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ON SOME RELATIONS BETWEEN LINEAR AND HOLOMORPHIC MAPPINGS ON BANACH SPACES

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Abstract. *Three aspects of the theory of holomorphic mappings between Banach spaces are presented in connection with the theory of continuous linear operators: (1) the behavious of $H(E,F)$ versus the approximation property on E; (2) extension theorems for holomorphic mappings (holomorphic Hahn-Banach theorems); (3) holomorphic mappings and quotients.*

1. INTRODUCTION

Our purpose in this survey is to examine some relations between the space $L(E,F)$ of all continuous linear operators from E into F and the space $H(E,F)$ of all holomorphic mappings from E into F. We will try to be reasonably clear even for people who are not specialists in infinite dimensional holomorphy. No completeness is claimed. Although most of the definitions and results which appear here are true for more general settings, we will be concerned with Banach spaces. We cover only a few extensions of the linear theory to the holomorphic case. For instance, we omit the holomorphic classification of locally convex spaces, which has had a big development in the last ten years. The results of this theory concerning Banach spaces can be found in a nice self-contained survey by Barroso, Matos and Nachbnin [9].

2. NOTATIONS AND DEFINITIONS

Let $E, F, G, \ldots$ be Banch spaces over $\mathbb{K}, B_E$ be the closed unit ball of E and $id_E : E \to E$ be the identity mapping. For every mapping $f : E \to F$ and $A \subset E$ $\|f\|_A := \sup\{\|f(x)\| : x \in A\}$. For $n = 1, 2, \ldots, L(^nE, F)$ is the space of all continuous n-linear mappings from E into F. The space $\mathcal{P}(^nE, F)$ of all continuous n-homogeneous polynomials from E into F is defined by $\mathcal{P}(^nE, F) := \{\hat{A}: x \in E \mapsto A(x, \ldots, x) \in F |\ A \in L(^nE, F)\}$. If $n = 0, L(^0E, F) = \mathcal{P}(^0E, E)$ is the space of all constant mappings from E into F and can be identified with F in a natural fashion. If $n = 1, L(^1E, F)$ coincides with $\mathcal{P}(E, F)$ and is denoted by $L(E, F)$. The space $\mathcal{P}(^nE, F)$ normed by $P \mapsto \|P\| := \sup\{\|P(x)\| : \|x\| \leq 1\}$ is denoted by $\mathcal{P}(^nE, F)_\beta$. In particular, $(L(E,F), \| \|)$ stands for $\mathcal{P}(^1E, F)_\beta$. It is well known that $\mathcal{P}(^nE, F)_\beta$ is a Banach space for all $n \in \mathbf{N}$. Let U be an open subset of E. A mapping $f : U \to F$ is called G-holomorphic (Gateaux-holomorphic) on U if for every $a \in U$ and $b \in E$ and for all $\varphi \in F'$ the function $\lambda \in \{\lambda \in \mathbb{C} : a + \lambda b \in U\} \mapsto \varphi \circ f(a + \lambda b) \in \mathbb{C}$ is analytic in the open set $\{\lambda \in \mathbb{C} : a + \lambda b \in U\}$. A mapping $f : U \to F$ is holomorphic on U if it is G-holomorphic and continuous on U. The spaces of all holomorphic mappings from U into F is denoted by $H(U, F)$. It is well known that $f \in H(U, F)$ if and only if there exists a power series $\sum_{n=0}^{\infty} P_n(y-x)$ with $P_n \in \mathcal{P}(^nE, F)$ for each $n \in \mathbb{N}$, which converges uniformly to $f(y)$ in a neighbourhood of x. Such a series is necessarily unique and for each n, P_n is called the n^{th} Taylor series coefficient of f at x and is denoted by $(1/n!)\,\hat{d}^n f(x)$. The compact open topology on $H(U,F)$, denoted by $\mathcal{T}_0$, is the locally convex topology generated by the seminorms $f \mapsto ||f||_K$ where K ranges over the compact subset of U. A seminorm p on $H(U,F)$ is said to be ported by the compact subsets K of U if, for every open set $V, K \subset V \subset U$, there exists $c(V) > 0$ such that $p(f) \leq c(V)||f||_V$ for all $f \in H(U,F)$. The $\mathcal{T}_\omega$ topology on $H(U,F)$ is the locally convex topology generated by all seminorms ported by compact subsets of U. The symbol $H_b(U,F)$ denotes the space of all $f \in H(U,F)$ such that f is bounded on bounded sets, with the topology $\mathcal{T}_b$ of uniform convergence on bounded sets. As usual, the dual E' of E endowed with the strong topology is denoted by E'_β. By definition $E'' := (E'_\beta)'_\beta$. The symbol E'_{co} stands for the dual of E endowed with the topology of uniform convergence on all absolutely convex compact sets in E.

