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CURVES IN SUBSPACES OF RECURRENT FINSLER SPACES

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1. INTRODUCTION

In this paper we examine subspaces of recurrent and non-recurrent Finsler spaces as well as curves in these subspaces. Following A. Moór [13] a recurrent Finsler space $\overline{F}_N$ is defined as a Finsler space in which there exist vector fields λ and μ, the vector of recurrency of the metric tensor with respect to the covariant differentiation $|$ and $\left.\right|$, which are given below (see (2.2), (2.3)). In such a space two families of vector fields, $B^\alpha_a(x,\dot{x}), a = 1,2,\dots,M$ and $N^\alpha_k(x,\dot{x}), k = M+1,\dots,N$ are given which are mutually orthogonal (satisfing (2.8)). In $\overline{F}_N$ the generalized Cartan connections are obtained. Vectors dx and $\dot{x}$ are decomposed in the direction of $B^\alpha_a(x,\dot{x})$ and $N^\alpha_a(x,\dot{x})$. Decomposing the vectors DB^α_a and DN^α_k in the direction of B^α_a and N^α_k by using the method of O. Varga in [18], we introduce connection coefficients $\overline{\Gamma}^*$ and $\overline{A}$ (see (2.19)-(2.21) and (2.27), (2.28)). The existence of subspaces to which $B^\alpha_a(x,\dot{x})(N^\alpha_k(x,\dot{x}))$ are associated should not be assumed at this stage.

Let us denote by $T_H(T_V)$ the vector space spanned by $B^\alpha_a(x,\dot{x})(N^\alpha_k(x,\dot{x}))$. In §3 the conditions under which $DB^\alpha_a(DN^\alpha_k)$ are in T_H, T_V or equal to zero are examined. One of the interesting cases, examined in §5, will be determined as the so-called subspace of the third kind.

In §4 we investigate some special cases when the vector fields $B^\alpha_a(x)$ and $N^\alpha_k(x)$ are associated to subspaces, in the sense that they are the tangent vectors of these subspaces. The case when in a recurrent Finsler space $\overline{F}_N$ $B^\alpha_a = B^\alpha_a(x)$ and $dv^k = 0, v^k = 0$ (case 4.1.a) hold, will be denoted by $(\overline{F}_N, A)$. In this case $B^\alpha_a(x), a = 1,2,\dots M$, under special circumstances determine the subspaces, and the so-called induced connection coefficients $\overline{\Gamma}$ and $\overline{A}$ really reduce here to induced connection coefficients $\widetilde{\Gamma}$ and $\widetilde{A}$ of the subspace.

The space $(\overline{F}_N, B)$ is such a case of $(\overline{F}_N, A)$ in which the vector of recurrency μ is equal to zero, i.e. $g_{\alpha\beta}\big|_\gamma = 0$. $(\overline{F}_N, C)$ is such a case of $(\overline{F}_N, B)$ in which $\lambda = 0$ i.e. $g_{\alpha\beta|\gamma} = 0$. This means that $(\overline{F}_N, C)$ is the non-recurrent Finsler space F_N supplied with Cartan connection coefficients, in which the vector fields $B^\alpha_a(x)$ determine subspaces. The induced connection coefficients $\widetilde{\Gamma}$ and $\widetilde{A}$ are obtained in these cases.

In all former investigations concerning the subspace of the Finsler space, the equation $x^\alpha = x^\alpha(u^1, u^2, \dots, u^M)$ of the subspace was given. Here we will give the differential equation $\partial x^\alpha / \partial u^a = B^\alpha_a(x)$ of the subspace and the results of this paper are valid for any subspace $x^\alpha = x^\alpha(u^1, \dots, u^M, C_{M+1,0}, \dots, C_{N_o})$ which is the solution of the former differential equation. It is a generalization of the former problem, but there are also some restrictions. Here, in §5, in the investigation of curves in the subspace the element of support is $(u, du/ds)$, where $\ell^a = du^a/ds$ is the tangent vector to the curve $u^a = u^a(s)$ of the subspace. From this it follows that the definition of the path and h-path as given in [10] and [11] coincides both in the recurrent and in the non-recurrent Finsler space. From this it follows that here the subspaces of the first and second kind are also the same. We restrict our investigation only to the case when the enveloping space $(\overline{F}_N, A)$ is supplied with generalized Cartan connection coefficients.

The curvature vector of the curve in the subspace is defined as $D\ell^\alpha/ds$ and its part in $T_H(T_V)$ is the tangent (normal) curvature vector. The tangent curvature K_b (the normal curvature K_k) is the projection of $D\ell^\alpha/ds$ on B_b^β (on N_k) as given in Definition 5.1. In §5 relations between these curvatures and curvature vectors in the cases $(\overline{F}_N, A)$, $(\overline{F}_N, C)$ are obtained.

Definition of subspaces of the first and third kind are given and a condition for a subspace to be such is given as well. The definitions given here are generalizations of defintions given in [10] and [11], which are related to the hypersurfaces of the non-recurrent Finsler space F_N. The generalization concerns the dimension of subspace and the kind of enveloping space.

2. INDUCED CONNECTION COEFFICIENTS IN A RECURRENT FINSLER SPACE

In a Finsler space let the metric function be $L(x,\dot{x})$ and, then, the metric tensor is determined as usually by

$$(2.1) \qquad g_{\alpha\beta}(x,\dot{x}) = \dot{\partial}_\alpha\dot{\partial}_\beta F(x,\dot{x}), \quad F(x,\dot{x}) = 2^{-1}L^2(x,\dot{x})$$

$$\alpha,\beta,\gamma,\ldots = 1,2,\ldots,N.$$

Definition 1.1. *The recurrent Finsler space, $\overline{F}_N$, is a Finsler space in which there exist vector fields $\lambda_\gamma(x,\dot{x})$ and $\mu_\gamma(x,\dot{x})$ homogeneous of degree zero in $\dot{x}$, such that*

$$(2.2) \qquad g_{\alpha\beta|\gamma} = \lambda_\gamma g_{\alpha\beta}$$

$$(2.3) \qquad g_{\alpha\beta}\big|_\gamma = \mu_\gamma g_{\alpha\beta}.$$

As

$$Dg_{\alpha\beta} = g_{\alpha\beta|\gamma}dx^\gamma + g_{\alpha\beta}\big|_\gamma Dl^\gamma \quad (l^\gamma = L^{-1}(x,\dot{x})\cdot\dot{x}^\gamma)$$

from (2.2) and (2.3), we obtain

$$Dg_{\alpha\beta} = K(x,\dot{x},dx,Dl)\,g_{\alpha\beta},$$

where

$$K(x,\dot{x},dx,Dl)=\lambda_\gamma(x,\dot{x})\,dx^\gamma+\mu_\gamma(x,\dot{x})\,Dl^\gamma.$$

The absolute differential of $g_{\alpha\beta}$ is determined by

$$Dg_{\alpha\beta}=dg_{\alpha\beta}-(\Gamma^{*\delta}_{\alpha\gamma}g_{\delta\beta}+\Gamma^{*\delta}_{\beta\gamma}g_{\alpha\delta})\,dx^\gamma-(A^{\delta}_{\alpha\gamma}g_{\delta\beta}+A^{\delta}_{\beta\gamma}g_{\alpha\delta})\,Dl^\gamma,$$

where

$$Dl^\alpha=dl^\alpha+\Gamma^{*\alpha}_{0\gamma}dx^\gamma+A^{\alpha}_{0\gamma}Dl^\gamma. \tag{2.4}$$

The connection coefficients are determined in [3] under conditions

$$\text{(a) } \Gamma^*_{\alpha\beta\gamma}=\Gamma^*_{\gamma\beta\alpha} \qquad \text{(b) } A_{\alpha\beta\gamma}=A_{\gamma\beta\alpha}. \tag{2.5}$$

A. Moór, who first introduced the recurrent Finsler spaces, had the condition $A_{\alpha\beta\gamma}=A_{\beta\alpha\gamma}$ ([13]) instead of (2.5.b).

Let us define for any quantity $T_{\alpha\beta\gamma}$ the following expression

$$\{T_{\gamma\alpha\beta}\}=T_{\gamma\alpha\beta}+T_{\alpha\beta\gamma}-T_{\beta\gamma\alpha}.$$

Then we have

$$\begin{aligned}
&\text{(a) } 2\Gamma^*_{\alpha\beta\gamma}=\{\partial_\gamma g_{\alpha\beta}-L\dot{\partial}_\delta g_{\alpha\beta}\Gamma^{*\delta}_{0\gamma}-\lambda_\gamma g_{\alpha\beta}\}\\
&\text{(b) } 2\Gamma^*_{0\beta\gamma}=2\gamma_{0\beta\gamma}-L\dot{\partial}_\delta g_{\beta\gamma}\Gamma^{*\delta}_{00}-(\lambda_\gamma l_\beta+\lambda_0 g_{\beta\gamma}-\lambda_\beta l_\gamma)\\
&\text{(c) } 2\Gamma^*_{0\beta0}=2\gamma_{0\beta0}-(2\lambda_0 l_\beta-\lambda_\beta),
\end{aligned} \tag{2.6}$$

where $\gamma_{\alpha\beta\gamma}$ is the Christoffel symbol, and «0» means the contraction by l. Further, we obtain

$$\begin{aligned}
&\text{(a) } 2A_{\alpha\beta\gamma}=\{L\dot{\partial}_\alpha g_{\beta\gamma}-L\dot{\partial}_\delta g_{\beta\gamma}A^{\delta}_{0\alpha}-\mu_\alpha g_{\beta\gamma}\}\\
&\text{(b) } 2A_{0\beta\gamma}=-L\dot{\partial}_\delta g_{\beta\gamma}A^{\delta}_{00}-(\mu_0 g_{\beta\gamma}+\mu_\gamma l_\beta-\mu_\beta l_\gamma)\\
&\text{(c) } 2A_{0\beta0}=-(2\mu_0 l_\beta-\mu_\beta).
\end{aligned} \tag{2.7}$$

As Dl^α is given by (2.4), the connection coefficients determined by (2.6) and (2.7) are the generalization of Cartan connections in a non-recurrent Finsler space (where $\lambda_\gamma=0$ and $\mu_\gamma=0$). The non-recurrent Finsler space ($\lambda_\gamma=0,\mu_\gamma=0$) supplied with Cartan connection coefficients will be denoted by F_N.

In the space of $\overline{F}_N$ let us introduce M vector fields $B^\alpha_a(x,\dot{x})$ and $N-M$ vector fields $N^\alpha_k(x,\dot{x})$

$$a,b,c,d,e,f,\ldots = 1,2,\ldots,M$$

$$k,l,m,n,\ldots = M+1,\ldots,N,$$

homogeneous of degree zero in $\dot{x}$, linearly independent at each $(x,\dot{x})$, satisfying the relation:

$$(2.8) \qquad g_{\alpha\beta}(x,\dot{x})B^\alpha_a(x,\dot{x})N^\alpha_k(x,\dot{x}) = 0$$

for each $a = 1,2,\ldots,M, k = M+1,\ldots,N$.

Let us define

$$(2.9) \qquad \text{(a) } g_{ab} = g_{\alpha\beta}B^\alpha_a B^\beta_b \qquad \text{(b) } g_{kl} = g_{\alpha\beta}N^\alpha_k N^\beta_l$$

$$(2.10) \qquad \text{(a) } B^b_\beta = g^{ab}g_{\alpha\beta}B^\alpha_a \qquad \text{(b) } N^k_\alpha = g^{km}g_{\alpha\beta}N^\beta_m,$$

where (g^{ab}) and (g^{km}) are inverse matrices of (g_{ab}) and (g_{km}) respectively. From (2.9) and (2.10) we have

$$\text{(a) } N^k_\alpha N^\alpha_p = g^{kl}g_{\alpha\beta}N^\beta_l N^\alpha_p = g^{kl}g_{lp} = \delta^k_p$$

$$\text{(a) } N^k_\alpha N^\alpha_p = g^{kl}g_{\alpha\beta}N^\beta_l N^\alpha_p = g^{kl}g_{lp} = \delta^k_p$$

$$\text{(b) } B^a_\alpha B^\alpha_b = g^{ac}g_{\alpha\beta}B^\beta_c B^\alpha_b = g^{ac}g_{cb} = \delta^a_b$$

As usual,

$$(2.12) \qquad \delta^\alpha_\beta = B^\alpha_a B^a_\beta + N^\alpha_k N^k_\beta.$$

If $\xi^\alpha(x,\dot{x})$ is a vector field in $\overline{F}_N$ homogeneous of degree zero in $\dot{x}$, then it can be decomposed in the following form:

$$\xi^\alpha = B^\alpha_a\xi^a + N^\alpha_k\xi^k.$$

We may write

$$(2.13) \qquad dx^\alpha = B^\alpha_a du^a + N^\alpha_k dv^k$$

$$(2.14) \qquad \dot{x}^\alpha = B^\alpha_a\dot{u}^a + N^\alpha_k\dot{v}^k.$$

Let us denote the absolute differential which corresponds to the motion from $(x, \dot{x})$ to $(x + dx, \dot{x} + d\dot{x})$ by D.

