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SEQUENCES OF IDEAL NORMS

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Dedicated to the memory of Professor Gottfried Köthe

Abstract. *There is a host of possibilities to associate with every (bounded linear) operator T, acting between Banach spaces, a scalar sequence*

$$\| T \| = A_1(T) \leq A_2(T) \leq \dots$$

such that all maps $A_n : T \to A_n(T)$ are ideal norms. The asymptotic behaviour of $A_n(T)$ as $n \to \infty$ can be used to define various subclasses of operators. The most simple condition is that

$$\sup_n n^{-\rho} A_n(T) < \infty,$$

where $\rho \geq 0$. This yields a 1-parameter scale of Banach operator ideals.

In what follows, this construction will be applied in some concrete cases. In particular, we let

$$H_n(T) := \sup \left\{ \| T J_M^E | \mathfrak{H} \| : M \subseteq E, \ \dim(M) \leq n \right\},$$

where J_M^E denotes the canonical embedding from the subspace M into E. Note that (H_n) is the natural dimensional gradation of the Hilbertian operator norm $\| \cdot | \mathfrak{H} \|$ in the sense of A. Pełczyński ([30], p. 165) and N. Tomczak-Jägermann ([46] and [48], p. 175). Taking the infimum over all $\rho \geq 0$ with

$$\sup_n n^{-\rho} H_n(T) < \infty,$$

we get an index $h(T) \in [0, 1/2]$ which can be used to measure the «Hilbertness» of the operator T.

Our main purpose is to show that several sequences of concrete ideal norms have the same asymptotic behaviour. This solves a problem posed in ([48], p. 210). We also give some applications to the geometry of Banach spaces.

Concerning the basic definitions and various results from the theory of operator ideals, the reader is referred to my monographs [31] and [32]. The notation is adopted from the latter.

The present paper is a revised and extended version of my preprint [36]. This revision became necessary when I observed that its main result was already contained in Remark 13.4 of G. Pisier's book [43]; see 5.3 below.

1. IDEAL NORMS

1.1. Let E and F be (real or complex) Banach spaces. We denote by $\mathfrak{L}(E,F)$ the collection of all (bounded linear) operators T from E into F. Recall that $\mathfrak{L}(E,F)$ becomes a Banach space with the norm

$$\| T \| := \sup \{\| Tx \| : x \in U_E\},$$

where U_E is the closed unit ball of E. For $a_0 \in E'$ (dual space) and $y_0 \in F$, we let $a_0 \otimes y_0 : x \to \langle x, a_0 \rangle y_0$.

1.2. A function A which assigns to every operator T between arbitrary Banach spaces a non-negative number $A(T)$ is called an *ideal norm* (on $\mathfrak{L}$) if the following conditions are satisfied:

$$(N_1) \quad A(a \otimes y) = \| a \| \, \| y \| \quad \text{for } a \in E' \text{ and } y \in F.$$

$$(N_2) \quad A(S+T) \le A(S) + A(T) \quad \text{for } S, T \in \mathfrak{L}(E,F).$$

$$(N_3) \quad A(YTX) \le \| Y \| \, A(T) \, \| X \| \quad \text{for } X \in \mathfrak{L}(E_0, E),$$
$$T \in \mathfrak{L}(E,F),\ Y \in \mathfrak{L}(F, F_0).$$

Note that we always have $\| T \| \le A(T)$.

1.3. An ideal norm A is said to be *symmetric* if $A(T') = A(T)$ for all $T \in \mathfrak{L}(E,F)$, where $T' \in \mathfrak{L}(F', E')$ denotes the dual operator.

1.4. An ideal norm A is called *injective* if $A(JT) = A(T)$ for all $T \in \mathfrak{L}(E,F)$ and all metric injections $J \in \mathfrak{L}(F, F_0)$; see ([31], B.2.6 and D.1.14).