For further notations and basic results we refer to [16].

3. THE APPROXIMATION PROPERTY

Grothendieck states in [18] the following

Definition 1. *A Banach space E has the* approximation property *(a.p.) if for each compact subset K of E and each $\varepsilon > 0$ there exists a $T \in E' \otimes E$ such that $||Tx - x|| < \varepsilon$ for all $x \in K$.*

The study of spaces having this Grothendieck's approximation property was started and developed in [18], chap. I, §5 and plays a very important role in functional analysis. Most of the locally convex spaces occuring in analysis do have the a.p.

The first example of a (Banach) space failing to have the a.p. is due to P. Enflo (cf. [17]) and involves a quite difficult construction. We recall the following results concerning the a.p. in linear functional analysis on Banach spaces, whose proofs can be found in [18]:

Theorem 2. *For a Banach space E, the following are equivalent:*

(a) E has the approximation property.

(b) $E' \otimes F$ is dense in $(L(E,F), \mathcal{T}_0)$ for all Banach spaces F.

(c) E'_{co} has the approximation property.

(d) $F' \otimes E$ is dense in $(L(F,E), \mathcal{T}_0)$ for all Banach spaces F.

The a.p. is a basic tool in functional analysis. It is well known that $(C(E), \mathcal{T}_0)$ has the a.p., since there are continuous partitions of unity. This does not occur with $(H(E), \mathcal{T}_0)$. Aron and Schottenloher prove in [7] that $(H(E), \mathcal{T}_0)$ has the a.p. if and only if E has the

a. p. (This is the holomorphic analogue of the equivalence between (a) and (c) in Theorem 2). They establish also in [8] an exact analogy between the question of approximation of the identity of E by elements of $E' \otimes E$, uniformly on compact sets, and the question of approximation of the identity of E by elements of $H(E) \otimes E$, uniformly on compact sets. More precisely, the following Theorem 3, due to Aron and Schottenloher (cf. [8]), gives the holomorphic analogue of Theorem 2:

Theorem 3. *For a Banach space E, the following are equivalent:*

(a) For each compact subset K of E and each $\varepsilon > 0$ there exists a (finite rank) holomorphic mapping $g \in H(E) \otimes E$ such that $\|g(x) - x\| < \varepsilon$ for all $x \in K$.

(b) $H(E) \otimes F$ is dense in $(H(E,F), \mathscr{T}_0)$, for all Banach spaces F.

(c) $(H(E), \mathscr{T}_0)$ has the approximation property.

(d) E has the approximation property.

(e) $H(V) \otimes E$ is dense in $(H(V,E), \mathscr{T}_0)$, for all Banach spaces F and non empty open subsets $V \subset F$.

The proof of Theorem 3 uses the isomorphism $(H(E), \mathscr{T}_0)\varepsilon F \cong (H(E,F), \mathscr{T}_0)$ which was proved, independently, by Aron [1] and Schottenloher [29].

Definition 4. *A mapping f from E into F is* compact *if for each $x \in E$, there is a neighbourhood V of x such that $f(V)$ is relatively compact in F.*

We denote by $\mathfrak{F}_K(E,F)$ the vector space of all compact mappings from E into F. The intersections of $\mathfrak{F}_K(E,F)$ with $L(E,F), \mathscr{P}(^nE,F)$ and $H(E,F)$ will be denoted respectively by $L_K(E,F), \mathscr{P}_K(^nE,F)$ and $H_K(E,F)$.