The so-called induced differentials are defined by

$$(2.15) \qquad \text{(a) } \overline{D}\xi^a = B^a_\alpha D\xi^\alpha, \qquad \text{(b) } \overline{D}\xi^k = N^k_\alpha D\xi^\alpha$$

and

$$(2.16) \qquad D\xi^\alpha = B^\alpha_a \overline{D}\xi^a + N^\alpha_k \overline{D}\xi^k.$$

We shall use the notation

$$l^\alpha = L^{-1}\dot{x}^\alpha = L^{-1}(B^\alpha_a \dot{u}^a + N^\alpha_k \dot{v}^k) = B^\alpha_a l^a + N^\alpha_k l^k,$$

where

$$l^a = L^{-1}\dot{u}^a, \quad l^k = L^{-1}\dot{v}^k, \quad L^{-1} = (L(x, \dot{x}))^{-1}.$$

From (2.16) we have

$$(2.17) \qquad Dl^\alpha = B^\alpha_a \overline{D}l^a + N^\alpha_k \overline{D}l^k.$$

The vectors dx^γ and Dl^γ are not linearly independent. As it is known, in any space F_N from $g_{\alpha\beta}l^\alpha l^\beta = 1$ it follows that $l_\alpha Dl^\alpha = 0$. In the recurrent Finsler space $\overline{F}_N$ from $g_{\alpha\beta}l^\alpha l^\beta = 1$ we obtain, using (2.2)-(2.4),

$$(2.18) \qquad \lambda_\beta dx^\beta + (\mu_\beta + 2l_\beta)Dl^\beta = 0.$$

Since in the entire paper, we will need the formula for induced connection coefficients, we will write them here as they were determined in [4], using relation (2.18). The absolute differential of B^α_a and N^α_k may be decomposed in the direction of these two classes of vector fields in the following way:

$$(2.19) \qquad DB^\alpha_a = \overline{w}^d_a(d)B^\alpha_d + \overline{w}^m_a(d)N^\alpha_m$$

$$(2.20) \qquad DN^\alpha_k = \overline{w}^d_k(d)B^\alpha_d + \overline{w}^m_k(d)N^\alpha_m,$$

where

$$(2.21) \qquad \begin{gathered} \overline{w}^x_y(d) = \overline{\Gamma}^{*x}_{yb}du^b + \overline{\Gamma}^{*x}_{yk}dv^k + \overline{A}^x_{yb}\overline{D}l^b + \overline{A}^x_{yk}\overline{D}l^k, \\ x = d \quad or \quad x = m, \qquad y = a \quad or \quad y = k. \end{gathered}$$

From (2.2), (2.3) and (2.8), we have

$$D(g_{\alpha\beta}B^{\alpha}_{a}N^{\beta}_{k}) = g_{\alpha\beta}(DB^{\alpha}_{a})N^{\beta}_{k} + g_{\alpha\beta}B^{\alpha}_{a}DN^{\beta}_{k} = 0.$$

Substituting (2.19) and (2.20) into the above equation and using (2.8), we obtain in $\overline{F}_N$

$$\overline{w}_{ak} = -\overline{w}_{ka} \Leftrightarrow g_{km}\overline{w}^{m}_{a} = -g_{ab}\overline{w}^{b}_{k},$$

an equation of the same type as in F_N. If we express DB^{α}_{a} by the connection coefficients of the space $\overline{F}_N$ and use (2.13), (2.17), we get

$$DB^{\alpha}_{a} = (B^{\alpha}_{a|\beta}du^{b} + B^{\alpha}_{a}|_{\beta}\overline{D}l^{b})B^{\beta}_{b} + (B^{\alpha}_{a|\beta}dv^{k} + B^{\alpha}_{a}|_{\beta}\overline{D}l^{k})N^{\beta}_{k}, \tag{2.23}$$

where

$$\begin{aligned} &\text{(a)}\ B^{\alpha}_{a|\beta} = \partial_{\beta}B^{\alpha}_{a} - \dot{\partial}_{\gamma}B^{\alpha}_{a}\Gamma^{*\gamma}_{\beta} + \Gamma^{*\alpha}_{\gamma\beta}B^{\gamma}_{a} \qquad (\Gamma^{*\gamma}_{\beta} = \Gamma^{*\gamma}_{\alpha\beta}\dot{x}^{\alpha}) \\ &\text{(b)}\ B^{\alpha}_{a}|_{\beta} = L\dot{\partial}_{\delta}B^{\alpha}_{a}(\delta^{\delta}_{\beta} - A^{\delta}_{0\beta}) + A^{\alpha}_{\delta\beta}B^{\delta}_{a} \qquad (A^{\delta}_{0\beta} = A^{\delta}_{\alpha\beta}l^{\alpha}). \end{aligned} \tag{2.24}$$

If we substitute (2.13) and (2.17) into (2.18) using the notation

$$\lambda_{b} = B^{\beta}_{b}\lambda_{\beta}, \lambda_{k} = N^{\beta}_{k}\lambda_{\beta}, \mu_{b} = B^{\beta}_{b}\mu_{\beta}, \mu_{k} = N^{\beta}_{k}\mu_{\beta}, \tag{2.25}$$

we obtain

$$0 = \theta^{\alpha}_{a}(x,\dot{x})[\lambda_{b}du^{b} + \lambda_{k}dv^{k} + (\mu_{b} + 2l_{b})\overline{D}l^{b} + (\mu_{k} + 2l_{k})\overline{D}l^{k}], \tag{2.26}$$

where θ^{α}_{a} is any parameter homogeneous of degree zero in $\dot{x}$. If we equate the right-hand side of (2.19) with the sum of the right-hand sides of (2.23) and (2.26), we get the equation where on both sides terms with factors $du^{b}, dv^{k}, \overline{D}l^{b}$ and $\overline{D}l^{k}$ are present. Equating the corresponding coefficients after multiplying by $g_{\alpha\gamma}B^{\gamma}_{c}$ and $g_{\alpha\gamma}N^{\gamma}_{n}$ and then using the notation

$$\theta_{ac} = \theta^{\alpha}_{a}g_{\alpha\gamma}B^{\gamma}_{c}, \qquad \theta_{an} = \theta^{\alpha}_{a}g_{\alpha\gamma}N^{\gamma}_{n},$$

we obtain

(2.27)
$$
\begin{aligned}
&(a)\ \overline{\Gamma}^*_{acb} = g_{\alpha\gamma}B^\gamma_c B^\beta_b B^\alpha_{a|\beta} + \theta_{ac}\lambda_b\\
&(b)\ \overline{\Gamma}^*_{ack} = g_{\alpha\gamma}B^\gamma_c N^\beta_k B^\alpha_{a|\beta} + \theta_{ac}\lambda_k\\
&(c)\ \overline{A}_{acb} = g_{\alpha\gamma}B^\gamma_c B^\beta_b B^\alpha_a|_\beta + \theta_{ac}(\mu_b + 2l_b)\\
&(d)\ \overline{A}_{ack} = g_{\alpha\gamma}B^\gamma_c N^\beta_k B^\alpha_a|_\beta + \theta_{ac}(\mu_k + 2l_k)\\
&(e)\ \overline{\Gamma}^*_{anb} = g_{\alpha\gamma}N^\gamma_n B^\beta_b B^\alpha_{a|\beta} + \theta_{an}\lambda_b\\
&(f)\ \overline{\Gamma}^*_{ank} = g_{\alpha\gamma}N^\gamma_n N^\beta_k B^\alpha_{a|\beta} + \theta_{an}\lambda_k\\
&(g)\ \overline{A}_{anb} = g_{\alpha\gamma}N^\gamma_n B^\beta_b B^\alpha_a|_\beta + \theta_{an}(\mu_b + 2l_b)\\
&(h)\ \overline{A}_{ank} = g_{\alpha\gamma}N^\gamma_n N^\beta_k B^\alpha_a|_\beta + \theta_{an}(\mu_k + 2l_k)
\end{aligned}
$$

In a similar manner, using the expression for DN^α_k and the notations

$$\nu_{kc} = \nu^\alpha_k g_{\alpha\gamma}B^\gamma_c, \qquad \nu_{kn} = \nu^\alpha_k g_{\alpha\gamma}N^\gamma_n,$$

where $\nu^\alpha_k(x,\dot{x})$ is any parameter homogeneous of degree zero in $\dot{x}$, we obtain:

(2.28)
$$
\begin{aligned}
&(a)\ \overline{\Gamma}^*_{kcb} = g_{\alpha\gamma}B^\gamma_c B^\beta_b N^\alpha_{k|\beta} + \nu_{kc}\lambda_b\\
&(b)\ \overline{\Gamma}^*_{kcl} = g_{\alpha\gamma}B^\gamma_c N^\beta_l N^\alpha_{k|\beta} + \nu_{kc}\lambda_l\\
&(c)\ \overline{A}_{kcb} = g_{\alpha\gamma}B^\gamma_c B^\beta_b N^\alpha_k|_\beta + \nu_{kc}(\mu_b + 2l_b)\\
&(d)\ \overline{A}_{kcl} = g_{\alpha\gamma}B^\gamma_c N^\beta_l N^\alpha_k|_\beta + \nu_{kc}(\mu_l + 2l_l)\\
&(e)\ \overline{\Gamma}^*_{knb} = g_{\alpha\gamma}N^\gamma_n B^\beta_b N^\alpha_{k|\beta} + \nu_{kn}\lambda_b\\
&(f)\ \overline{\Gamma}^*_{knl} = g_{\alpha\gamma}N^\gamma_n N^\beta_l N^\alpha_{k|\beta} + \nu_{kn}\lambda_l\\
&(g)\ \overline{A}_{knb} = g_{\alpha\gamma}N^\gamma_n B^\beta_b N^\alpha_k|_\beta + \nu_{kn}(\mu_b + 2l_b)\\
&(h)\ \overline{A}_{knl} = g_{\alpha\gamma}N^\gamma_n N^\beta_l N^\alpha_k|_\beta + \nu_{kn}(\mu_l + 2l_l)
\end{aligned}
$$

The parameters θ^α_a and ν^α_k cannot be chosen arbitrarily because of (2.22), from which we obtain

(2.29)
$$
\begin{aligned}
&\overline{\Gamma}^*_{akb} = -\overline{\Gamma}^*_{kab} \qquad \overline{\Gamma}^*_{akl} = -\overline{\Gamma}^*_{kal}\\
&\overline{A}_{akb} = -\overline{A}_{kab} \qquad \overline{A}_{akl} = -\overline{A}_{kal}.
\end{aligned}
$$

Substituting the connection coefficients from (2.27) and (2.28) into (2.29) and using the relation

$$g_{\alpha\beta}B^{\alpha}_{a|\beta}N^{\gamma}_{k} + g_{\alpha\gamma}B^{\alpha}_{a}N^{\gamma}_{k|\beta} = 0$$

and a similar one with $|_{\beta}$, we obtain

$$\theta_{ak} = -\nu_{ka}. \tag{2.30}$$

From (2.14) and (2.17) we have

$$Dl^{\alpha} = B^{\alpha}_{a}\overline{D}l^{a} + N^{\alpha}_{k}\overline{D}l^{k} = (DB^{\alpha}_{a})l^{a} + (DN^{\alpha}_{m})l^{m} + B^{\alpha}_{a}dl^{a} + N^{\alpha}_{k}dl^{k}. \tag{2.31}$$

If we substitute from (2.19) and (2.20) the expression for DB^{α}_{a} and DN^{α}_{k} by using the notation

$$\overline{\Gamma}^{*x}_{0y} = \overline{\Gamma}^{*x}_{ay}l^{a} + \overline{\Gamma}^{*x}_{my}l^{m} \tag{2.32}$$

$$\overline{A}^{x}_{0y} = \overline{A}^{x}_{ay}l^{a} + \overline{A}^{x}_{my}l^{m} \qquad \begin{matrix} x = d & \text{or} & x = m \\ y = b & \text{or} & y = k \end{matrix} \tag{2.33}$$

we obtain

$$\overline{D}l^{x} = dl^{x} + \overline{\Gamma}^{*x}_{0b}du^{b} + \overline{\Gamma}^{*x}_{0k}dv^{k} + \overline{A}^{x}_{0b}\overline{D}l^{b} + \overline{A}^{x}_{0k}\overline{D}l^{k} \tag{2.34}$$

$$x = d \text{ or } x = m.$$

3. SOME SPECIAL VECTOR FIELDS IN $\overline{F}_N$

Let us denote by T_H, T_V the subspace of the tangent space of the differentiable manifold M spanned by vectors $B^{\alpha}_{a}(x,\dot{x}), N^{\alpha}_{k}(x,\dot{x})$ respectively. We will examine the special cases where the vector fields $B^{\alpha}_{a}(x,\dot{x})$ and $N^{\alpha}_{k}(x,\dot{x})$ satisfy some special conditions such as:

Case 1. $DB^{\alpha}_{a} \in T_H$ **Case 1a.** $DN^{\alpha}_{k} = 0$ **Case 1b.** $DB^{\alpha}_{a} = 0$,

Case 2. $DB^{\alpha}_{a} \in T_V$

Case 3. $DN^{\alpha}_{k} \in T_H$

Case 4. $(DB^{\alpha}_{a} \in T_V) \wedge (DN^{\alpha}_{k} \in T_H)$,

where DB^{α}_{a} and DN^{α}_{k} are determined by (2.19), (2.20), (2.21), (2.27) and (2.28).

