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A STRUCTURE THEOREM FOR ECHELON KOTHE SPACES

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INTRODUCTION

In [9], Lopez-Molina defined the echelon Köthe spaces $\Lambda^p(X,\beta,\mu,g_k)$, which provide a suitable generalization of the echelon sequence spaces $\lambda^p(a_n^k)$ to a general measure space (X,β,μ) (see also [5]).

In this paper, we show that the structure of the separable echelon Köthe spaces is «nicely» close to the structure of the echelon sequence spaces. Namely, our main result is:

Let $\Lambda^p(X,\beta,\mu,g_k)$ be a separable echelon Köthe space of order $p, 1 \leq p < \infty$, with (X,β,μ) purely non-atomic. Then, there is an echelon sequence space λ^p so that Λ^p is isomorphic to the space $\lambda^p(L^p)$ of all λ^p-summable sequences in L^p.

An an application we show that a separable echelon Köthe space has a basis (by a basis we mean a Schauder basis), which is unconditional if $p > 1$.

DEFINITIONS AND NOTATIONS

(X,β,μ) denotes a measure space. Given a function $f : X \to R$, we define the support of f as

$$S(f) = \{x \in X; f(x) \neq 0\}$$

Given a sequence $(g_k)_k$ of β-measurable R-valued functions defined on X, so that

$$0 \leq g_k(x) \leq g_{k+1}(x) \quad \mu - ae, \ \forall k \in N$$

and

$$\mu(\{x \in X; \quad g_k(x) = 0, \quad \forall k \in N\}) = 0$$

we define the echelon Köthe space of order $p, 1 \leq p < \infty$ (this condition is always assumed on p), as the space of all (equivalence classes of) β-measurable functions $f : X \to R$, such that

$$||f||_k = \left(\int_X |f|^p g_k \mathrm{d}\mu\right)^{1/p} < \infty, \quad \forall k \in N$$

This space will be denoted by $\Lambda^p(X,\beta,\mu,g_k)$ or by $\Lambda^p(g_k)$ (or by Λ^p if there is no risk of confusion) and it is endowed with the natural Fréchet topology defined by the seminorms $||\cdot||_k, k \in N$. Given a subset $A \in \beta, \chi_A$ denotes the characteristic function of A, and the sectional subspace of Λ^p by A is defined as

$$\Lambda^p/A = \{f \in \Lambda^p; S(f) \subset A\}$$

Denoting by β_A, μ_A and g_k/A, the restriction of β to A, the restriction of μ to β_A and the restriction of g_k to A, respectively, it is clear that Λ^p/A, with the induced topology, is isomorphic to the echelon Köthe space $\Lambda^p(A, \beta_A, \mu_A, g_k/A)$.

In this paper, we shall always assume that the echelon Köthe spaces $\Lambda^p(X, \beta, \mu, g_k)$ are separable. Thus, (X, β, μ) will be σ-finite (cf. [2]). Then, X is the disjoint union of a purely non-atomic set, say A, and a purely atomic set, B. It is clear that Λ^p is isomorphic to $\Lambda^p/A \oplus \Lambda^p/B$. is either finite dimensional or isomorphic to a sequence space λ^p, so its structure is well known. This is the reason why we assume from now on that (X, β, μ) is purely non-atomic.

Let I denote an index set (in this paper, I will be either $\mathbf{N}$ or $\mathbf{N}^2$), $(a_i^k)_{i \in I}$ a positive Köthe matrix on I (i.e., $0 \leq a_i^k \leq a_i^{k+1}, \forall i \in I, k \in N$, and for each $i \in I$, there is $k \in N$ so that $a_i^k > 0$), and E a Banach space; we fix the norm $\|\cdot\|$ which induces the topology of E and put

$$\lambda^p(E) = \lambda^p(a_i^k, E) := \{(x_i)_i \in E^I; \|(x_i)\|_k := \left(\sum_{i \in I} \|x_i\|^p a_i^k\right)^{1/p} < \infty, k \in N\}.$$

With the topology induced by the increasing sequence $(\|\cdot\|_k)_k$ of seminorms, $\lambda^p(E)$ is the Fréchet space of all λ^p-summable generalized sequences $(x_i)_{i \in I}$ in E.