The following equivalence are well known in the linear functional analysis and can be found, for instance, in [18]:

(1) A Banach space E has the a.p. if and only if $F' \otimes E$ is a dense subspace of $(L_K(F,E), \|\ \|)$ for every Banach space F.

(2) If E is a Banach space, E' has the a.p. if and only if $E' \otimes F$ is a dense subspace of $(L_K(E,F), \|\ \|)$ for every Banach space F.

In the holomorphic case, Aron and Schottenloher prove in [8] the following:

Theorem 5. *For a Banach space E, the following results hold:*

(a) E has the a.p. if and only if $H(F) \otimes E$ is dense in $(H_K(E,F), \mathscr{T}_\omega)$ for every Banach space F.

(b) $(H(E), \mathscr{T}_\omega)$ has the approximation property if and only if $H(E) \otimes F$ is dense in $(H_K(E,F), \mathscr{T}_\omega)$ for every Banach space F.

In order to prove Theorem 5, Aron and Schottenloher establish a topological isomorphism between $(H_K(E,F), \mathscr{T}_\omega)$ and $(H(E), \mathscr{T}_\omega)\varepsilon F$.

4. THE HAHN-BANACH THEOREM

Given a closed subspace E of a Banach space G the initial conjecture was: is every element of $H(E)$ the restriction to E of an element of $H(G)$? This conjecture is contained in a more general question of Dineen in [15] and concerns an attempt to find a holomorphic Hahn-Banach extension theorem. In this paragraph we will see that it is not possible to set a general Hahn-Banach theorem in the case of closed subspaces of Banach spaces (even in the case $E \subset G = E''$) but it is reasonable to look for suitable rich classes of holomorphic mappings on E where such extension theorems are true.

Dineed proves in [14] the following very useful result:

Proposition 6. *If E is a Banach space and $(\varphi_n)_{n\in\mathbb{N}} \subset E'$ then $\sum_{n=1}^{\infty} \varphi_n^n \in H(E)$ if and only if $\varphi_n(x) \to 0$ as $n \to \infty$ for every $x \in E$ (i.e. $\varphi_n \to 0$ in the $\sigma(E', E)$ -topology).*

In 1975 the following theorem was proved independently by B. Josefson [20] and A. Nissenzweig [27]:

Theorem 7. *If E is an infinite Banach space, then there exists a sequence $(\varphi_n)_{n\in\mathbb{N}} \subset E'$ such that $\|\varphi_n\| = 1$ for all $n \in \mathbb{N}$ and $\varphi_n(x) \to 0$ as $n \to \infty$ for every $x \in E$.*

A nice proof of Theorem 7 can be find in [11] Cap. XII, p. 219.

This deep result is interesting for many fields of functional analysis. Concerning holomorphic mappings, Dineen proves (cf. [14], Proposition 5 and [20] Corollary 2) the following consequence of Theorem 7 and Proposition 6:

Corollay 8. *If E is an infinite dimensional Banach space then there exists $f \in H(E)$ such that $r_f(0) := \sup\{|\lambda| : \lambda \in \mathbb{C}$ and $\|f\|_{\lambda B_E} < \infty\} = 1$.*

Corollary 8 shows that if E is an infinite dimensional Banach space there exists a function $f \in H(E)$ which is not bounded on the bounded subsets of E. Sets $A \subset E$ such that $\|f\|_A < \infty$ for all $f \in H(E)$ arise naturally in problems of analytic continuation, construction of envelope of holomorphy and in problems concerning topologies on $H(U)$. Such sets are called bounding sets. More exactly we have

Definition 9. *Let U be an open subset of a Banach space E. A subset A of U is said to be bounding for U if $\|f\|_A < \infty$ for every $f \in H(U)$.*

Schottenloher has written a nice survey article on bounding sets (see [28]). We refer also to Dineen [16] §4.2 and for recent results to [30] and [31].