Lemma 3.1. *In the recurrent Finsler space $\overline{F}_N$ condition $DB^\alpha_a \in T_H$ for $a = 1,2,\dots,M$ is equivalent to condition $DN^\alpha_k \in T_V$ for $k = M+1,\dots,N$.*

Proof. Let us suppose that $DB^\alpha_a \in T_H$ for $a = 1,2,\dots,M$. From (2.2), (2.3) and (2.8) follows

$$(3.1)\quad D(g_{\alpha\beta}B^\alpha_a N^\beta_k) = (\lambda_\gamma dx^\gamma + \mu_\gamma Dl^\gamma) g_{\alpha\beta}B^\alpha_a N^\beta_k + g_{\alpha\beta}(DB^\alpha_a)N^\beta_k + g_{\alpha\beta}B^\alpha_a DN^\alpha_k = 0$$

Since the first summand on the right-hand side is zero by (2.8), the second is also zero by the assumption $DB^\alpha_a \in T_H$ and (2.8), so, from (3.1) we obtain

$$g_{\alpha\beta}B^\alpha_a DN^\beta_k = 0, \qquad for \qquad a = 1,2,\dots,M,$$

which proves that $DN^\beta_k = h^n_k N^\beta_n$ i.e. $DN^\alpha_k \in T_V$ for $k = M+1,\dots,N$.

The proof, that from $DN^\alpha_k \in T_V$ for $k = M+1,\dots,N$ it follows $DB^\alpha_a \in T_H$ for $a = 1,2,\dots,M$, is similar.

Remark. Lemma 3.1 is valid in any Finsler space F_N.

Lemma 3.2. *In $\overline{F}_N$ the relation $DB^\alpha_a \in T_H$ is true iff*

$$(3.2)\quad \begin{aligned} &\text{(a)}\ N^m_\alpha B^\beta_b B^\alpha_{a|\beta} = -\theta^m_a \lambda_b \\ &\text{(b)}\ N^m_\alpha N^\beta_k B^\alpha_{a|\beta} = -\theta^m_a \lambda_k \\ &\text{(c)}\ N^m_\alpha B^\beta_b B^\alpha_a|_\beta = -\theta^m_a(\mu_b + 2l_b) \\ &\text{(d)}\ N^m_\alpha N^\beta_k B^\alpha_a|_\beta = -\theta^m_a(\mu_k + 2l_k) \\ &\text{for } m,k = M+1,\dots,N \qquad \text{and} \qquad b = 1,2,\dots,M. \end{aligned}$$

Proof. From (2.19) and condition $DB_a \in T_H$ follows

$$(3.3)\qquad DB^\alpha_a = \overline{w}^d_a(d)\, B^\alpha_a$$

i.e.

$$\overline{w}^m_a(d)\, N^\alpha_m = 0.$$

As the vector N^α_m are linearly independent, from the above equation follows $\overline{w}^m_a(d) = 0$ for $m = M+1,\dots,N$. From (2.21) we have

$$(3.4)\qquad \overline{w}^m_a(d) = \overline{\Gamma}^{*m}_{ab} du^b + \overline{\Gamma}^{*m}_{ak} dv^k + \overline{A}^m_{ab}\overline{D}l^b + \overline{A}^m_{ak}\overline{D}l^k.$$

From (3.4) follows that $\overline{w}_a^m(d) = 0$ for every $du^b, dv^k, \overline{D}l^b, \overline{D}l^k$ and $m = M+1, \dots, N$ which satisfy (2.13) and (2.17) iff

$$(3.5)\qquad \begin{aligned} &\overline{\Gamma}^{*m}_{ab} = 0, \overline{\Gamma}^{*m}_{ak} = 0, \overline{A}^m_{ab} = 0, \overline{A}^m_{ak} = 0 \\ &\text{for } m, k = M+1, \dots, N \quad \text{and} \quad b = 1, 2, \dots, M. \end{aligned}$$

From (2.27) it follows that (3.5) and (3.2) are equivalent. We have proved that $(DB_a^\alpha \in T_H) \Rightarrow (3.2)$. On the other hand,

$$(3.2) \Leftrightarrow (3.5), (3.5) \wedge (3.4) \Rightarrow \overline{w}_a^m(d) = 0 \quad \text{for} \quad m = M+1, \dots, N,$$

$$(\overline{w}_a^m(d) = 0 \quad \text{for} \quad m = M+1, \dots, N) \wedge (2.19) \Rightarrow DB_a^\alpha \in T_H.$$

Lemma 3.3. *In $\overline{F}_N$ relation $DB_a^\alpha \in T_H$ is true iff*

$$(3.6)\qquad \begin{aligned} &(a)\ B_b^\beta B^\alpha_{a|\beta} = b^d_{ba} B^\alpha_d - \theta^\alpha_a \lambda_b \\ &(b)\ N_k^\beta B^\alpha_{a|\beta} = b^d_{ka} B^\alpha_d - \theta^\alpha_a \lambda_k \\ &(c)\ B_b^\beta B^\alpha_a\big|_\beta = \overline{b}^d_{ba} B^\alpha_d - \theta^\alpha_a(\mu_b + 2l_b) \\ &(d)\ N_k^\beta B^\alpha_a\big|_\beta = \overline{b}^d_{ka} B^\alpha_d - \theta^\alpha_a(\mu_k + 2l_k) \end{aligned}$$

for $b = 1, 2, \dots, M$ and $k = M+1, \dots, N$, where $b, \overline{b}$ and θ are any parameters homogeneous of degree zero in $\dot{x}$.

Proof. The proof follows directly from (2.8) and $\theta_a^m = \theta_a^\alpha N_\alpha^m$.

Lemma 3.4. *In any Finsler space F_N relation $DB_a^\alpha \in T_H$ holds iff*

$$(3.7)\qquad \begin{aligned} &(a)\ B_b^\beta B^\alpha_{a|\beta} = b^d_{ab} B^\alpha_d \\ &(b)\ N_k^\beta B^\alpha_{a|\beta} = b^d_{ak} B^\alpha_d \\ &(c)\ B_b^\beta B^\alpha_a\big|_\beta = \overline{b}^d_{ab} B^\alpha_d - 2\theta^\alpha_a l_b \\ &(d)\ N_k^\beta B^\alpha_{a|\beta} = \overline{b}^d_{ak} B^\alpha_d - 2\theta^\alpha_a l_k \end{aligned}$$

for $b = 1, 2, \dots, M$ and $k = M+1, \dots, N$.

Proof. The Finsler space F_N is a special case of a recurrent Finsler space $\overline{F}_N$ defined in the former way, when $\lambda_\gamma(x, \dot{x}) = 0$ and $\mu_\gamma(x, \dot{x}) = 0$. From $\lambda_\gamma = 0$, $\mu_\gamma = 0$ and (2.25), it follows that

$$(3.8)\qquad \lambda_b = 0, \lambda_k = 0, \mu_c = 0, \mu_k = 0.$$

It is easy to see that under conditions (3.8) the corresponding formulae of (3.6) and (3.7) are equivalent.

Lemma 3.5. *In $\overline{F}_N$ the relation $DN_k^\alpha \in T_V$ holds iff*

$$(3.9)\qquad \begin{aligned} &(a)\ B_\alpha^a B_b^\beta N_{k|\beta}^\alpha = -\nu_k^a \lambda_b \\ &(b)\ B_\alpha^a N_l^\beta N_{k|\beta}^\alpha = -\nu_k^a \lambda_l \\ &(c)\ B_\alpha^a B_b^\beta N_k^\alpha\big|_\beta = -\nu_k^a(\mu_b + 2l_b) \\ &(d)\ B_\alpha^a N_l^\beta N_k^\alpha\big|_\beta = -\nu_k^a(\mu_l + 2l_l) \end{aligned}$$

for $a = 1, 2, \ldots, M$ and $l = M+1, \ldots, N$.

Proof. In $\overline{F}_N$ from (2.20) and $DN_k \in T_V$ follows

$$(3.10)\qquad DN_k^\alpha = \overline{w}_k^m(d) N_m^\alpha \Leftrightarrow$$

$$(3.11)\qquad \overline{w}_k^a(d) B_a^\alpha = 0.$$

Since the vectors $B_a^\alpha, a = 1, 2, \ldots, M$ are linearly independent, so form (3.11) we have

$$(3.12)\qquad \overline{w}_k^a(d) = \overline{\Gamma}_{kb}^{*a} du^b + \overline{\Gamma}_{kl}^{*a} dv^l + \overline{A}_{kc}^a \overline{D}l^c + \overline{A}_{kl}^a \overline{D}l^l = 0,$$

for $a = 1, 2, \ldots, M$. The above relation is satisfied for every $du^b, dv^l, \overline{D}l^c, \overline{D}l^l$ iff

$$(3.13)\qquad \overline{\Gamma}_{kb}^{*a} = 0, \overline{\Gamma}_{kl}^{*a} = 0, \overline{A}_{kc}^a = 0, \overline{A}_{kl}^a = 0,$$

for $a, b = 1, 2, \ldots, M$ and $l = M+1, \ldots, N$. From (2.28) it is evident that (3.13) and (3.9) are equivalent. On the other hand,

$$(3.9) \Leftrightarrow (3.13) \Leftrightarrow (3.12) \Leftrightarrow \overline{w}_k^a(d) = 0 \quad \text{for} \quad a = 1, 2, \ldots, M,$$

$$(\overline{w}_k^a(d) = 0 \quad \text{for} \quad a = 1, 2, \ldots, M) \wedge (2.20) \Rightarrow DN_k^\alpha \in T_V.$$

Lemma 3.6. *In $\overline{F}_N$ the relation $DN_k^\alpha \in T_V$ holds iff*

$$(3.14)\qquad \begin{aligned} &(a)\ B_b^\beta N_{k|\beta}^\alpha = n_{kb}^m N_m^\alpha - \nu_k^\alpha \lambda_b \\ &(b)\ N_l^\beta N_{k|\beta}^\alpha = n_{kl}^m N_m^\alpha - \nu_k^\alpha \lambda_l \\ &(c)\ B_b^\beta N_k^\alpha\big|_\beta = \overline{n}_{kb}^m N_m^\alpha - \nu_k^\alpha(\mu_b + 2l_b) \\ &(d)\ N_l^\beta N_k^\alpha\big|_\beta = \overline{n}_{kl}^m N_m^\alpha - \nu_k^\alpha(\mu_l + 2l_l) \end{aligned}$$

for $b = 1, 2, \ldots, M$ and $l = M+1, \ldots, N$ where $n, \overline{n}$ and ν are any parameters homogeneous of degree zero in $\dot{x}$.

Proof. The proof follows from (2.8) and $\nu_k^a = \nu_k^\alpha B_\alpha^a$.

Lemma 3.7. *In the Finsler space F_N the relation $DN^\alpha_k \in T_V$ is true iff*

$$
\begin{aligned}
&\text{(a) } B^\beta_b N^\alpha_{k|\beta} = n^m_{kb} N^\alpha_m \\
&\text{(b) } N^\beta_l N^\alpha_{k|\beta} = n^m_{kl} N^\alpha_m \\
(3.15)\qquad &\text{(c) } B^\beta_b N^\alpha_k|_\beta = \overline{n}^m_{kb} N^\alpha_m - 2\nu^\alpha_k l_b \\
&\text{(d) } N^\beta_l N^\alpha_k|_\beta = \overline{n}^m_{kl} N^\alpha_m - 2\nu^\alpha_k l_l \\
&\text{for } b = 1,2,\dots,M, \quad \text{and} \quad l = M+1,\dots,N,
\end{aligned}
$$

where $n, \overline{n}$ and ν are parameters homogeneous of degree zero in $\dot{x}$.

Proof. In the Finsler space F_N we have $\lambda_\gamma = 0$ and $\nu_\gamma = 0$. From (2.25) follows

$$\lambda_b = 0, \lambda_l = 0, \nu_b = 0, \nu_l = 0.$$

(3.15) is the obvious consequence of the above relation and (3.14).

Lemma 3.8. *In $\overline{F}_N$ the following relations are equivalent:*

$$
\begin{aligned}
&N^m_\alpha B^\beta_b B^\alpha_{a|\beta} = -\theta^m_a \lambda_b \Leftrightarrow B^a_\alpha B^\beta_b N^\alpha_{k|\beta} = -\nu^a_k \lambda_b \\
&N^m_\alpha N^\beta_k B^\alpha_{a|\beta} = -\theta^m_a \lambda_k \Leftrightarrow B^a_\alpha N^\beta_l N^\alpha_{k|\beta} = -\nu^a_k \lambda_l \\
&N^m_\alpha B^\beta_b B^\alpha_a|_\beta = -\theta^m_a(\mu_b + 2l_b) \Leftrightarrow B^a_\alpha B^\beta_b N^\alpha_k|_\beta = -\nu^a_k(\mu_b + 2l_b) \\
&N^m_\alpha N^\beta_k B^\alpha_a|_\beta = -\theta^m_a(\mu_k + 2l_k) \Leftrightarrow B^a_\alpha N^\beta_k N^\alpha_k|_\beta = -\nu^a_k(\mu_l + 2l_l).
\end{aligned}
$$

Proof. The proof is a direct consequence of Lemma 3.1. On the other hand, it follows directly from (2.8) and (2.30) or from (2.29).