The usual Lebesgue space on [0, 1] will be denoted by L^p.

For other standard notation and concepts we refer to [7] and [8].

RESULTS

Lemma 1 is the main tool in this paper.

Lemma 1. *Let $\Lambda^p(X, \beta, \mu, g_k)$ be a separable echelon Köthe space, so that*

(1) $$S(g_1) = X \quad \mu - ae$$

(2) $$g_k(x) < g_{k+1}(x) \quad \mu - ae, \quad \forall k \in N$$

and

(3) $$g_k(x) \leq M_k$$

for some $M_k \in R$ and each $k \in N, x \in X$.

Then, there is a sequence $(A_n)_n$ of β-measurable, pairwise disjoint sets, and for each $n \in N$, there is a sequence $(B_{nj})_{j \in N}$ of β-measurable, pairwise disjoint sets, and there are *scalars* $\alpha_n, a^k_{nj} \in R^+$, *with* $a^k_{nj} \leq a^{k+1}_{nj}$ *for every* $n, j, k \in N$, *such that*

$$(4) \qquad X = \bigcup_{n=1}^{\infty} A_n, \quad A_n = \bigcup_{j=1}^{\infty} B_{nj} \quad \mu - ae, \forall n \in N$$

$$(5) \qquad \operatorname*{essinf}_{x \in A_n} g_1(x) \geq \alpha_n > 0, \forall n \in N$$

and such that the functions

$$h_k = \sum_{n,j=1}^{\infty} a^k_{nj} \chi_{B_{nj}}$$

fulfil the following conditions, for each $k \in N$

$$(6) \qquad h_k \leq g_{k+1}, \quad \mu - ae$$

$$(7) \qquad h_k \leq g_k \quad \mu - ae \quad on \quad X - \bigcup_{i=1}^{k-1} A_i$$

$$(8) \qquad h_k \chi_{A_i} = h_i \quad i = 1, \ldots, k-1$$

and $\Lambda^p(X, \beta, \mu, g_k)$ *is isomorphic to*

$$(9) \qquad \Lambda^p(X, \beta, \mu, h_k)$$

Proof. By (1), it can be found a sequence $(A_n)_n \subset \beta$, of pairwise disjoint sets, and scalars $\alpha_n \in R^+, n \in N$, such that

$$\operatorname*{essinf}_{x \in A_n} g_1(x) \geq \alpha_n > 0$$

and $X = \bigcup_{n \geq 1} A_n$. Now, le us fix $i \in N$, and take A_i, then from (2) and (3) it is not hard to find, for each $k \in N$, β-measurable pairwise disjoint sets $B^k_{ij}, j \in N$, and a function

$$s_{k,i} = \sum_{j=1}^{\infty} a'^k_{ij} \chi_{B^k_{ij}}$$

so that $A_i = \bigcup_{j\geq 1} B^k_{ij}, \mu$-ae, and $g_k(x) < s_{k,i}(x) \leq g_{k+1}(x)\mu$-ae on A_i (cf. [6], (11.35)). Now, for each $k \in N$, the function h_k we are looking for is

$$h_k = \sum_{i=1}^{k} s_{i,i} + \sum_{i=k+1}^{\infty} s_{k,i}$$

and, for each $n \in N$, the sequence $(B_{nj})_{j\in N}$ can be found by arranging the following set

$$\Omega_n = \{B^1_{ni_1} \cap B^2_{ni_2} \cap \ldots \cap B^n_{ni_n};\ (i_1, i_2, \ldots, i_n) \in N^n\}$$

The reader can readily see that conditions (4) to (8) hold. Tho check (9), use (6) and note that

$$g_k \leq h_k \max\{m_k/\alpha_i; i = 1, \ldots, k-1\}$$

from (3), (5), (7) and (8). ∎

The next lemma, which we shall need below, has been proved in [4].