It is clear that every relatively compact subset of E is bounding. The first example of a non-relatively compact bounding subset of a Banach space is due to Dineen. He shows in

[12] that if $u_n := (\delta_{nm})_m$ for each $n \in \mathbb{N}$ and $A = \bigcup_{n=1}^{\infty}\{u_n\}$, then A is a closed bounded non-compact subset of ℓ_∞ which is a bounding subset of ℓ_∞. His proof motivated Josefson to undertake a deep analysis of the geometry of ℓ_∞. He states in [21], Theorem 1 & Corollaries 1 and 4, the following:

Theorem 10. *(a) If A is a bounded subset of ℓ_∞ then the following conditions are equivalent:*

(i) A is a bounding subset of ℓ_∞,

(ii) every sequence $(\varphi_n)_n \subset \ell'_\infty$ which converges pointwise to zero converges uniformly to zero on A,

(iii) there is no sequence $(a_n)_n$ in A which is equivalent to the unit vector basis in ℓ_1,

(iv) there is no continuous linear mapping $T : \ell_1 \to \ell_\infty$ with continuous inverse $T^{-1} : T(\ell_1) \to \ell_1$, such that $T(B) \subset \overline{Convex\ Hull\ of\ A}$ where B is the unit ball of ℓ_1.

(b) The convex hull of a bounding subset of ℓ_∞ is bounding.

(c) Every bounded subset of c_0 is a bounding subset of ℓ_∞.

From Theorem 10 (c) we infer that a general holomorphic Hahn-Banach theorem is not possible in the case of Banach spaces. More precisely, if E is a closed subspace of a Banach space G, it may happen that there exists a holomorphic function on E which can not be extended to a holomorphic function on G, even in the case $G = E'' \supset E$. Indeed, for $E = c_0$ and $G = \ell_\infty$, Theorem 10 (c) and Corollary 8 imply that there exists $f \in H(c_0)$ which can not be the restriction to c_0 of an element of $H(\ell_\infty)$.

The holomorphic Hahn-Banach theorem in the case of Banach spaces was studied first by Aron and Berner in [3] and Aron in [2]. The main result in [3] states that given a pair of Banach spaces $E \subset G$, the existence of extensions for various types of holomorphic mappings is equivalent to the existence of a continuous extension mapping from E' to G', which is equivalent to the existence of a continuous linear mapping $S : G \to E''$ such that $S|_E = id_E$ (cf. [3], Theorem 1.1). As a consequence of Theorem 1.1 of [3], Aron and Berner show that a holomorphic function $f : c_0 \to \mathbb{C}$ can be extended to a holomorphic function $g : \ell_\infty \to \mathbb{C}$ if and only if f is bounded on every bounded subset of c_0 (cf. [3], Proposition 1.1). Therefore, it is reasonable to look for suitable "rich" classes of holomorphic mappings on E where an extension theorem is true.

Let $\mathscr{P}_f({}^nE)$ be the span of $\{\varphi^n : x \in E \mapsto (\varphi(x))^n \mid \varphi \in E'\}$. The closure of $\mathscr{P}_f({}^nE)$ in $\mathscr{P}({}^nE)_\beta$ is denoted by $\mathscr{P}_c({}^nE)$. For details, we refer to Gupta [19]. By definition $\mathscr{P}_c({}^nE, F)$ is the closure of $\mathscr{P}_f({}^nE) \otimes F$ in $\mathscr{P}({}^nE, F)_\beta$. Now, if U is an open subset of E let $H_c(U, F) := \{f \in H(U, F) : \hat{d}^n f(x) \in \mathscr{P}_c({}^nE, F)$ for all $n \in \mathbb{N}$ and $x \in U\}$. Finally, we set $H_{cb}(E, F) := H_c(E, F) \cap H_b(E, F)$. Aron and Berner prove in [3] the following

Proposition 11. *Let E and F be Banach spaces. If $U \subset E$ is open and non empty and $f \in H_c(U,F)$ then there exists an open set $W \subset E''$ and $\tilde{f} \in H_c(W,F)$ such that $U \subset W$ and $\tilde{f}|_U = f$. Furthermore, there is a strict morphism $T : H_{cb}(E,F) \to H_{cb}(E'',F)$ such that $T_f/E = f$ for all $f \in H_{cb}(E,F)$.*

Aron and Berner's general approach is to extend to the whole space G the n-homogeneous polynomial $\hat{d}f(y)$ defined on E, and then use local Taylor representations to extend the holomorphic function locally. It is necessary to show that the local extensions are «coherent on the overlaps». This can be done if there is a linear and continuous mapping extending the elements of $\mathcal{P}_c({}^nE,F)$ to elements of $\mathcal{P}_c({}^nG,F)$.