1.5. An ideal norm A is called *surjective* if $A(TQ) = A(T)$ for all $T \in \mathfrak{L}(E,F)$ and all metric surjections $Q \in \mathfrak{L}(E_0, E)$; see ([31], B.2.8 and D.1.15).

2. THE IDEAL NORMS H_n

2.1. An operator $T \in \mathfrak{L}(E,F)$ is *Hilbertian* if it factors through a Hilbert space H. This means that

$$T : E \xrightarrow{A} H \xrightarrow{Y} F.$$

The *Hilbertian norm* is defined by

$$\| T | \mathfrak{H} \| := \inf \ \| Y \| \, \| A \|,$$

where the infimum ranges over all possible factorizations. The collection of Hilbertian operators is a Banach ideal, denoted by $\mathfrak{H}$; see ([31], 6.6.2).

2.2. For $T \in \mathfrak{L}(E,F)$ and $n = 1, 2, \ldots$, we define

$$H_n(T) := \sup \left\{ \| TJ_M^E | \mathfrak{H} \| : M \subseteq E,\ \dim(M) \leq n \right\} .$$

Remark. Recall that J_M^E denotes the canonical embedding from the subspace M into E.

2.3. The non-trivial part of the following proposition is an immediate consequence of a famous result of F. John [11].

Proposition. *H_n is an ideal norm. Moreover, for $T \in \mathfrak{L}(E,F)$, we have*

$$\| T \| = H_1(T) \leq \ldots \leq H_n(T) \leq \ldots \quad \textit{and} \quad H_n(T) \leq n^{1/2} \| T \| .$$

2.4. We now state a classical result which goes back to J.I. Joichi [12]. Its proof is based on compactness arguments or ultraproduct techniques; see ([24], Prop. 7.1) and ([41], Prop. 2.3).

Criterion. *An operator $T \in \mathfrak{L}(E,F)$ is Hilbertian if and only if the sequence $(H_n(T))$ is bounded. In this case,*

$$\| T | \mathfrak{H} \| = \sup_n H_n(T) .$$

2.5 Proposition. *The ideal norms H_n are injective.*

Proof. This follows from the injectivity of the Hilbertian operator norm $\| \cdot | \mathfrak{H} \|$; see ([31], 8.4.9).

2.6. I do not know whether the ideal norms H_n are symmetric. Therefore it is of interest to describe the dual norms.

Proposition. *Let $T \in \mathfrak{L}(E,F)$. Then*

$$H_n(T') = \sup \left\{ \| Q_N^F T | \mathfrak{H} \| : N \subseteq F,\ \operatorname{codim}(N) \leq n \right\} .$$

Remark. Recall that Q_N^F denotes the quotient map from F onto F/N.

2.7 Proposition. *The following problems are equivalent:*

(1) Is H_n symmetric?

(2) Is H_n surjective?

(3) Is it true that $H_n(T) = \| T | \mathfrak{H} \|$ whenever rank$(T) \leq n$*?*

Proof.

(1) $\rightarrow$ (2): Let $T \in \mathfrak{L}(E,F)$. If $Q \in \mathfrak{L}(E_0,E)$ is a metric surjection, then $Q' \in \mathfrak{L}(E',E_0')$ is a metric injection. Hence, assuming the symmetry of H_n, we conclude from Proposition 2.5 that

$$H_n(TQ) = H_n(Q'T') = H_n(T') = H_n(T).$$

(2) $\rightarrow$ (3): Let $\operatorname{rank}(T) \leq n$, and write $N := \{x \in E : Tx = o\}$. Then $\dim(E/N) \leq n$. Consider the canonical factorization $T = T_0 Q_N^E$, where Q_N^E denotes the quotient map from E onto E/N and $T_0 \in \mathfrak{L}(E/N, F)$. Assuming the surjectivity of H_n and using the surjectivity of $\| \cdot | \mathfrak{H} \|$, we now obtain

$$H_n(T_0 Q_N^E) = H_n(T_0) = \| T_0 | \mathfrak{H} \| = \| T_0 Q_N^E | \mathfrak{H} \| .$$