Lemma 3.9. *If N^α_k are parallel vector fields in $\overline{F}_N$, for $k = M+1,\dots,N$, then in the same space $DB^\alpha_a \in T_H$ for $a = 1,2,\dots,M$.*

Proof. From $DN^\alpha_k = 0$ for $k = M+1,\dots,N$ and (2.20) follows $\overline{w}^d_k(d) = 0, \overline{w}^m_k(d) = 0$. Further, from (2.19) and (2.22) we have

$$DB^\alpha_a = \overline{w}^d_a(d) B^\alpha_d,$$

which proves the Lemma.

Lemma 3.10. *If B^α_a are parallel vector fields in $\overline{F}_N$ for $a = 1,2,\ldots,M$, then in the same space $DN^\alpha_k \in T_V$ for $k = m+1,\ldots,n$.*

Proof. From $DB^\alpha_a = 0$ for $a = 1,2,\ldots,M$ and (2.19), we have $\overline{w}^d_a = 0, \overline{w}^m_a = 0$ for $d = 1,2,\ldots,M$ and $m = M+1,\ldots,N$. From (2.20) and (2.22) we obtain

$$DN^\alpha_k = \overline{w}^m_k(d)N^\alpha_m,$$

which proves the Lemma.

Lemma 3.11. *In $\overline{F}_N$ relation $DB^\alpha_a \in T_V$ holds iff $\overline{w}^d_a(d) = 0$ for $d = 1,2,\ldots,M$, which is equivalent to the conditions*

$$\overline{\Gamma}^{*d}_{ab} = 0, \overline{\Gamma}^{*d}_{ak} = 0, \overline{A}^d_{ab} = 0, \overline{A}^d_{ak} = 0$$

$$\textit{for} \quad b,d = 1,2,\ldots,M \quad \textit{and} \quad k = M+1,\ldots,N.$$

Proof. The proof is the direct consequence of (2.19) and (2.21).

Lemma 3.12. *In $\overline{F}_N$ the relation $DN_k \in T_H$ holds iff $\overline{w}^m_k(d) = 0$ for $m = M+1,\ldots,N$ which is equivalent to the conditions*

$$\overline{\Gamma}^{*m}_{kb} = 0, \overline{\Gamma}^{*m}_{kl} = 0, \overline{A}^m_{kb} = 0, \overline{A}^m_{kl} = 0$$

$$\textit{for} \quad m,l = M+1\ldots,N \quad \textit{and} \quad b = 1,2,\ldots,M.$$

Proof. The proof is a direct consequence of (2.20) and (2.21).

Remark. Lemmas 3.9-3.12 are valid also in the space F_N.

4. VECTOR FIELDS IN $\overline{F}_N$ WHICH DETERMINE SUBSPACES

Here we will examine special vector fields $B^\alpha_a(x,\dot{x})$ and $N^\alpha_k(x,\dot{x})$ which determine subspaces in $\overline{F}_N$, in the sense that they are the tangent vector of these subspaces. In what follows we will consider some special cases having an interesting geometric interpretation:

$$(4.1) \qquad \begin{aligned} &(a)\ (\dot{\partial}_\beta B^\alpha_a = 0) \wedge (\dot{v}^k = 0)\\ &(b)\ (\dot{\partial}_\beta N^\alpha_k = 0) \wedge (\dot{u}^a = 0)\\ &(c)\ (\dot{\partial}_\beta B^\alpha_a = 0) \wedge (\dot{\partial}_\beta N^\alpha_k = 0)\\ &\text{for } \alpha,\beta = 1,2,\ldots,N, \quad a = 1,2,\ldots,M, \quad k = M+1,\ldots,N. \end{aligned}$$

Case 4.1.a. In this case when $dv^k = 0$ for $k = M + 1, \dots, N$ (2.13) and (2.14) reduce to the form

$$dx^\alpha = B^\alpha_a(x)\, du^a \qquad \dot{x}^\alpha = B^\alpha_a(x)\, \dot{u}^a.$$

As

$$B_a(x) = B^\alpha_a(x)\frac{\partial}{\partial x^\alpha}, \quad B_b(x) = B^\beta_b(x)\frac{\partial}{\partial x^\beta},$$

we have

$$[B_a(x), B_b(x)] = B^\alpha_x(x)\partial_\alpha B^\beta_b(x)\frac{\partial}{\partial x^\beta} - B^\beta_b(x)\partial_\beta B^\alpha_a(x)\frac{\partial}{\partial x^\alpha} =$$
$$= (B^\beta_a \partial_\beta B^\alpha_b - B^\beta_b \partial_\beta B^\alpha_a)\frac{\partial}{\partial x^\alpha}.$$

If we suppose that the relations

$$B^\beta_a \partial_\beta B^\alpha_b = \partial_a B^\alpha_b = \partial_b B^\alpha_a = B^\beta_b \partial_\beta B^\alpha_a \tag{4.2}$$

for $a, b = 1, 2, \dots, M$ are valid, then the Frobenius integrability conditions are satisfied and the vector fields $B^\alpha_a(x), a = 1, 2, \dots, M$ span T_H, which is the tangent space of the submanifold in M. The system of partial differential equations

$$\frac{\partial x^\alpha}{\partial u^a} = B^\alpha_a(x) \quad a = 1, 2, \dots, M, \quad \alpha = 1, 2, \dots, N \tag{4.3}$$

is satisfied by the family of subspaces of the form

$$x^\alpha = f^\alpha(u^1, \dots, u^M, C_{M+1}, \dots, C_N) \quad \alpha = 1, 2, \dots, N, \tag{4.4}$$

where

$$\det J = \det\left[\frac{\partial(x^1, \dots, x^N)}{\partial(u^1, \dots, u^M, C_{M+1}, \dots, C_N)}\right] \neq 0$$

and $C_{M+1}, \dots, C_N$ are arbitrary constants. It is obvious from $\det J \neq 0$, that (4.4) may be solved in the form

$$u^a = u^a(x^1, \dots, x^N) \qquad a = 1, 2, \dots, M,$$
$$C_k = C_k(x^1, \dots, x^N) \qquad k = M + 1, \dots, N.$$

Using these equations, we obtain

$$\frac{\partial x^\alpha}{\partial u^a} = \partial_a f^\alpha(u^1,\ldots,u^M,C_{M+1},\ldots,C_N) = \overline{B}_a^\alpha(u^1,\ldots,u^M,C_{M+1},\ldots,C_N) =$$
$$= \overline{B}_a^\alpha[u^1,(x^1,\ldots,x^N),\ldots,C_N(x^1,\ldots,x^N)] = B_a^\alpha(x^1,\ldots,x^N).$$

For all M-dimensional subspaces determined by (4.4), $B_a^\alpha(x)$ are tangent vectors to the coordinate curves and, according to (2.8), $N_k^\alpha(x,\dot{x})$ are the normal vectors.

Case 4.1.b. In this case for $du^a = 0$, (2.13) and (2.14) have the form

$$dx^\alpha = N_k^\alpha(x)dv^k \qquad \dot{x}^\alpha = N_k^\alpha(x)\dot{v}^k.$$

If the relations

$$N_m^\beta\partial_\beta N_k^\alpha = \partial_m N_k^\alpha = \partial_k N_m^\alpha = N_k^\beta\partial_\beta N_m^\alpha \tag{4.5}$$

for $k = M+1,\ldots,N$ are valid, then the conditions of the Frobenius theorem are satisfied and the vector fields $N_k^\alpha(x), k = M+1,\ldots,N$, which form the vector space T_V, are tangent vectors to the submanifold of M, i.e. the distribution T_V is integrable. The system of partial differential equations

$$\frac{\partial x^\alpha}{\partial v^k} = N_k^\alpha(x), \quad k = M+1,\ldots,N, \alpha = 1,2,\ldots,N$$

is satisfied by the family of $N-M$ dimensional subspaces

$$x^\alpha = g^\alpha(C_1,\ldots,C_M,v^{M+1},\ldots,v^N) \quad \alpha = 1,2,\ldots,N.$$

As follows from (2.8), for this family of subspaces, $N_k^\alpha(x) = \overline{N}_k^\alpha(C_1,\ldots,C_M,v^{M+1},\ldots,v^N)$ $k = M+1,\ldots,N$ are tangent vectors and $B_a^\alpha(x,\dot{x})\, a = 1,2,\ldots,M$ are normal vectors.

Case 4.1.c. Let us suppose that the metric tensor $g_{\alpha\beta}(x,\dot{x})$ of $\overline{F}_N$ allows vector fields $B_a^\alpha(x)$ and $N_k^\alpha(x)$, such that

$$g_{\alpha\beta}(x,\dot{x})B_a^\alpha(x)N_k^\alpha(x) = 0. \tag{4.6}$$

It is clear that $g_{\alpha\beta}(x,\dot{x})$ should have a special form when (4.6) is satisfied. In this case, the system of differential equations

$$\frac{\partial x^\alpha}{\partial u^a} = B_a^\alpha(x) \qquad \left(\frac{\partial x^\alpha}{\partial v^k} = N_k^\alpha(x)\right)$$

under conditions (4.2) (and (4.5)) are satisfied by the family of subspaces

$$x^\alpha = x^\alpha(u^1, \dots, u^M, C_{M+1}, \dots, C_N)$$
$$(x^\alpha = x^\alpha(C_1, \dots, C_M, v^{M+1}, \dots, v^N)),$$

for which $B^\alpha_a(x)(N^\alpha_k(x))$ are tangent vectors and $N^\alpha_k(x)(B^\alpha_a(x))$ are normal vectors. According to (4.6), these two families of subspaces are mutually orthogonal.

Connection coefficients for case 4.1.a. If we fix the parameters $C_{M+1}, \dots, C_N$ in the equation $x^\alpha = x^\alpha(u^1, \dots, u^M, C_{M+1}, \dots, C_N)$, we obtain one subspace. The induced metric tensor on the subspace is

$$g_{ab}(u, \dot{u}) = g_{\alpha\beta}(x(u), \overline{B}_a \dot{u}^a)\overline{B}^\alpha_a(u)\overline{B}^\beta_b(u). \tag{4.7}$$

It is known that this is the same as the intrinsic metric tensor, obtained by

$$g_{ab}(u, \dot{u}) = 2^{-l}\dot{\partial}_a\dot{\partial}_b L^2(x(u), \overline{B}_a \dot{u}^a).$$

A vector field ξ defined on the subspace is given by

$$\xi^\alpha(x, \dot{x}) = \xi^\alpha(x(u), \overline{B}_a \dot{u}^a) = \overline{B}^\alpha_a(u)\xi^a(u, \dot{u})$$
$$(B^\alpha_a(x) = \overline{B}^\alpha_a(u) = \partial_a x^\alpha),$$

from which we have

$$\xi^k = 0, \partial_k \xi^a = 0, \dot{\partial}_k \xi^a = 0 \left(\partial_k = \frac{\partial}{\partial v^k}, \ \dot{\partial}_k = \frac{\partial}{\partial \dot{v}^k}\right). \tag{4.8}$$

Definition 4.1. *A recurrent Finsler space $\overline{F}_N$ in which the conditions*

$$B^\alpha_a = B^\alpha_a(x), \dot{v}^m = 0, dv^k = 0, \partial_b B^\alpha_a = \partial_a B^\alpha_b \quad a, b = 1, 2, \dots, M \tag{A}$$

are satisfied will be denoted by $(\overline{F}_N, A)$.

In the following we will examine the induced connection coefficients in $(\overline{F}_N, A)$. From (A) follows

$$l^m = 0 \Rightarrow dl^m = 0 \quad \text{for} \quad m = M + 1, \dots, N, \tag{4.9}$$

and by substitution (4.9) into (2.34), we obtain

$$\overline{D}l^m = \overline{\Gamma}^{*m}_{0b} du^b + \overline{A}^m_{0b}\overline{D}l^b + \overline{A}^m_{0k}\overline{D}l^k \Rightarrow \tag{4.10}$$

$$I^m_k \overline{D}l^k = \overline{\Gamma}^{*m}_{0b} du^b + \overline{A}^m_{0b}\overline{D}l^b, \tag{4.11}$$

where

$$I^m_k = \delta^m_k - \overline{A}^m_{0k}. \tag{4.12}$$

From (4.9), (2.32) and (2.33) follows that in $(\overline{F}_n,)$

$$\overline{\Gamma}^{*x}_{0y} = \overline{\Gamma}^{*x}_{ay} l^a, \overline{A}^x_{0y} = \overline{A}^x_{ay} l^a \quad x = d \text{ or } x = m, \quad y = b \text{ or } y = k. \tag{4.13}$$

If we suppose that the matrix $[I^m_k]$ is regular, then from (4.11) follows

$$\overline{D}l^n = J^n_m(\overline{\Gamma}^{*m}_{0k} du^b + \overline{A}^m_{0b}\overline{D}l^b), \tag{4.14}$$

where $[J^n_m]$ is the inverse matrix of $[I^m_k]$. Substituting $\overline{D}l^n$ form (4.14) and $dv^k = 0$ into (2.21), we obtain in $(\overline{F}_N, A)$

$$\begin{aligned} \overline{w}^x_y(d) &= \overline{\Gamma}^{*x}_{yb} du^b + \overline{A}^x_{yb}\overline{D}l^b + \overline{A}^x_{yn} J^n_m(\overline{\Gamma}^{*m}_{0b} du^b + \overline{A}^m_{0b}\overline{D}l^b) = \\ &= \widetilde{\Gamma}^{*x}_{yb} du^b + \widetilde{A}^x_{yb}\overline{D}l^b = \widetilde{w}^x_y(d) \\ & x = d \quad \text{or} \quad x = m, \qquad y = b \quad \text{or} \quad y = k, \end{aligned} \tag{4.15}$$

where

$$\widetilde{\Gamma}^{*x}_{yb} = \overline{\Gamma}^{*x}_{yb} + \overline{A}^x_{yn} J^n_m \overline{\Gamma}^{*m}_{0b} \tag{4.16}$$

$$\widetilde{A}^x_{yb} = \overline{A}^y_{xb} + \overline{A}^x_{yn} J^n_m \overline{A}^m_{0b}. \tag{4.17}$$