Moraes defines in [22] the space $\mathcal{P}_{f^*}({}^nE'')$ as the span of $\{\varphi^n : x \in E' \mapsto (\varphi(x))^n \mid \varphi \in E'\}$ and $\mathcal{P}_{c^*}({}^nE'',F)$ as the closure of $\mathcal{P}_{f^*}({}^nE'') \otimes F$ in $\mathcal{P}({}^nE'',F)_\beta$. In particular, $\mathcal{P}_{c^*}({}^nE'')$ is the closure of $\mathcal{P}_{f^*}({}^nE'')$ in $\mathcal{P}({}^nE'')_\beta$. Finally, if W is an open subset E'', let $H_{c^*}(W,F) := \{f \in H(W,F) : \hat{d}^nf(x) \in \mathcal{P}_{c^*}({}^nE,F)$ for all $n \in \mathbb{N}$ and $x \in W\}$ and $H_{c^*b}(E'',F) := H_{c^*}(E'',F) \cap H_b(E'',F)$. In [22] Moraes proves that for every $n \in \mathbb{N}$ there exists an isomorphism onto $T_n : \mathcal{P}_c({}^nE,F) \to \mathcal{P}_{c^*}({}^nE'',F)$ such that $T_nP)/E = P$ for every $P \in \mathcal{P}_c({}^nE,F)$. Using the same techniques as Aron and Berner [3], Lemma, Moraes improves in [23] Proposition 2.2 of [3] by proving the

Proposition 12. *Let E and F be Banach spaces. If $U \subset E$ is open and non empty and $f \in H_c(U,F)$, then there exists an open set $W \subset E''$ and a* unique *$\tilde{f} \in H_{c^*}(W,F)$ such that $U \subset W$ and $\tilde{f}/U = f$. Furthermore, there is an* isomorphism *$T_n : H_{cb}(E,F) \to H_{c^*b}(E'',F)$ such that $(Tf)|_E = f$ for all $f \in H_{cb}(E,F)$.*

Consider now the spaces $\mathcal{P}_w({}^nE,F) := \{P \in \mathcal{P}({}^nE,F) : P/B$ is $\sigma(E,E')$-continuous for every $B \subset E$ bounded$\}$ and $\mathcal{P}_{wu}({}^nE,F) := \{P \in \mathcal{P}({}^nE,F) : P/B$ is $\sigma(E,E')$-uniformly continuous for every $B \subset E$ bounded$\}$. Aron, Hervés and Valdivia prove in [4] that $\mathcal{P}_w({}^nE,F) = \mathcal{P}_{wu}({}^nE,F)$ if E and F are Banach spaces (cf. Theorem 2.9 of [4]). If we define $P(x) := \sum_{n=1}^{\infty} x_n^2$ for all $x = (x_n) \in \ell_2$, it is easy to see that $P \in \mathcal{P}({}^2\ell_2) \setminus \mathcal{P}_w({}^2\ell_2)$.