(3) $\rightarrow$ (1): Given $\varepsilon > 0$, we choose a subspace M of F' such that

$$H_n(T') - \varepsilon \leq \| T' J_M^{F'} | \mathfrak{H} \| \quad \text{and} \quad \dim(M) \leq n.$$

Note that M is the polar N^0 of a subspace N of F with $\dim(F/N) \leq n$. Moreover, $J_{N^0}^{F'} = (Q_N^F)'$. By assumption, we have $\| Q_N^F T | \mathfrak{H} \| = H_n(Q_N^F T)$. Now the symmetry of $\| \cdot | \mathfrak{H} \|$ implies that

$$H_n(T') - \varepsilon \leq \| T' J_{N^0}^{F'} | \mathfrak{H} \| = \| Q_N^F T | \mathfrak{H} \| = H_n(Q_N^F T) \leq H_n(T).$$

Letting $\varepsilon \rightarrow 0$ yields $H_n(T') \leq H_n(T)$. Next, replacing T by T', we obtain $H_n(T'') \leq H_n(T')$. Since $H_n(T) \leq H_n(T'')$ is obvious, it follows that

$$H_n(T) \leq H_n(T'') H_n(T') \leq H_n(T).$$

3. THE IDEAL NORMS K_n

3.1. For $T \in \mathfrak{L}(E,F)$ and $n = 1, 2, \ldots$, we define

$$K_n(T) := \inf c,$$

where the infimum is taken over all constants $o \geq 0$ such that

$$\left(\sum_{j=1}^{n} \| \sum_{i=1}^{n} \sigma_{ij} T x_i \|^2 \right)^{1/2} \leq c \left(\sum_{i=1}^{n} \| x_i \|^2 \right)^{1/2}$$

for $x_1, \ldots, x_n \in E$ and any unitary (n,n)-matrix $S = (\sigma_{ij})$.

3.2. First, we state some elementary facts.

Proposition. *K_n is an ideal norm. Moreover, for $T \in \mathfrak{L}(E,F)$, we have*

$$\| T \| = K_1(T) \leq \ldots \leq K_n(T) \leq \ldots \quad \textit{and} \quad K_n(T) \leq n^{1/2} \, \| T \| .$$

Proof. Let $x_1, \ldots, x_n \in E$, and let $S = (\sigma_{ij})$ be unitary. Then

$$\| \sum_{i=1}^{n} \sigma_{ij} T x_i \| \leq \sum_{i=1}^{n} |\sigma_{ij}| \;\; \| T x_i \| \leq \| T \| \left(\sum_{i=1}^{n} |\sigma_{ij}|^2 \right)^{1/2} \left(\sum_{i=1}^{n} \| x_i \|^2 \right)^{1/2}$$

Hence

$$\sum_{j=1}^{n} \| \sum_{i=1}^{n} \sigma_{ij} T x_i \|^2 \leq n \, \| T \|^2 \sum_{i=1}^{n} \| x_i \|^2 .$$

This proves that $K_n(T) \leq n^{1/2} \, \| T \|$. The other assertions are even more obvious.

3.3. Next a famous result of S. Kwapień ([19]. Prop. 3.1) is reformulated in terms of the ideal norms K_n. See also ([24], Theorem 7.3), ([41], Cor. 2.5) and ([48], Theorem 13.11).

