem 2.2. In fact if, for ν sufficiently large, y_ν denotes a solution of the problem

$$\begin{cases} y' = \sigma(t, y) + \frac{1}{\nu} \\ y(0) = \frac{1}{\nu} \end{cases}$$

moreover, we have

$$y_\nu(t) = \int_0^t \sigma(\tau, y_\nu(\tau))\,\mathrm{d}\tau + \frac{1}{\nu}(t+1).$$

Then, put $K(t, \nu) = \sigma(t, \nu)$, I, II and IV of Theorem 2.2 are true.

Remark 3. Assumptions of Theorem 2.2 are true for the functions defined in the introduction.

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