Aron and Prolla prove in [6] that E' has the Grothendieck approximation property if and only if for all Banach spaces F and $m \in \mathbb{N}$, $\mathcal{P}_w({}^mE,F) = \mathcal{P}_c({}^mE,F)$ (cf. Proposition 2.9 of [6]). Moraes defines in [22] the space $\mathcal{P}_{w^*u}({}^nE'') := \{P \in \mathcal{P}({}^nE'') : P/B^{oo}$ is $\sigma(E'',E')$-uniformly continuous for every $B \subset E$ bounded$\}$. It is clear that in the case of Banach spaces, $P \in \mathcal{P}_{w^*u}({}^nE'')$ if and only if $P \in \mathcal{P}({}^nE'')$ and P/X is $\sigma(E'',E')$-uniformly continuous for every $X \subset E''$ bounded. In [22], Moraes proves that for every $n \in$

$\mathbf{N}$ there exists an isomorphism onto $T_n : \mathscr{P}_{wu}({}^nE) \to \mathscr{P}_{w^*u}({}^nE'')$ such that $(T_nP)/E = P$ for every $P \in \mathscr{P}_{wu}({}^nE)$. If we define $H_{wu}(E) := \{f \in H(E) : \hat{d}^n f(x) \in \mathscr{P}_{wu}({}^nE)$ for all $n \in \mathbf{N}\}$ and $H^{wu}(E) := \{f \in H(E) : f/B$ is $\sigma(E, E')$-uniformly continuous for every $B \subset E$ bounded $\}$, the Cauchy inequalities imply $H^{wu}(E) \subset H_{wu}(E)$. Analogously Moraes defines in [23] and [24] the spaces $H_{w^*u}(E'') := \{f \in H(E'') : \hat{d}^n f(x) \in \mathscr{P}_{w^*u}({}^nE'')$ for all $n \in \mathbf{N}\}$ and $H^{w^*u}(E'') := \{f \in H(E'') : f/X$ is $\sigma(E'', E')$-uniformly continuous for every $X \subset E''$ bounded$\}$. It is easy to see that $H^{wu}(E) \subset H_b(E)$ and $H^{w^*u}(E'') \subset H^{wu}(E'') \subset H_b(E'')$. Using this fact and Lemma 3 of [24] we get $H^{w^*u}(E'') = H_{w^*u}(E'')$. Analogously, $H^{wu}(E) = H_{wu}(E)$. By Proposition 8 of [23] and Lemma 3 of [24] we get:

Proposition 13. *If E is a Banach space, there exists a continuous isomorphism $T : (H^{wu}(E), \mathscr{T}_b) \to (H^{w^*u}(E''), \mathscr{T}_b)$ such that $(Tf)/E = f$ for all $f \in H^{wu}(E)$.*

Remark 14. *It is clear that in Proposition 13 we may substitute E'' by any Banach space G such that $E \subset G$ and there exists $S : G \to E''$ linear continuous with $S/E = id_E$. In particular we know that E is an $\mathscr{L}_\infty$-space in the sense of Lindenstrauss and Pelczynski if and only if for every G which contains E as a subspace there exists $S : G \to E''$ linear, continuous such that $S/E = id_E$ (cf. [25], example 2(c)). The spaces $c_0, \ell_\infty, L_\infty(\mu)$ and $C(K)$ are examples of such spaces.*

5. HOLOMORPHIC FUNCTIONS AND QUOTIENTS

Let F be a subspace of the Banach space E. In this paragraph we are concerned with the generalization, to the nonlinear setting of holomorphic functions on E, of the following basic identities relating the quotient space with the whole space:

$$E'/F^{\perp} \cong F' \tag{1}$$

and

$$(E/F)' \cong F^{\perp}. \tag{2}$$

Let $H^{\perp}(F)$ and $H_b^{\perp}(F)$ denote, respectively, the space of all elements of $H(E)$ such that $f/F = 0$ and the space of all elements of $H_b(E)$ such that $f/F = 0$.

As a consequence of Theorem 1.1 of [3], Moraes states the isomorphism $H_b(E)/H_b^{\perp}(F) \cong H_b(F)$ in the case when F is a closed subspace of E such that there exists a continuous linear mapping $S : E \to F''$ with $S/F = id_F$ (cf. Proposition 1 of [25]). This result and the following proposition are holomorphic versions of (1) due to Moraes:

Proposition 15. *Let E be a Banach space and suppose that F is a closed subspace of E such that there exists a continuous linear mapping $S : E \to F''$ with $S/F = id_F$. We consider on $H(E)$ the topology of uniform convergence on the bounding subsets of E. Then the following are equivalent:*

(1) every bounded subset of F is a bounding subset for E.