Lemma A. *Let $\Lambda^p(X, \beta, \mu, g_k)$ be a separable echelon Köthe space. Then, there is an echelon Köthe space $\Lambda^p(X, \beta, \mu, \Phi_k)$, isomorphic to $\Lambda^p(g_k)$, such that Φ_k is bounded for each $k \in N$ and*

$$\Phi_k(x) < \Phi_{k+1}(x) \quad \mu - ae \quad on \quad S(\Phi_{k+1}),\ \forall k \in N$$

We can now state and prove our main result

Theorem 2. *Given a separable echelon Köthe space $\Lambda^p(X, \beta, \mu, g_k)$ there is an echelon sequence space λ^p so that Λ^p is isomorphic to $\lambda^p(L^p)$.*

Proof. In the first place, let us assume that Λ^p has a continuous norm. This is equivalent to say that there is $k \in N$ (we can, and we shall, assume $k = 1$), such that $S(g_1) = X, \mu$-ae. At the end of the proof it will be showed how this assumption can be removed. Furthermore, from Lemma A we can assume, without loss of generality, that the sequence $(g_k)_k$ even fulfils conditions (2) and (3) in Lemma 1. Thus, by this Lemma, there are subsets $B_{nj}, n, j \in N$, and functions $h_k = \sum_{n,j=1}^{\infty} a^k_{nj} \chi_{B_{nj}}$, verifying (9).

Let us consider the echelon sequence space $\lambda^p = \lambda^p(a^k_{nj})$, and show that $\Lambda^p(X, \beta, \mu, h_k)$ is isomorphic to $\lambda^p(L^p)$.

Indeed, it is clear that $\Lambda^p(h_k)/B_{nj}$ is normable for each $n, j \in N$, and the norm $\left(\int_{B_{nj}} |f|^p \mathrm{d}\mu\right)^{1/p}$ defines its topology. Thus, since Λ^p is separable and (X, β, μ) is purely non-atomic (hence, the restricted measure space $(B_{nj}, \beta_{B_{nj}}, \mu_{B_{nj}})$ also is purely non-atomic) there is an isomorphism $\Phi_{nj} : \Lambda^p(h_k)/B_{nj} \to L^p$, such that

$$\|\Phi_{nj}(g)\|^p = \int_{B_{nj}} |g|^p \mathrm{d}\mu, \quad \forall g \in \Lambda^p(h_k)/B_{nj}$$

Let us now define the linear mapping

$$\Phi : \Lambda^p(h_k) \to \lambda^p(L^p), f \to (\Phi_{nj}(f\chi_{B_{nj}}))$$

For each $k \in N$, we get

$$\|\Phi(f)\|_k = \|(\Phi_{nj}(f\chi_{B_{nj}}))\|_k = \left(\sum_{n,j=1}^{\infty} \|\Phi_{nj}(f\chi_{B_{nj}})\|^p a_{nj}^k\right)^{1/p} =$$

$$= \left(\sum_{n,j=1}^{\infty} \int_{B_{nj}} |f|^p a_{nj}^k \mathrm{d}\mu\right)^{1/p} = \left(\int_X |f|^p \left(\sum_{n,j=1}^{\infty} a_{nj}^k \chi_{B_{nj}}\right)\right)^{1/p} = \|f\|_k$$

and it is straightforward that Φ is onto, so it is an isomorphism.

To finish the proof let us remove the earlier assumption on $S(g_1)$. If no continuous norm can be defined on Λ^p, then, by choosing a subsequence if it is necessary, we have that

$$\mu(S(g_k) - S(g_{k-1})) > 0, \quad \forall k \in N$$

where $g_0 \equiv 0$. Let us denote $S(g_i) - S(g_{i-1})$ by S_i, for each $i \in N$; we can then apply this theorem to Λ^p/S_i, hence there is an echelon sequence space $\lambda_i^p(b_{ni}^k)$, so that Λ^p/S_i is isomorphic to $\lambda_i^p(L^p)$, for each $i \in N$. Then, we define the Köthe matrix $(a_{nj}^k)_{n,j \in N}$, as $a_{nj}^k = b_{nj}^k$, if $j \le k$, and $a_{nj}^k = 0$ if $j > k, n, j, k \in N$. The proof is finished by checking the following (natural) isomorphism which are left to the reader

$$\Lambda^p \simeq \prod_{i=1}^{\infty} \Lambda^p/S_i, \quad \lambda^p(L^p) \simeq \prod_{i=1}^{\infty} \lambda_i^p(L^p)$$

∎

Notice that, in particular, for $p = 1$, a separable echelon Köthe space $\Lambda^1(X, \beta, \mu, g_k)$ is isomorphic to $\lambda^1 \hat{\otimes}_\pi L^1$, for some space λ^1 ([8], II, §41, 7.a), and, if $p = 2$, we get that every separable space $\Lambda^2(X, \beta, \mu, g_k)$ is isomorphic to some echelon sequence space λ^2.