Substituting $dv^k = 0$ and $\overline{D}l^n$ from (4.14) into (2.34) we get in $(\overline{F}_N, A)$

$$\overline{D}l^d = dl^d + \widetilde{\Gamma}^{*d}_{0b} du^b + \widetilde{A}^d_{0b}\overline{D}l^b, \tag{4.18}$$

where index «0» means contraction with l i.e.

$$\widetilde{\Gamma}^{*x}_{0b} = \widetilde{\Gamma}^{*x}_{ab} l^a \quad \widetilde{A}^x_{0b} = \widetilde{A}^x_{ab} l^a \qquad x = d \text{ or } x = m.$$

In $(\overline{F}_N, A)$, (2.19) and (2.20) are reduced to the form

$$DB^\alpha_a = \tilde{w}^d_a(d) B^\alpha_d + \tilde{w}^m_a(d) N^\alpha_m \tag{4.19}$$

$$DN^\alpha_k = \tilde{w}^d_k(d) B^\alpha_d + \tilde{w}^m_k(d) N^\alpha_m, \tag{4.20}$$

where $\tilde{w}^y_x(d)$ is given by (4.15). As in $(\overline{F}_N, A)\, l^\alpha = B^\alpha_a l^a$, so from (4.19) we have

$$Dl^\alpha = (DB^\alpha_a) l^a + B^\alpha_a dl^a = B^\alpha_d(dl^d + \tilde{w}^d_a(d) l^a) + N^\alpha_k w^k_a(d) l^a. \tag{4.21}$$

From (4.21) and (4.15) it follows

$$Dl^\alpha = B^\alpha_d(dl^d + \widetilde{\Gamma}^{*d}_{0b} du^b + \widetilde{A}^d_{0b}\overline{D}l^b) + N^\alpha_k(\widetilde{\Gamma}^k_{0b} du^b + \widetilde{A}^k_{0b}\overline{D}l^b).$$

Using (4.18) and introducing the notation

$$\widetilde{D}l^k = \widetilde{\Gamma}^{*k}_{0b} du^b + \widetilde{A}^k_{0b}\overline{D}l^b, \tag{4.22}$$

(4.21) has the form

$$Dl^\alpha = B^\alpha_d \overline{D}l^d + N^\alpha_k \widetilde{D}l^k. \tag{4.23}$$

Comparing (4.23) and (2.31), we obtain taht in $(\overline{F}_N, A)$ these two formulae are consistent iff $\widetilde{D}l^k = \overline{D}l^k$.

Lemma 4.1. *In* $(\overline{F}_n, A)$,

$$\widetilde{D}l^k = \overline{D}l^k \tag{4.24}$$

is always true, where $\widetilde{D}l^k$ *is defined by (4.22) and* $\overline{D}l^k$ *by (4.14).*

Proof. From (4.22), (4.14), further, from (4.16) and (4.17), we get

$$\widetilde{D}l^k - \overline{D}l^k = (\overline{\Gamma}^{*m}_{0b} du^b + \overline{A}^m_{0b}\overline{D}l^b)(\delta^k_m + A^k_{0n} J^n_m - J^k_m). \tag{4.25}$$

Multiplying (4.12) with J_l^k we get

$$\delta_l^m - J_l^m + A_{0k}^m J_l^k = 0 .$$

Substitution of the above relation into (4.25) after the changing of indices proves (4.24).

If we denote $\overline{D}l^d$ defined with (4.18) by $\widetilde{D}l^d$, then in $(\overline{F}_n, A)$ (4.23) has the form

$$Dl^\alpha = B_d^\alpha \widetilde{D}l^d + N_k^\alpha \widetilde{D}l^k.$$

If we suppose that the matrix

$$[\widetilde{I}_d^c] = [\delta_d^c - \widetilde{A}_{0d}^c]$$

is regular and introduce the notation

$$[\widetilde{J}_d^c] = [\widetilde{I}_d^c]^{-1} = [\delta_d^c - \widetilde{A}_{0d}^c]^{-1} \tag{4.26}$$

from (4.18), (4.22) and (4.24) we have

$$\widetilde{D}l^c = \widetilde{J}_d^c[dl^d + \widetilde{\Gamma}_{0b}^{*d} du^b] \tag{4.27}$$

$$\widetilde{D}l^k = \widetilde{\Gamma}_{0b}^{*k} du^b + \widetilde{A}_{0c}^k \widetilde{J}_d^c [dl^d + \widetilde{\Gamma}_{0b}^{*d} du^b]. \tag{4.28}$$

Definition 4.2. *The special case of a Finsler space $(\overline{F}_N, A)$ in which beside (A) the condition $\mu_\gamma = 0$ holds will be denoted by $(\overline{F}_N, B)$, where (B) is given by*

$$\begin{aligned} & g_{\alpha\beta|\gamma} = \lambda_\gamma g_{\alpha\beta}, \ \ \mu_\gamma = 0 \Leftrightarrow g_{\alpha\beta}\big|_\gamma = 0 \Leftrightarrow (\mu_k = 0) \wedge (\mu_b = 0), \\ (B) \qquad & B_a^\alpha = B_a^\alpha(x), dv^k = 0, \dot{v}^k = 0, l^k = 0, \partial_a B_b^\alpha = \partial_b B_a^\alpha \\ & for\ \forall \alpha, \beta, \gamma = 1, 2, \ldots, N, \ \ a = 1, 2, \ldots, M, \ \ k = M+1, \ldots, N. \end{aligned}$$

Lemma 4.2. *In $(\overline{F}_N, B)$ the following relations hold:*

$$B_a^\alpha\big|_\beta l^a = 0, \qquad A_{0\beta}^\alpha = 0 \tag{4.29}$$

$$\overline{A}_{0k}^x = 0 \tag{4.30}$$

(4.31) $$I^m_k = \delta^m_k = J^m_k$$

(4.32) $$\widetilde{\Gamma}^{*x}_{0b} = \overline{\Gamma}^{x}_{0b}$$

(4.33) $$\widetilde{A}^x_{0b} = \overline{A}^x_{0b}$$

$$x = d \quad \text{or} \quad x = m.$$

Proof. From (2.24b) we can see that the first term on the right-hand side is zero, because $\dot{\partial}_\delta B^\alpha_a = 0$. In the case $\mu_\gamma = 0$ and $l^\alpha = B^\alpha_a l^a$, we have from (2.7b) and (2.7c) that $A^\alpha_{0\beta} = 0$, so from (2.24b) follows (4.29). (4.30) follows from (2.27d), (2.27h) and $B^\alpha_a|_\beta l^a = 0$; (4.31) follows from (4.12) and (4.30); (4.32) and (4.33) follow from (4.16), (4.17) and (4.30).

Lemma 4.3. *In* $(\overline{F}_N, B)$, *the following relations are valid:*

(4.34) $$\widetilde{A}^x_{0b}\widetilde{D}l^b = -\theta^x_0\lambda_b du^b \quad for\ x = d\ or\ x = m$$

(4.35) $$\overline{D}l^k = \widetilde{D}l^k = (\widetilde{\Gamma}^{*k}_{0b} - \theta^k_0\lambda_b)\,du^b$$

(4.36) $$\overline{D}l^d = \widetilde{D}l^d = dl^d + (\widetilde{\Gamma}^{*d}_{0b} - \theta^d_0\lambda_b)\,du^b.$$

Proof. From (2.27c) and (2.27g) follows

(4.37) $$\widetilde{A}^x_{0b}\widetilde{D}l^b = 2\,\theta^x_0 l_b\widetilde{D}l^b.$$

As in $(\overline{F}_N, B)$ the formula (2.26) has the form

(4.38) $$2\,l_b\widetilde{D}l^b = -\lambda_b du^b,$$

the substitution of (4.38) into (4.37) gives (4.34). Formulae (4.35) and (4.36) follow directly from (4.27), (4.28) and (4.34).

Lemma 4.4. *In* $(\overline{F}_N, B)$ *we have*

$$\tilde{\Gamma}^{*k}_{0b} - \theta^k_0 \lambda_b = N^k_\alpha B^\beta_b (\partial_\beta B^\alpha_a + \Gamma^{*\alpha}_{\gamma\beta} B^\gamma_a) l^a \tag{4.39}$$

$$\tilde{\Gamma}^{*d}_{0b} - \theta^d_0 \lambda_b = B^d_\alpha B^\beta_b (\partial_\beta B^\alpha_a + \Gamma^{*\alpha}_{\gamma\beta} B^\gamma_a) l^a. \tag{4.40}$$

Proof. Taking into account (4.32), (2.27a) and (2.27c) the proof follows immediately. In (4.39) and (4.40), the connection coefficients Γ^* are functions of the metric tensor g and the vector λ.

Definition 4.3. *The special case of a Finsler space* $(\overline{F}_N, B)$ *in which* $\lambda_\gamma = 0$ *will be denoted by* $(\overline{F}_N, C)$, *where the conditions (C) are given by*

$$(C) \quad \begin{aligned} & B^\alpha_a = B^\alpha_a(x), \partial_a B^\alpha_b = \partial_b B^\alpha_a, dv^k = 0, \dot{v}^k = 0, \\ & g_{\alpha\beta|\gamma} = 0 \Leftrightarrow \lambda_\gamma = 0 \Leftrightarrow (\lambda_b = 0) \wedge (\lambda_k = 0) \\ & g_{\alpha\beta}|\gamma = 0 \Leftrightarrow \mu_\gamma = 0 \Leftrightarrow (\mu_b = 0) \wedge (\mu_k = 0) \\ & for\ \alpha, \beta, \gamma = 1, 2, \ldots, N, a, b = 1, 2, \ldots, M \quad and \quad k = M+1, \ldots, N. \end{aligned}$$

Lemma 4.5. *In* $(\overline{F}_N, C)$ *we have*

$$\overline{A}^d_{0k} = 0, \quad \overline{A}^m_{0k} = 0 \tag{4.41}$$

$$\tilde{\Gamma}^{*k}_{0b} = \overline{\Gamma}^{*k}_{0b} = N^k_\alpha B^\beta_b (\partial_\beta B^\alpha_a + \Gamma^{*\alpha}_{\gamma\beta} B^\gamma_a) l^a \tag{4.42}$$

$$\tilde{\Gamma}^{*d}_{0b} = \overline{\Gamma}^{*d}_{0b} = B^d_\alpha B^\beta_b (\partial_\beta B^\alpha_a + \Gamma^{*\alpha}_{\gamma\beta} B^\gamma_a) l^a \tag{4.43}$$

$$\overline{D} l^k = \tilde{D} l^k = \tilde{\Gamma}^{*k}_{0b} du^b \tag{4.44}$$

$$\overline{D} l^d = \tilde{D} l^d = dl^d + \tilde{\Gamma}^{*d}_{0b} du^b, \tag{4.45}$$

where the $\Gamma^{*\alpha}_{\beta\gamma}$ *are determined by (2.6) in which* $\lambda_\gamma = 0$.

Proof. Lemma 4.5. follows from Lemma 4.4 and Lemma 4.3, if we put $\lambda_b = 0$, which characterizes $(\overline{F}_N, C)$.

Lemma 4.6. *In* $(\overline{F}_N, A)$, *the relation* $DB_a^\alpha \in T_H$ *for* $a = 1,2,\ldots,M$ *holds iff one of the following equivalent conditions is valid:*

$$\text{(a)}\ \widetilde{w}_a^k = 0$$

$$\text{(b)}\ DN_k^\alpha \in T_V$$

$$\text{(c)}\ \widetilde{\Gamma}_{ab}^{*k} = \overline{\Gamma}_{ab}^{k} + \overline{A}_{an}^{k} J_m^n \overline{\Gamma}_{0b}^{*m} = 0$$

$$\widetilde{A}_{ab}^k = \overline{A}_{ab}^k + \overline{A}_{an}^k J_m^n \overline{A}_{0b}^m = 0$$

$$for\ a,b = 1,2,\ldots,M \quad and \quad k = M+1,\ldots,N.$$

Proof. Condition (a) follows directly from (4.19) and $DB_a^\alpha \in T_H$. From $\widetilde{w}_{am} = -\widetilde{w}_{ma}$, which is valid in $(\overline{F}_N, A)$ and (4.20), (b) follows. As

$$\widetilde{w}_x^y = \widetilde{\Gamma}_{xb}^{*y} du^b + \widetilde{A}_{xb}^y \widetilde{D} l^b, \tag{4.46}$$

(c) follows from (a). The proof that from (a) or (b) or (c) follows $DB_a^\alpha \in T_H$ is trivial.

In the space $(\overline{F}_N, C)$ condition (c) reduces to

$$\text{(c}'\text{)}\ \widetilde{\Gamma}_{ab}^{*k} = \overline{\Gamma}_{ab}^{*k} + \overline{A}_{an}^k \overline{\Gamma}_{0b}^{*n} = 0$$

$$\widetilde{A}_{ab}^k = \overline{A}_{ab}^k + \overline{A}_{an}^k \overline{A}_{0b}^n = 0,$$

because in $(\overline{F}_N, C)$ we have from (4.41) that $\overline{A}_{0k}^x = 0$, so $J_m^n = \delta_m^n$.