Criterion. *An operator $T \in \mathfrak{L}(E,F)$ is Hilbertian if and only if the sequence $(K_n(T))$ is bounded. In this case,*

$$\| T | \mathfrak{H} \| = \sup_n K_n(T) .$$

3.4 Proposition. *The ideal norms K_n are symmetric, injective and surjective.*

Proof. Fix $b_1, \ldots, b_n \in F'$, and let $S = (\sigma_{ij})$ be any unitary (n,n) matrix. Given $\varepsilon > 0$, we choose $x_1, \ldots, x_n \in E$ such that

$$\left\langle x_i, \sum_{j=1}^{n} \sigma_{ij} T' b_j \right\rangle = \| \sum_{j=1}^{n} \sigma_{ij} T' b_j \|^2$$

and

$$\| x_i \| \leq (1 + \varepsilon) \, \| \sum_{j=1}^{n} \sigma_{ij} T' b_j \| .$$

Then it follows that

$$\sum_{i=1}^{n} \| \sum_{j=1}^{n} \sigma_{ij} T' b_j \|^2 = \sum_{i=1}^{n} \left\langle x_i, \sum_{j=1}^{n} \sigma_{ij} T' b_j \right\rangle = \sum_{j=1}^{n} \left\langle \sum_{i=1}^{n} \sigma_{ij} T x_i, b_j \right\rangle \leq$$

$$\leq \sum_{j=1}^{n} \| \sum_{i=1}^{n} \sigma_{ij} T x_i \| \| b_j \| \leq$$

$$\leq \left(\sum_{j=1}^{n} \| \sum_{i=1}^{n} \sigma_{ij} T x_i \|^2 \right)^{1/2} \left(\sum_{j=1}^{n} \| b_j \|^2 \right)^{1/2} \leq$$

$$\leq K_n(T) \left(\sum_{i=1}^{n} \| x_i \|^2 \right)^{1/2} \left(\sum_{j=1}^{n} \| b_j \|^2 \right)^{1/2} \leq$$

$$\leq (1+\varepsilon) K_n(T) \left(\sum_{i=1}^{n} \| \sum_{j=1}^{n} \sigma_{ij} T' b_j \|^2 \right)^{1/2} \left(\sum_{j=1}^{n} \| b_j \|^2 \right)^{1/2} .$$

Hence

$$\left(\sum_{i=1}^{n} \| \sum_{j=1}^{n} \sigma_{ij} T' b_j \|^2 \right)^{1/2} \leq (1+\varepsilon) K_n(T) \left(\sum_{j=1}^{n} \| b_j \|^2 \right)^{1/2}$$

This means that $K_n(T') \leq (1+\varepsilon) K_n(T)$. Letting $\varepsilon \to 0$ yields $K_n(T') \leq K_n(T)$. Next, replacing T by T', we obtain $K_n(T'') \leq K_n(T')$. Since $K_n(T) \leq K_n(T'')$ is obvious, it follows that

$$K_n(T) \leq K_n(T'') \leq K_n(T') \leq K_n(T).$$

Therefore, K_n is indeed symmetric.

The injectivity of K_n is trivial, and the surjectivity can easily be seen by passing to dual operators.

3.5 Problem. *Find the stabilization index*

$$\sigma(n) := \min \{ m : K_m(T) = \| T | \mathfrak{H} \| \textit{ whenever } \operatorname{rank}(T) \leq n \} .$$

Remark. In view of the Tomczak-Jägermann theorem (Remark at the end of 5.4), using a similar argument as in the proof of 5.3, we see that $\sigma(n) \leq \frac{1}{2} n(n+1)$ in the real case and $\sigma(n) \leq n^2$ in the complex case.

3.6. Let $l_2^n(E)$ denote the Banach space of all E-valued n-tuples (x_i) equipped with the norm

$$\| (x_i) | l_2^n \| := \left(\sum_{i=1}^{n} \| x_i \|^2 \right)^{1/2} .$$

In what follows, we use the complex interpolation method

$$\Phi_\Theta : (E_0, E_1) \to [E_0, E_1]_\Theta$$

which yields an exact functor of type Θ for $0 < \Theta < 1$; see ([49], p. 59). Note that the Banach spaces

$$[l_2^n(E_0), l_2^n(E_1)]_\Theta \text{ and } l_2^n([E_0, E_1]_\Theta)$$

can be identified isometrically; ([49], p. 121).