APPLICATIONS

In this section we are mainly concerned with showing that a separable echelon Köthe space Λ^p has a basis, which is unconditional if $p > 1$. This will follow from Theorem 2 and the next Lemma

Lemma 3. *Let λ^p be an echelon sequence space, $1 \leq p \leq \infty$, and let E be a Banach space with a basis (resp. an unconditional basis). Then $\lambda^p(E)$ has a basis (resp. an unconditional basis).*

Proof. Let $(x_n)_n$ be a basis in E. The key to the proof is to consider on E, a norm $P(.)$ (equivalent to the norm of E) for which $'x_n)_n$ is a monotone basis. Then, it readily follows that the sequence: $h_1 = (x_1, 0, \ldots), h_2 = (x_2, 0, \ldots), h_3 = (0, x_1, 0, \ldots), h_4 = (x_3, 0, \ldots), h_5 = (0, x_2, 0, \ldots), h_6 = (0, 0, x_1, 0, \ldots), \ldots$ and so on, is a basis in $\lambda^p(E)$. If $(x_n)_n$ is an unconditional basis, then one consider on E a norm $Q(.)$ for which the basis $(x_{\pi(n)})_{n \in N}$ is a monotone basis for every permutation π of the positive integers (cf. [1], IV, §4, Th. 1). Then one gets that $(h_n)_n$ is an unconditional basis in $\lambda^p(E)$. The details are left to the reader. ■

As a consequence, we state the next

Corollary 4. *A separable echelon Köthe space $\Lambda^p(X, \beta, \mu, g_k)$ has a basis, which is unconditional if $p > 1$.*

Remark. The author has proved in [2] that a separable Fréchet space F possesses an unconditional basis only if it contains no copy of L^1. So, the spaces $\lambda^1(L^1)$ have no unconditional basis.

To finish this paper we shall give a further application of Theorem 2. Before starting, we need the following definition. Given a Fréchet space F and fixed a sequence $(\|.\|_k)_k$ of seminorms generating the topology of F, we define

$$l^p(F) = \{(x_n) \in F^N; (\|x_n\|_k)_n \in l^p, \ \forall k \in N\}.$$

$l^p(F)$ is a Fréchet space with the topology induced by the seminorms $\|(x_n)\|_k = \left(\sum_{n=1}^{\infty} \|x_n\|_k^p\right)^{1/p}$, $k \in N$. It is easy to see that $l^p(l^p(F)) \simeq l^p(F) \simeq F \oplus l^p(F)$, and $l^p(F \oplus G) \simeq l^p(F) \oplus l^p(G)$, F and G being Fréchet spaces.

Proposition 5. *Let $\Lambda^p(X, \beta, \mu, g_k)$ be a separable echelon Köthe space. Then $l^p(\Lambda^p)$ is isomorphic to Λ^p.*

Proof. Indeed, from Theorem 2, Λ^p is isomorphic to $\lambda^p(L^p)$ for some echelon sequence space λ^p. Then, we get

$$l^p(\Lambda^p) \simeq l^p(\lambda^p(L^p)) \simeq \lambda^p(l^p(L^p)) \simeq \lambda^p(L^p) \simeq \Lambda^p$$

∎

Now, the Pelczynski's decomposition method and Proposition 4 allow us to show the following

Corollary 6. *Let F be a complemented subspace of a separable echelon Köthe space $\Lambda^p(X, \beta, \mu, g_k)$. Then F is isomorphic to Λ^p if it has a complemented copy of the whole space.*

Remark. We point out that, in general, an echelon Köthe space Λ^p is not primary (cf. [3]).

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