Lemma 4.7. *In* $(\overline{F}_N, A)$ *relation* $DB_a^\alpha \in T_V$ *is valid iff one of the following equivalent conditions holds:*

$$\text{(a)}\ \widetilde{w}_a^b = 0$$

$$\text{(b)}\ \widetilde{\Gamma}_{ac}^{*b} = \overline{\Gamma}_{ac}^{*b} + \overline{A}_{an}^b J_m^n \overline{\Gamma}_{0c}^{*m} = 0$$

$$\widetilde{A}_{ac}^b = \overline{A}_{ac}^b + \overline{A}_{an}^b J_m^n \overline{A}_{0c}^m = 0$$

$$for\ b,c = 1,2,\ldots,M.$$

Proof. Condition (a) follows from (4.19) and $DB_a \in T_V$, (b) follows from (a) and (4.16). The proof in the opposite direction is trivial. In the space $(\overline{F}_N, C)$ condition (b) of Lemma 4.7 takes the form

$$\text{(b}'\text{)}\ \widetilde{\Gamma}_{ac}^{*b} = \overline{\Gamma}_{ac}^{*b} + \overline{A}_{an}^b \overline{\Gamma}_{0c}^{*n} = 0$$

$$\widetilde{A}_{ac}^b = \overline{A}_{ac}^b + \overline{A}_{an}^b \overline{A}_{0c}^n = 0.$$

Lemma 4.8. *In* $(\overline{F}_N, A)$ *the relation* $DN_k^\alpha \in T_H$ *holds iff*

$$(a)\ \widetilde{w}_k^m = 0 \qquad or$$

$$(b)\ \widetilde{\Gamma}_{kb}^{*m} = \overline{\Gamma}_{kb}^{*m} + \overline{A}_{kn}^m J_m^n \overline{\Gamma}_{0b}^{*m} = 0$$

$$\widetilde{A}_{kb}^m = \overline{A}_{kb}^m + \overline{A}_{kn}^m J_m^n \overline{A}_{0b}^m = 0$$

$$for\ m = M+1, \dots, N \quad and \quad b = 1, 2, \dots, M.$$

Condition (b) for $(\overline{F}_N, C)$ *reduces to*

$$(b')\ \widetilde{\Gamma}_{kb}^{*m} = \overline{\Gamma}_{kb}^{*m} + \overline{A}_{kn}^m \overline{\Gamma}_{0b}^{*n} = 0$$

$$\widetilde{A}_{kb}^m = \overline{A}_{kb}^m + \overline{A}_{kn}^m \overline{A}_{0b}^n = 0$$

$$for\ m = M+1, \dots, N \quad and \quad b = 1, 2, \dots, M.$$

Proof. Condition (a) follows from (4.20) and the condition $DN_k^\alpha \in T_H$. (b) follows from (a) and (4.46). The proof in the opposite direction is trivial. The explicit expressions for $\widetilde{\Gamma}_{xb}^{*y}$ and $\widetilde{A}_{xb}^y$ in $(\overline{F}_N, C)$ are given in [4] with formulae (4.12) and (4.16).

5. CURVATURE, THE NORMAL AND TANGENT CURVATURE OF A CURVE IN THE SUBSPACE OF THE RECURRENT FINSLER SPACE $(\overline{F}_N, A)$

According to Definition 4.1 we will restrict our attention to the case when the differential equation of the subspace in the recurrent Finsler space is given by

$$dx^\alpha = B_a^\alpha(x)\, du^a, \tag{5.1}$$

and where

$$\dot{x}^\alpha = B_a^\alpha \dot{u}^a,\, B_a^\alpha = \frac{\partial x^\alpha}{\partial u^a},\, dv^k = 0,\, \dot{v}^k = 0 \Leftrightarrow l^\alpha = B_a^\alpha l^a \tag{5.2}$$

i.e when dx and $\dot{x}$ are in T_H, spanned by $B_a^\alpha (a = 1, 2, \dots, M)$. On any subspace

$$x^\alpha = x^\alpha(u^l, \dots, u^M, C_{M+1,0}, C_{N,0}) \tag{5.3}$$

which is the solution of (5.1), the normal vectors and the metric tensors are given by

$$\begin{aligned}
&(a)\ N_k^\alpha(x, \dot{x}) = N_k^\alpha(x(u), \overline{B}_a(u)\dot{u}^a) = \overline{N}_k^{\dot{\alpha}}(u, \dot{u}) \\
&(b)\ g_{\alpha\beta}(x, \dot{x}) = g_{\alpha\beta}(x(u), \overline{B}_a(u)\dot{u}^a) = \overline{g}_{\alpha\beta}(u, \dot{u}) \\
&(c)\ g_{ab}(u, \dot{u}) = \overline{g}_{\alpha\beta}(u, \dot{u})\overline{B}_a^\alpha(u)\overline{B}_b^\beta(u) \\
&(d)\ g_{kn}(u, \dot{u}) = \overline{g}_{\alpha\beta}(u, \dot{u})\overline{N}_k^\alpha(u, \dot{u})\overline{N}_n^\beta(u, \dot{u}).
\end{aligned} \tag{5.4}$$

If (g^{ab}) and (g^{kn}) are inverse matrices of (g_{ab}) and (g_{kn}) respectively, then

$$\overline{N}^{k}_{\alpha}(u,\dot{u}) = g^{kn}(u,\dot{u})\overline{g}_{\alpha\beta}(u,\dot{u})\overline{N}^{\beta}_{n}(u,\dot{u}) \tag{5.5}$$

$$\overline{B}^{b}_{\beta}(u,\dot{u}) = g^{ab}(u,\dot{u})\overline{g}_{\alpha\beta}(u,\dot{u})B^{\alpha}_{a}(u). \tag{5.6}$$

The equation of any curve on the subspace (5.3) is given by

$$u^{a} = u^{a}(s) \qquad a = 1,2,\dots,M, \tag{5.7}$$

where s is the arclength.

The curvature of the curve $u^a = u^a(s)$ in the subspace of $(\overline{F}_N, A)$, further the tangent and the normal curvature of the curve will be defined in a similar manner as in the Euclidean space. The supporting element in all formulae connected with the curvature is supposed to be $(u,\dot{u})$, where

$$\dot{u}^{a} = l^{a} = \frac{du^{a}}{ds} \qquad\qquad g_{ab}(u,\dot{u})l^{a}l^{b} = 1$$

Definition 5.1. *If we take in* $(\overline{F}_N, A)$

$$l^{\alpha} = B^{\alpha}_{a}\frac{du^{a}}{ds} = \frac{dx^{\alpha}}{ds} \qquad\qquad \overline{g}_{\alpha\beta}(u,\dot{u})l^{\alpha}l^{\beta} = 1 \tag{5.8}$$

i.e. l^α *is the unit tangent vector to the curve (5.7) of the subspace (5.3), and if we denote by* N^α *the unit normal vector in the direction of* Dl^α/ds *and by* K *the curvature of the curve, then*

$$\frac{Dl^{\alpha}}{ds} = KN^{\alpha}. \tag{5.9}$$

The normal curvature K_k *of the curve in the direction of the normal vector* $N^{\beta}_{k}(u,\dot{u})$ *is defined by*

$$K_{k} = g_{\alpha\beta}\frac{Dl^{\alpha}}{ds}N^{\beta}_{k}, \tag{5.10}$$

and the tangent curvature K_b *of the same curve in the direction of* B^{β}_{b} *is given by*

$$K_{b} = g_{\alpha\beta}\frac{Dl^{\alpha}}{ds}B^{\beta}_{b}. \tag{5.11}$$

Theorem 5.1. *The curvature K and the normal and tangent curvatures K_n and K_b of the curve $u^a = u^a(s)$ of the subspace (5.3) in $(\overline{F}_N, A)$ are connected by the formulae*

$$KN^\alpha = K^d B_d^\alpha + K^k N_k^\alpha = \frac{Dl^\alpha}{ds}, \tag{5.12}$$

where

$$K_b = g_{bd}K^d \quad K_n = g_{kn}K^k \tag{5.13}$$

$$K^d = \frac{d^2u^d}{ds^2} + \widetilde{\Gamma}^{*d}_{ac}\frac{du^a}{ds}\frac{du^c}{ds} + \widetilde{A}^d_{ac}\frac{du^a}{ds}\frac{\widetilde{D}l^c}{ds} \tag{5.14}$$

$$K^k = \widetilde{\Gamma}^{*k}_{ab}\frac{du^a}{ds}\frac{du^b}{ds} + \widetilde{A}^k_{ab}\frac{du^a}{ds}\frac{\widetilde{D}l^b}{ds}. \tag{5.15}$$

Proof. From (5.8) we obtain

$$\frac{Dl^\alpha}{ds} = \frac{DB_a^\alpha}{ds}l^a + B_a^\alpha\frac{d^2u^a}{ds^2}. \tag{5.16}$$

Substituting DB_a^α from (4.19), then $\widetilde{\omega}^x_y$ from (4.15) into (5.16), we get

$$\frac{Dl^\alpha}{ds} = B_d^\alpha\left(\frac{d^2u^d}{ds^2} + \widetilde{\omega}^d_a l^a\right) + N_k^\alpha\widetilde{\omega}^k_a l^a = B_d^\alpha K^d + N_k^\alpha K^k,$$

which proves (5.12). (5.13) follows directly from Definition 5.1, formulae (5.10) and (5.11).

Considering l^α as a vector in the space $(\overline{F}_N, A)$, we have

$$\begin{aligned} &\text{(a)}\ \frac{Dl^\alpha}{ds} = \frac{dl^\alpha}{ds} + \Gamma^{*\alpha}_{\beta\gamma}l^\beta\frac{dx^\gamma}{ds} + A^\alpha_{\beta\gamma}l^\beta\frac{Dl^\gamma}{ds} \Rightarrow \\ &\text{(b)}\ \frac{Dl^\alpha}{ds} = J^\alpha_\delta\left[\frac{d^2x^\delta}{ds^2} + \Gamma^{*\delta}_{\beta\gamma}\frac{dx^\beta}{ds}\frac{dx^\gamma}{ds}\right], \end{aligned} \tag{5.17}$$

where

$$[J^\alpha_\delta] = [I^\alpha_\delta]^{-1} \quad I^\alpha_\delta = \delta^\alpha_\delta - A^\alpha_{0\delta} \quad \det\,[I^\alpha_\delta] \neq 0. \tag{5.18}$$

Lemma 5.1. *In $(\overline{F}_N, A)$ the normal curvature K_k of the curve $u^a = u^a(s)$ in the subspace (5.3) is given by*

$$K_k = \widetilde{\Gamma}^*_{0k0} + \widetilde{A}_{0kc}\widetilde{J}^c_d\left(\frac{dl^d}{ds} + \widetilde{\Gamma}^{*d}_{00}\right) = g_{\alpha\beta}N^\beta_k J^\alpha_\delta\left[\frac{d^2x^\delta}{ds^2} + \Gamma^{*\delta}_{\beta\gamma}\frac{dx^\beta}{ds}\frac{dx^\gamma}{ds}\right]. \tag{5.19}$$

Proof. From (5.13) and (5.15) we have

$$K_k = \widetilde{\Gamma}^*_{0k0} + \widetilde{A}_{0kc}\frac{\widetilde{D}l^c}{ds}.$$

The first part of (5.19) follows from the above equation and the second part from (5.17) and (5.10).

Lemma 5.2. *The normal curvature K_k of the curve $u^a = u^a(s)$ on the subspace (5.3) in $(\overline{F}_N, A)$ is equal to zero iff one of the following equivalent conditions hold*

$$\begin{aligned}&\text{(a)}\ \widetilde{\Gamma}^*_{0k0} = -\widetilde{A}_{0kc}\frac{\widetilde{D}l^c}{ds}\\ &\text{(b)}\ \frac{Dl^\alpha}{ds} = J^\alpha_\delta\left[\frac{d^2x^\delta}{ds^2} + \Gamma^{*\delta}_{\beta\gamma}\frac{dx^\beta}{ds}\frac{dx^\gamma}{ds}\right] = c^a B^\alpha_a,\end{aligned} \tag{5.20}$$

where $c^a = c^a(u, \dot{u})$ is any scalar field homogeneous of degree zero in $\dot{u}$.

Proof. The proof follows from (5.19) and (2.8).