Proposition. *Let* (E_0, E_1) *and* (F_0, F_1) *be interpolation couple, and assume that* $T \in \mathfrak{L}(E_0 + E_1, F_0 + F_1)$ *transforms* E_0 *into* F_0 *and* E_1 *into* F_1. *Then*

$$K_n(T : [E_0, E_1]_\Theta \to [F_0, F_1]_\Theta) \le K_n(T : E_0 \to F_0)^{1-\Theta} K_n(T : E_1 \to F_1)^\Theta .$$

Proof. For every unitary (n, n)-matrix $S = (\sigma_{ij})$, we define the operator

$$S \otimes T : (x_i) \to \left(\sum_{i=1}^{n} \sigma_{ij} T x_i \right) .$$

The desired result now follows from

$$\begin{aligned} &\| S \otimes T : l_2^n([E_0, E_1]_\Theta) \to l_2^n([F_0, F_1]_\Theta) \| = \\ &= \| S \otimes T : [l_2^n(E_0), l_2^n(E_1)]_\Theta \to [l_2^n(F_0), l_2^n(F_1)]_\Theta \| \le \\ &\le \| S \otimes T : l_2^n(E_0) \to l_2^n(F_0) \|^{1-\Theta} \| S \otimes T : l_2^n(E_1) \to l_2^n(F_1) \|^\Theta \le \\ &\le K_n(T : E_0 \to F_0)^{1-\Theta} K_n(T : E_1 \to F_1)^\Theta . \end{aligned}$$

4. THE IDEAL NORMS G_n

4.1. An operator $T \in \mathfrak{L}(E, F)$ is called 2-*summing* if there exists a constant $c \ge 0$ such that

$$\left(\sum_{i=1}^{n} \| T x_i \|^2 \right)^{1/2} \le c \sup \left\{ \left(\sum_{i=1}^{n} |\langle x_i, a \rangle|^2 \right)^{1/2} : \| a \| \le 1 \right\}$$

for all finite families of elements $x_1, \ldots, x_n \in E$ and $n = 1, 2, \ldots$. The *2-summing norm* is defined by

$$\| T | \mathfrak{P}_2 \| := \inf c,$$

where the infimum ranges over all admissible constants. The collection of *2-summing* operators is a Banach ideal, denoted by $\mathfrak{P}_2$; see ([31], 17.1.2) and ([32], 1.2.3).

4.2. An operator $T \in \mathfrak{L}(E, F)$ is *dual 2-summing* if $T' \in \mathfrak{P}_2(F', E')$. In this case, we let $\| T | \mathfrak{P}_2' \| := \| T' | \mathfrak{P}_2 \|$. The collection of these operators is also a Banach ideal, denoted by $\mathfrak{P}_2'$.

4.3. For $T \in \mathfrak{L}(E, F)$ and $n = 1, 2, \ldots$, we define the *approximation number*

$$a_n(T) := \inf \{ \| T - L \| : L \in \mathfrak{L}(E, F),\ rank(L) < n \} .$$

The *Schatten-von Neumann norms* of $T \in \mathfrak{L}(l_2^n)$ are given by

$$\| T | \mathfrak{S}_p \| := \left(\sum_{k=1}^{n} a_k(T)^p \right)^{1/p}, \quad \text{where } 1 \leq p < \infty .$$

Note that $\| T | \mathfrak{S}_1 \|$ and $\| T | \mathfrak{S}_2 \|$ coincide with the nuclear norm and the 2-summing norm of T, respectively. Moreover, we have

$$\| T | \mathfrak{S}_1 \| \leq n^{1/2} \| T | \mathfrak{S}_2 \| .$$

4.4. For every (m, m)-matrix $S = (\sigma_{ij})$, we denote by $\| S \|$ the norm of the operator

$$S : (\xi_j) \to \left(\sum_{j=1}^{m} \sigma_{ij} \xi_j \right)$$

defined on l_2^m.