(2) $H(E)/H^{\perp}(F) \cong H_b(F)$.

(3) $H(E)/H^{\perp}(F)$ is a Fréchet space.

The proof of Proposition 15 uses Theorem 1.1 of [3] and Baire's theorem and can be found in [25]. As a consequence of Proposition 15 and Theorem 10 (c) we get $H(\ell_\infty)/H^{\perp}(c_0) \cong H_b(c_0) \cong H_b(\ell_\infty)/H_b^{\perp}(c_0)$. More generally this remains true if we substitute ℓ_∞ with any Banach space E which contains ℓ_∞ as a subspace (since c_0 is an $\mathscr{L}_\infty$-space).

Now let $\pi : E \to E/F$ denote the canonical quotient mapping. If U is an open subset of E, we say that a holomorphic mapping $f : U \to \mathbb{C}$ factors through the open set $\pi(U)$ if there exists a holomorphic mapping $g : \pi(U) \to \mathbb{C}$ such that $f = g \circ \pi$. Aron, Moraes and Ryan prove in [5] the following

Proposition 16. *Let U be a balanced non-empty subset of E, and let $f \in H(U)$. The following are equivalent:*

(1) f factors through $\pi(U)$

(2) $\mathrm{d}f(x) \in F^{\perp}$ for every $x \in U$.

Some properties of a holomorphic function carry over to its quotient. For $g \in H(E/F)$ let $f := g \circ \pi \in H(E)$. It is easy to see that $g \in H_b(E/F)$ if (and only if) $f \in H_b(E)$. In [5], Theorem 2.6 it is proved that $g \in H_{wu}(E/F)$ if $f \in H_{wu}(E)$; in particular, if $P \in \mathscr{P}_{wu}(^nE)$ and $P = Q \circ \pi$ for some $A \in \mathscr{P}(^nE/F)$, then $Q \in \mathscr{P}_{wu}(^nE/F)$. The space $H_{wu}(E)$ is contained in the space $H_{wsc}(E)$ of all holomorphic functions on E which are weakly sequentially continuous, and this inclusion is, in general, strict. The space $H_{wsc}(E)$ can behave badly under factorization. Indeed, if π is the quotient mapping of ℓ_1 onto ℓ_2, and we define $Q(x) := \sum_{n=1}^{\infty} x_n^2$ for all $x = (x_n)_n \in \ell_2$, it is easy to see that $Q \notin \mathscr{P}_{wsc}(^2\ell_2)$ unless $Q \circ \pi \in \mathscr{P}_{wsc}(^2\ell_1)$.

Topological questions are considered in paragraph 3 of [5]. The main result of this paper states:

Theorem 17. *Let U be a non-empty open subset of E.*

(i) The mapping $f \mapsto f \circ \pi$ is an isomorphism of $(H(\pi(U)), \mathscr{T}_0)$ onto a closed subspace of $(H(U), \mathscr{T}_0)$.

(ii) If U is balanced, the mapping $f \mapsto f \circ \pi$ is an isomorphism of $(H(\pi(U)), \mathscr{T}_\omega)$ onto a closed subspace of $(H(U), \mathscr{T}_\omega)$.

Here we have the following open problem:

Problem. *If U is a balanced open subset of E, is $f \mapsto f \circ \pi$ an isomorphism of $(H(\pi(U)), \mathcal{T}_\delta)$ onto a closed subspace of $(H(U), \mathcal{T}_\delta)$? (see [16] for the definition of $\mathcal{T}_\delta$).*

By a theorem of Dineen [13] we infer that $(H(U), \mathcal{T}_\omega)$ is bornological if U is a balanced subset of ℓ_1 and, since every separable Banach space is isomorphic to a quotient of ℓ_1, a positive answer to this problem would imply (via Theorem 3.4 and Corollary 3.5 of [5]) a positive answer to the conjecture that $\mathcal{T}_\omega$ is a bornological topology for balanced domains in separable Banach spaces [10].

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