Lemma 5.3. *The normal curvature K_k of the curve $u^a = u^a(s)$ on the subspace (5.3) in $(\overline{F}_N, A)$ is equal to zero when*

$$\left(\frac{Dl^c}{ds} = 0\right) \wedge (B^\alpha_{a|\beta}l^a = 0) \wedge (\theta_{0k} = 0). \tag{5.21}$$

Proof. From (4.16) and (4.17) we obtain

$$\widetilde{\Gamma}^*_{0k0} = \overline{\Gamma}^*_{0k0} + \overline{A}_{0kn}J^n_m\overline{\Gamma}^{*m}_{00} \tag{5.22}$$

$$\widetilde{A}_{0kc} = \overline{A}_{0kc} + \overline{A}_{0kn}J^n_m\overline{A}^m_{0c}, \tag{5.23}$$

and from (2.27e), (2.27g) and (2.27h) follow

$$
(5.24)\qquad
\begin{aligned}
&(a)\ \overline{\Gamma}^{*}_{0k0} = g_{\alpha\gamma} N^{\gamma}_{k} B^{\beta}_{b} l^{b} B^{\alpha}_{a|\beta} l^{a} + \theta_{0k}\lambda_{0}\\
&(b)\ \overline{A}_{0kc} = g_{\alpha\gamma} N^{\gamma}_{k} B^{\beta}_{c} B^{\alpha}_{a}\big|_{\beta} l^{a} + \theta_{0k}\mu_{c}\\
&(c)\ \overline{A}_{0kn} = g_{\alpha\gamma} N^{\gamma}_{k} N^{\beta}_{n} B^{\alpha}_{a}\big|_{\beta} l^{a} + \theta_{0k}\mu_{n}
\end{aligned}
$$

Further we have

$$(5.24.a) \wedge (B^{\alpha}_{a|\beta} l^{a} = 0) \wedge (\theta_{0k} = 0) \Rightarrow \overline{\Gamma}_{0k0} = 0$$

$$(5.22) \wedge (\overline{\Gamma}^{*}_{0k0} = 0) \Rightarrow \widetilde{\Gamma}_{0k0} = 0$$

$$(5.19) \wedge (\widetilde{\Gamma}^{*}_{0k0} = 0) \wedge \left(\frac{Dl^{c}}{ds} = 0\right) \Rightarrow K_{k} = 0\,.$$

Remark. *It is clear that the conditions (5.21) of Lemma 5.3 are satisfied if we take the stronger conditions*

$$
(5.25)\qquad
\begin{aligned}
&(a)\ \frac{d^{2}u^{c}}{ds^{2}} + \widetilde{\Gamma}^{*c}_{ab}\frac{du^{a}}{ds}\frac{du^{b}}{ds} = 0\\
&(b)\ B^{\alpha}_{a|\beta} = \partial_{\beta} B^{\alpha}_{a} + \Gamma^{*\alpha}_{\gamma\beta} B^{\gamma}_{a} = 0 \qquad (\dot{\partial}_{\delta} B^{\alpha}_{a} = 0)\\
&(c)\ \theta_{0k} = 0\,.
\end{aligned}
$$

This means that the sufficient conditions in $(\overline{F}_{N}, A)$ for the normal curvature to be zero are that the unit tangent vectors $\dfrac{du^{c}}{ds}$ to the curve $u^{a} = u^{a}(s)$ be parallel in the subspace with respect to the induced connection coefficients (in which $\theta_{0k} = 0$) and the tangent vectors $B^{\alpha}_{a}(a = 1, 2, \ldots, M)$ to the coordinate curves

$$x^{\alpha} = x^{\alpha}(u^{l}_{0}, \ldots, u^{a-1}_{0}, u^{a}, u^{a+1}_{0}, \ldots, u^{M}_{0}, C_{M+1,0}, \ldots, C_{N,0})$$

be covariant constant with respect to $|$ and the connection coefficients of the surrounding space $(\overline{F}_{N}, A)$.

Let us examine the normal curvature of the curve in the subspace of $(\overline{F}_{N}, B)$ i.e. where

$$g_{\alpha\beta}\big|_{\gamma} = 0\,, \partial_{b} B^{\alpha}_{a} = \partial_{a} B^{\alpha}_{b}\,, B^{\alpha}_{a} = B^{\alpha}_{a}(x)\,, dv^{k} = 0\,, \dot{v}^{k} = 0 \Leftrightarrow l^{k} = 0\,.$$

Here we have

Lemma 5.4. *The normal curvature K_k of the curve $u^a = u^a(s)$ in the subspace (5.3) of $(\overline{F}_N, B)$ has the form*

$$K_k = g_{kn} N^n_\alpha B^\beta_b (\partial_\beta B^\alpha_a + \Gamma^{*\alpha}_{\gamma\beta} B^\gamma_a) l^a l^b = g_{\alpha\beta} N^\beta_k \left[\frac{d^2 x^\alpha}{ds^2} + \Gamma^{*\alpha}_{\beta\gamma} \frac{dx^\beta}{ds} \frac{dx^\gamma}{ds} \right]. \tag{5.26}$$

Proof. From (5.15) we have

$$K_k = g_{kn} \left(\widetilde{\Gamma}^{*n}_{00} + \widetilde{A}^n_{0c} \frac{\widetilde{D} l^c}{ds} \right).$$

Substituting from (4.34), (4.39) into the above formula, we obtain the first part of (5.26). The second part follows from (4.29) and (5.19) (in $(\overline{F}_N, B)$ from $A^\alpha_{0\beta} = 0 \Rightarrow J^\alpha_\delta = \delta^\alpha_\delta$). In (5.26) the connection Γ^* is determined by (2.6).

Lemma 5.5. *In $(\overline{F}_N, B)$ the normal curvature K_k of the curve $u^a = u^a(s)$ in the subspace (5.3) is equal to zero iff one of the equivalent conditions holds:*

$$\begin{aligned} &\text{(a)} \ B^\beta_b(\partial_\beta B^\alpha_a + \Gamma^{*\alpha}_{\gamma\beta} B^\gamma_a) l^a l^b = c^d B^\alpha_d, \\ &\text{(b)} \ \frac{d^2 x^\alpha}{ds^2} + \Gamma^{*\alpha}_{\beta\gamma} \frac{dx^\beta}{ds} \frac{dx^\gamma}{ds} = c^d B^\alpha_d, \end{aligned} \tag{5.27}$$

where $c^d = c^d(u, \dot{u})$ and $c^{d'} = c^{d'}(u, \dot{u})$ are scalar fields homogeneous of degree zero in $\dot{u}$.

Proof. The proof follows directly from (5.26) and (2.8).

Lemma 5.6. *In $(\overline{F}_N, B)$ the normal curvature K_k of the curve $u^a = u^a(s)$ in the subspace (5.3) is equal to zero when*

$$B^\alpha_{a|\beta} l^a = 0. \tag{5.28}$$

Remark. *The condition of Lemma 5.6 is satisfied when instead of (5.28) we take $B^\alpha_{a|\beta} = 0$.*

Proof. The proof follows from (5.26) and (2.24a) $(\dot{\partial}_\beta B^\alpha_a = 0)$.

Let us examine the normal curvature in $(\overline{F}_N, C)$ i.e. in the space where

$$g_{\alpha\beta|\gamma} = 0, g_{\alpha\beta}\big|_\gamma = 0, B^\alpha_a = B^\alpha_a(x), \partial_b B^\alpha_a = \partial_a B^\alpha_b, dv^k = 0, \dot{v}^k = 0.$$

Now, we have a Finsler space F_N in which the differential equation of the subspace $(\overline{F}_N, C)$ is given, as in the former case, by $\partial_a x^\alpha = B^\alpha_a(x)$. In $(\overline{F}_N, C)$ Lemma 5.4, Lemma 5.5, and

Lemma 5.6 are valid, but the connection coefficients $\Gamma^{*\alpha}_{\beta\gamma}$ in them are determined by (2.5), where $\lambda_\gamma = 0$.

It is not difficult to see that (5.26) is equivalent to (7.6) and (2.7) in [11], when we take

$$N^\alpha_\beta = \Gamma^{*\alpha}_{0\beta} = \Gamma^{*\alpha}_{\gamma\beta} l^\gamma = \Gamma^{*\alpha}_{\gamma\beta} B^\gamma_a l^a.$$

Let us now examine the tangent curvature of the curve in the subspace of $(\overline{F}_N, A)$. We have

Lemma 5.7. *The tangent curvature K_b of the curve $u^a = u^a(s)$ in the subspace (5.3) of $(\overline{F}_N, A)$*

$$(5.29)\quad \begin{aligned} K_b &= g_{\alpha\beta} J^\alpha_\theta \left(\frac{d^2 x^\theta}{ds^2} + \Gamma^{*\theta}_{\gamma\delta} \frac{dx^\gamma}{ds} \frac{dx^\delta}{ds} \right) B^\beta_b = \\ &= g_{db} \left[\frac{d^2 x^d}{ds^2} + \widetilde{\Gamma}^{*d}_{ac} \frac{du^a}{ds} \frac{du^c}{ds} + \widetilde{A}^d_{ac} \frac{du^a}{ds} \widetilde{J}^c_e \left(\frac{d^2 u^e}{ds^2} + \widetilde{\Gamma}^{*e}_{fg} \frac{du^f}{ds} \frac{du^g}{ds} \right) \right]. \end{aligned}$$

Proof. The first part of (5.29) follows from (5.11) and (5.17), the second part from (5.13), (5.14) and (4.27).

Lemma 5.8. *The tangent curvature K_b of the curve $u^a = u^a(s)$ in the subspace (5.3) of $(\overline{F}_N, A)$ is equal to zero iff*

$$(5.30)\quad \frac{Dl^\alpha}{ds} = J^\alpha_\theta \left(\frac{d^2 x^\theta}{ds^2} + \Gamma^{*\theta}_{\gamma\delta} \frac{dx^\gamma}{ds} \frac{dx^\delta}{ds} \right) = c^k N^\alpha_k,$$

where $c^k = c^k(u, \dot{u})$ is a scalar field homogeneous of degree zero in $\dot{u}$ (i.e. when $\frac{Dl^\alpha}{ds}$ is the vector field in the space spanned by N^α_k).

Proof. The proof follows from (5.29) and (2.8).

Lemma 5.9. *In $(\overline{F}_N, B)$ the tangent curvature K_b of any curve $u^a = u^a(s)$ of the subspace (5.3) has the form*

$$(5.31)\quad \begin{aligned} &\text{(a)}\ K_b = g_{\alpha\beta} \left(\frac{d^2 x^\alpha}{ds^2} + \Gamma^{*\alpha}_{\gamma\delta} \frac{dx^\gamma}{ds} \frac{dx^\delta}{ds} \right) B^\beta_b \\ &\text{(b)}\ K_b = g_{db} \left(\frac{d^2 u^d}{ds^2} + B^d_\alpha B^\alpha_{a|\beta} l^a l^\beta \right). \end{aligned}$$

Proof. In $(\overline{F}_N, B)$ we have

$$(\mu_\gamma = 0) \wedge (2.7.b) \wedge (2.7.c) \Rightarrow A^{\alpha}_{0\beta} = 0$$

$$(A^{\alpha}_{0\beta} = 0) \wedge (5.17.a) \Rightarrow J^{\alpha}_{\theta} = \delta^{\alpha}_{\theta}$$

$$(J^{\alpha}_{\theta} = \delta^{\alpha}_{\theta}) \wedge (5.31) \Rightarrow (5.31.a).$$

On the other hand, using (4.32), (4.33), (2.27) and (2.26) we have

$$\widetilde{\Gamma}^{*d}_{00} + \widetilde{A}^{d}_{0c}\frac{\widetilde{D}l^c}{ds} = \overline{\Gamma}^{*d}_{00} + \overline{A}^{d}_{0c}\frac{\overline{D}l^c}{ds} =$$

$$= B^d_\alpha B^\alpha_{a|\beta} l^a l^\beta + \theta^d_0\left(\lambda_b \frac{du^b}{ds} + 2\, l_b \frac{\overline{D}l^b}{ds}\right) = B^d_\alpha B^\alpha_{a|\beta} l^a l^\beta,$$

which proves (5.31.b).

Lemma 5.10. *In $(\overline{F}_N, B)$ the curve $u^a = u^a(s)$ of the subspace (5.3) has the tangent curvature K_b (for $b = 1, 2, \dots, M$) equal to zero iff its equation $x^\alpha = x^\alpha(u^a(s))$ in $(\overline{F}_N, B)$ satisfies the relation*

$$\frac{d^2 x^\alpha}{ds^2} + \Gamma^{*\alpha}_{\beta\gamma}\frac{dx^\beta}{ds}\frac{dx^\gamma}{ds} = c^k N^\alpha_k,$$

where $c^k = c^k(u, \dot{u})$ is any scalar field homogeneous of degree zero in $\dot{u}$.

Proof. The proof is obvious from (5.31.a).

Now, let us examine the space $(\overline{F}_N, C)$ (where $\lambda_\gamma = 0, \mu_\gamma = 0$). Here we have:

Lemma 5.11. *The tangent curvature K_b of any curve $u^a = u^a(s)$ of the subspace (5.3) in $(\overline{F}_N, C)$ is equal to zero iff one of the following conditions holds:*

$$\frac{d^2 u^d}{ds^2} + \widetilde{\Gamma}^{*d}_{ac}\frac{du^a}{ds}\frac{du^c}{ds} = 0,$$

$$\frac{d^2 x^\alpha}{ds^2} + \Gamma^{*\alpha}_{\gamma\delta}\frac{dx^\gamma}{ds}\frac{dx^\delta}{ds} = c^k N^\alpha_k,$$

where $c^k = c^k(u, \dot{u})$ is any scalar field homogeneous of degree zero in $\dot{u}$.

Proof. The proof follows from (5.31) and

$$\overline{\Gamma}^{*d}_{ac} l^a l^c = B^d_\alpha B^\beta_c B^\alpha_{a|\beta} l^a l^c,$$

where $\Gamma^{*\alpha}_{\gamma\delta}$ is determined by (2.6), in which $\lambda_\gamma = 0$.