4.5. The next result, which is closely related to ([37], Prop. 1), can be proved by routine arguments. See also ([48], Theorem 27.1).

Lemma. *Let $T \in \mathfrak{L}(E, F)$. Then, for every constant $c \geq 0$ and every fixed number $n = 1, 2, \ldots$, the following statements are equivalent:*

$$(G_1) \qquad \left(\sum_{j=1}^{m} \| \sum_{i=1}^{m} \sigma_{ij} T x_i \|^2 \right)^{1/2} \leq c \left(\sum_{i=1}^{m} \| x_i \|^2 \right)^{1/2}$$

for $x_1, \dots, x_m \in E$ *and any (m,m)-matrix* $S = (\sigma_{ij})$ *such that* $\| S \| \leq 1$ *and* $rank(S) \leq n$, *where* $m = n, n+1, \dots$

$$(G_2) \qquad |\sum_{i=1}^{m}\sum_{j=1}^{m}\sigma_{ij}\langle Tx_i, b_j\rangle| \leq c\left(\sum_{i=1}^{m}\| x_i \|^2\right)^{1/2}\left(\sum_{j=1}^{m}\| b_j \|^2\right)^{1/2}$$

for $x_1, \dots, x_m \in E$, $b_1, \dots, b_m \in F'$ *and any (m,m)-matrix* $S = (\sigma_{ij})$ *such that* $\| S \| \leq 1$ *and* $rank(S) \leq n$, *where* $m = n, n+1, \dots$

$$(G_3) \qquad |\sum_{i=1}^{m}\sum_{j=1}^{m}\sigma_{ij}\langle x''_i, T'b_j\rangle| \leq c\left(\sum_{i=1}^{m}\| x''_i \|^2\right)^{1/2}\left(\sum_{j=1}^{m}\| b_j \|^2\right)^{1/2}$$

for $x''_1, \dots, x''_m \in E''$, $b_1, \dots, b_m \in F'$ *and any (m,m)-matrix* $S = (\sigma_{ij})$ *such that* $\| S \| \leq 1$ *and* $rank(S) \leq n$, *where* $m = n, n+1, \dots$

$$(G_4) \qquad \| BTX | \mathfrak{S}_1 \| \leq c \| B | \mathfrak{P}_2 \| \, \| X | \mathfrak{P}'_2 \|$$

for $X \in \mathfrak{L}(l_2^n, E)$ *and* $B \in \mathfrak{L}(F, l_2^n)$.

$$(G_5) \qquad \| TX | \mathfrak{P}_2 \| \leq c \| X | \mathfrak{P}'_2 \| \quad for \quad X \in \mathfrak{L}(l_2^n, E).$$

Proof.

$(G_1) \to (G_2)$: Use the Cauchy-Schwarz inequality.

$(G_2) \to (G_3)$: Apply Helly's lemma ([31], 28.1.1).

$(G_3) \to (G_4)$: This implication can be shown by obvious modifications of the techniques used in the proofs of 5.3 to 5.5.

$(G_4) \to (G_5)$: Note that, by trace duality, we have

$$\| TX | \mathfrak{P}_2 \| = \sup \{ | \text{trace } (BTX) | : B \in \mathfrak{L}(F, l_2^n), \| B | \mathfrak{P}_2 \| \leq 1 \}.$$

$(G_5) \to (G_1)$: See the proof of Proposition 1 in [37].

4.6. For $T \in \mathfrak{L}(E, F)$ and $n = 1, 2, \dots,$ we define

$$G_n(T) := \inf c,$$

where the infimum ranges over all constants $c \geq 0$ for which the equivalent conditions $(G_1), \dots, (G_5)$ are satisfied.