From Lemma 5.11. follows

Lemma 5.12. *If $x^\alpha = x^\alpha(u^a(s))$ is a geodesic line of the subspace (5.3) with respect to the connection coefficients $\Gamma^{*\alpha}_{\gamma\delta}$, then $u^a = u^a(s)$ is a geodesic line of the subspace (5.3) with respect to the induced connection coefficients $\widetilde{\Gamma}^{*d}_{ac}$, and its tangent curvature K_b is equal to zero for $b = 1, 2, \ldots, M$.*

Theorem 5.2. *If the curve $u^a = u^a(s)$ of the subspace (5.3) in the space $(\overline{F}_N, A)$ satisfies the relation*

$$J^\alpha_\delta \left[\frac{d^2 x^\delta}{ds^2} + \Gamma^{*\delta}_{\beta\gamma}\frac{dx^\beta}{ds}\frac{dx^\gamma}{ds}\right] = 0 \tag{5.32}$$

then the curvature K of the curve, its tangent curvature K_b and the normal curvature K_k are equal to zero.

Proof. Form (5.17.b) follows that the left-hand side of (5.32) is equal to Dl^α/ds. From this fact and Definition 5.1 follows Theorem 5.2.

Theorem 5.3. *If the curve $u^a = u^a(s)$ of the subspace (5.3) in $(\overline{F}_N, A)$ satisfies the relation*

$$\frac{d^2 u^a}{ds^2} + \widetilde{\Gamma}^{*a}_{bc}\frac{du^b}{ds}\frac{du^c}{ds} = 0, \tag{5.33}$$

then

$$J^\alpha_\delta \left[\frac{d^2 x^\delta}{ds^2} + \Gamma^{*\delta}_{\beta\gamma}\frac{dx^\beta}{ds}\frac{dx^\gamma}{ds}\right] = N^\alpha_k \widetilde{\Gamma}^{*k}_{ab}\frac{du^a}{ds}\frac{du^b}{ds} \tag{5.34}$$

i.e. its curvature vector KN^α lies in the vector space spanned by the normal vectors N^α_k of the subspace.

Proof. Substituting (5.17), (5.14), (5.15) and (4.27) into (5.12), we obtain

$$\begin{aligned} KN^\alpha &= J^\alpha_\delta \left[\frac{d^2 x^\delta}{ds^2} + \Gamma^{*\delta}_{\beta\gamma}\frac{dx^\beta}{ds}\frac{dx^\gamma}{ds}\right] = \\ &= B^\alpha_d \left(\delta^d_a + \widetilde{A}^d_{bc}\frac{du^b}{ds}\widetilde{J}^c_a\right)\left(\frac{d^2 u^a}{ds^2} + \widetilde{\Gamma}^{*a}_{fg}\frac{du^f}{ds}\frac{du^g}{ds}\right) + \\ &+ N^\alpha_k \left[\widetilde{\Gamma}^{*k}_{ab}\frac{du^a}{ds}\frac{du^b}{ds} + \widetilde{A}^k_{ab}\frac{du^a}{ds}\widetilde{J}^b_c\left(\frac{d^2 u^c}{ds^2} + \widetilde{\Gamma}^{*c}_{fg}\frac{du^f}{ds}\frac{du^g}{ds}\right)\right]. \end{aligned} \tag{5.35}$$

In this formula Γ^* is determined by (2.6) and the induced connection coefficients $\widetilde{\Gamma}^*$ and $\widetilde{A}$, by (5.22)-(5.24). Substituting (5.33) into (5.35) we obtain (5.34).

Theorem 5.4. *If in the Finsler space $(\overline{F}_N, B)(\mu_\gamma = 0)$*

$$(\partial_\beta B^\alpha_a + \Gamma^{*\alpha}_{\gamma\beta} B^\gamma_a)\, l^a l^\beta = 0\,, \tag{5.36}$$

then

$$K N^\alpha = B^\alpha_d \frac{d^2 u^d}{ds^2}, \tag{5.37}$$

i.e. the curvature vector KN^α is in the tangent space of the subspace spanned by $B^\alpha_d, d = 1,2,\dots,M$.

Remark. *The condition (5.36) of Theorem 5.4. is satisfied when we take*

$$B^\alpha_{a|\beta} = \partial_\beta B^\alpha_a + \Gamma^{*\alpha}_{\gamma\beta} B^\gamma_a = 0\,, \tag{5.38}$$

i.e. when B^α_a are parallel vector fields in $(\overline{F}_N, B)$.

Proof. Substituting (5.26) and (5.31) into (5.12), we obtain in $(\overline{F}_N, B)$

$$\begin{aligned} KN^\alpha &= \frac{d^2 x^\alpha}{ds^2} + \Gamma^{*\alpha}_{\beta\gamma} \frac{dx^\beta}{ds}\, \frac{dx^\gamma}{ds} = \\ &= B^\alpha_d \left(\frac{d^2 u^d}{ds^2} + B^d_\delta B^\delta_{a|\beta} l^a l^\beta \right) + N^\alpha_n N^n_\delta B^\delta_{a|\beta} l^a l^\beta . \end{aligned} \tag{5.39}$$

Using (2.11) we obtain

$$KN^\alpha = B^\alpha_d \frac{d^2 u^d}{ds^2} + B^\alpha_{a|\beta} l^a l^\beta, \tag{5.40}$$

which proves Theorem 5.4.

Proposition 5.1. *If in $(\overline{F}_N, B)$ we have for the curve $u^a = u^a(s)$ of the subspace (5.3) the equations*

$$\frac{d^2 u^d}{ds^2} = 0 \quad and \quad B^\alpha_{a|\beta} = 0\,,$$

then $K = 0\,, K_b = 0\,, K_k = 0$. This is the consequence of (5.39).

Theorem 5.5. *In $(\overline{F}_N, C)$ (where $\lambda_\gamma = 0$, we have*

$$\begin{aligned} &\frac{d^2 x^\alpha}{ds^2} + \Gamma^{*\alpha}_{\beta\gamma} \frac{dx^\beta}{ds}\, \frac{dx^\gamma}{ds} = \\ &= B^\alpha_a \left[\frac{d^2 u^a}{ds^2} + \overline{\Gamma}^{*a}_{bc} \frac{du^b}{ds}\, \frac{du^c}{ds} \right] + N^\alpha_k \overline{\Gamma}^{*k}_{ab} \frac{du^a}{ds}\, \frac{du^b}{ds} . \end{aligned} \tag{5.41}$$

Proof. The proof follows from (5.35), (4.41), (4.42) and (4.43). In (5.41) $\Gamma^{*\alpha}_{\beta\gamma}$ is determined by (2.6), in which $\lambda_\gamma = 0$ and $\overline{\Gamma}^{*a}_{bc}, \overline{\Gamma}^{*k}_{ab}$ by (2.27.a), (2.27.c), where $\lambda_b = 0$.

It is easy to see that (5.40) is the generalization of (7.5) in [11] for the case of the M-dimensional subspace of a non-recurrent Finsler space supplied with Cartan connection coefficients and for $M = 1$ it coincides with (7.5) in [11].

Definition 5.2. *The curve $x^\alpha = x^\alpha(s)$ in the recurrent Finsler space $\overline{F}_n$ supplied with the generalized Cartan connection coefficients $(\Gamma^{*\alpha}_{\beta\gamma}, \Gamma^{*\alpha}_{0\gamma}, A^\alpha_{\beta\gamma})$ is a path in $\overline{F}_N$, if it satisfies the equation*

$$\frac{Dl^\alpha}{ds} = J^\alpha_\delta \left(\frac{d^2 x^\delta}{ds^2} + \Gamma^{*\delta}_{\beta\gamma}\left(x, \frac{dx}{ds}\right) \frac{dx^\beta}{ds}\frac{dx^\gamma}{ds} \right) = J^\alpha_\delta \left(\frac{d^2 x^\delta}{ds^2} + \Gamma^{*\delta}_{00} \right) = 0, \tag{5.42}$$

where $l^\alpha = dx^\alpha/ds$ and $\Gamma^{*\delta}_{\beta\gamma}, A^\delta_{\beta\gamma}, J^\alpha_\delta$ are determined by (2.6), (2.7) and (5.18) respectively.

Definition 5.3. *The curve $u^a = u^a(s)$ is a path in the subspace of $(\overline{F}_N, A)$ with respect to the induced connection coefficients $(\widetilde{\Gamma}^{*b}_{ac}, \widetilde{\Gamma}^{*b}_{0c}, \widetilde{A}^b_{ac})$ if it satisfies the equation*

$$\frac{\widetilde{D}l^c}{ds} = \widetilde{J}^c_d \left(\frac{d^2 u^d}{ds^2} + \widetilde{\Gamma}^{*d}_{ab}\frac{du^a}{ds}\frac{du^b}{ds} \right) = \widetilde{J}^c_d \left(\frac{d^2 u^d}{ds^2} + \widetilde{\Gamma}^{*d}_{00} \right) = 0 \tag{5.43}$$

where $\widetilde{\Gamma}^{*b}_{ac}, \widetilde{A}^b_{ac}, \widetilde{J}^c_d, \widetilde{D}l^c$ are given by (4.16), (4.17), (4.26), (4.27) respectively.

In equation (5.42) of the path, we used the supporting element $\left(x, \dfrac{dx}{ds}\right)$ which is the unit tangent vector to the curve, namely $l^\alpha = \dfrac{dx^\alpha}{ds}, l^a = \dfrac{du^a}{ds}$, so in our case the path and h-path as defined by (39.1) in [10] coincide.

This means that we have examined such curves in $(\overline{F}_N, A)$ or in its subspace for which the following is valid: if they are paths in $(\overline{F}_N, A)$ or in its subspace, then they are at the same time h-paths in the corresponding spaces.

Now, we can say that in this paper we have used such connection coefficients and such supporting elements that the paths and h-paths coincide both in $(\overline{F}_N, A)$ and in its subspace (5.3).

Definition 5.4. *The subspace (5.3) of the recurrent Finsler space $(\overline{F}_N, A)$ is a subspace of the first (second) kind, if each path (h-path) of the subspace with respect to the induced connection coefficients $(\overline{\Gamma}^{*b}_{ac}, \widetilde{\Gamma}^{*b}_{0c}, \widetilde{A}^b_{ac})$ is a path (h-path) of $(\overline{F}_N, A)$ with respect to the connection coefficients $(\Gamma^{*b\beta}_{\alpha\gamma}, \Gamma^{*\beta}_{0\gamma}, A^\beta_{\alpha\gamma})$ of the enveloping space $(\overline{F}_N, A)$.*

As in our cases paths and h-paths coincide, Definition 5.4. is at the same time the definition of subspaces of the first and second kind.

Theorem 5.6. *The subspace (5.3) in $(\overline{F}_N, A)$ is a subspace of the first kind, if for each curve $u^a = u^a(s)$ of the subspace for which*

$$\frac{d^2u^a}{ds^2} + \widetilde{\Gamma}^{*a}_{db}\frac{du^d}{ds}\frac{du^b}{ds} = 0, \tag{5.44}$$

the normal curvature is zero, i.e.

$$N^\alpha_k \widetilde{\Gamma}^{*k}_{ab}\frac{du^a}{ds}\frac{du^b}{ds} = 0. \tag{5.45}$$

Proof. It should be noted that (5.44) is a stronger condition as $Dl^a/ds = 0$, which can be seen from (5.43), because in $(\overline{F}_N, A)\ J^c_d \neq \delta^c_d$. For $(\overline{F}_N, A)$ the proof follows from (5.35).

Theorem 5.7. *In $(\overline{F}_N, C)$, i.e. in the non-recurrent Finsler space, the subspace (5.3) is of the first kind if for each curve $u^a = u^a(s)$ of the subspace (5.3) for which*

$$\frac{\overline{D}l^a}{ds} = \frac{d^2u^a}{ds^2} + \overline{\Gamma}^{*a}_{bc}\frac{du^b}{ds}\frac{du^c}{ds} = 0,$$

the normal curvature vector is zero i.e.

$$N^\alpha_k \overline{\Gamma}^{*k}_{ab}\frac{du^a}{ds}\frac{du^b}{ds} = 0.$$

Proof. The proof follows from (5.41).

Subspaces of the third kind in the recurrent Finsler space $(\overline{F}_N, A)$ similar to Definition 3 in [11] are determined by

Definition 5.5. *The subspace (5.3) of the recurrent Finsler space $(\overline{F}_N, A)$ is a subspace of the third kind, if its normal vectors N^α_k for $k = M+1, \dots, N$ are parallel vector fields along any curve $(u^a(s), l^a(s))$, $l^a = du^a/ds$ of the subspace.*

Theorem 5.8. *The subspace (5.3) of the recurrent Finsler space $(\overline{F}_N, A)$ is a subspace of the third kind iff one of the following equivalent conditions holds:*

$$(a)\ \widetilde{\omega}^d_k(d) = 0 \quad and \quad \widetilde{\omega}^m_k(d) = 0,$$

where

$$\widetilde{\omega}^x_k = \widetilde{\Gamma}^{*x}_{kb}du^b + \widetilde{A}^x_{kb}\widetilde{D}l^b,$$

or

$$\widetilde{\Gamma}^{*x}_{kb} = 0 \quad and \quad \widetilde{A}^x_{kb} = 0 \quad for \quad x = 0 \quad or \quad x = m.$$

Proof. The proof follows directly from (4.20), (4.15), (4.16) and (4.17).

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