4.7. Next we state an analogue of Propositions 2.3 and 3.2.

Proposition. *G_n is an ideal norm. Moreover, for $T \in \mathfrak{L}(E,F)$, we have*

$$\|T\| = G_1(T) \leq \ldots \leq G_n(T) \leq \ldots \quad \textit{and} \quad G_n(T) \leq n^{1/2}\,\|T\|\,.$$

Proof. The monotonicity is obvious. If $X \in \mathfrak{L}(l_2^n, E)$ and $B \in \mathfrak{L}(F, l_2^n)$, then

$$\begin{aligned}\| BTX|\mathfrak{S}_1 \| &\leq n^{1/2}\,\| BTX|\mathfrak{S}_2 \| = n^{1/2}\,\| BTX|\mathfrak{P}_2 \| \leq \\ &\leq n^{1/2}\,\| B|\mathfrak{P}_2 \|\,\| TX \| \leq n^{1/2}\,\| T \|\,\| B|\mathfrak{P}_2 \|\,\| X|\mathfrak{P}_2' \|\,.\end{aligned}$$

Hence $G_n(T) \leq n^{1/2}\,\|T\|$.

4.8. We now formulate an analogue of Criterions 2.4 and 3.3 which is due to S. Kwapień ([17] and [18], Cor. 1). See also ([31], 19.6.2), ([41], Cor. 2.10) and ([48], Theorem 13.11).

Criterion. *An operator $T \in \mathfrak{L}(E,F)$ is Hilbertian if and only if the sequence $(G_n(T))$ is bounded. In this case,*

$$\| T|\mathfrak{H} \| = \sup_n G_n(T)\,.$$

4.9 Proposition. *The ideal norms G_n are symmetric, injective and surjective.*

Proof. The symmetry follows from the equivalence $(G_2) \longleftrightarrow (G_3)$ established in 4.5. The injectivity is obvious by (G_1), and the surjectivity can be deduced by passing to dual operators.

4.10. The next result states that, for finite operators, the sequence $(G_n(T))$ stabilizes on the earliest possible moment.

Proposition. $G_n(T) = \| T|\mathfrak{H} \|$ *whenever* $\operatorname{rank}(T) \leq n$.

Proof. Let $X \in \mathfrak{L}(l_2^m, E)$ and $B \in \mathfrak{L}(F, l_2^m)$ with $m = 1, 2, \ldots$ In view of $\operatorname{rank}(T) \leq n$, there exist partial isometries $A \in \mathfrak{L}(l_2^m, l_2^n)$ and $Y \in \mathfrak{L}(l_2^m, l_2^n)$ such that $BTX = Y^*YBTXA^*A$. Now it follows from

$$\begin{aligned}&\| BTX|\mathfrak{S}_1 \| \leq \| YBTXA^*|\mathfrak{S}_1 \| \leq G_n(T)\,\| YB|\mathfrak{P}_2 \|\,\| XA^*|\mathfrak{P}_2' \| \leq \\ &\leq G_n(T)\,\| B|\mathfrak{P}_2 \|\,\| X|\mathfrak{P}_2' \|\end{aligned}$$

that

$$G_m(T) \leq G_n(T) \quad \text{for} \quad m = 1, 2, \ldots$$

Hence

$$\| T|\mathfrak{H} \| = \sup_m G_m(T) = G_n(T).$$

4.11. We now established an analogue of Proposition 3.6 which goes back to G. Pisier ([38], proof of Lemma 3.2). See also [39] and ([48], p. 223).

Proposition. *Let* (E_0, E_1) *and* (F_0, F_1) *be interpolation couples, and assume that* $T \in \mathfrak{L}(E_0 + E_1, F_0 + F_1)$ *transforms* E_0 *into* F_0 *and* E_1 *into* F_1. *Then*

$$G_n(T : [E_0, E_1]_\Theta \to [F_0, F_1]_\Theta) \le G_n(T : E_0 \to F_0)^{1-\Theta} G_n(T : E_1 \to F_1)^\Theta .$$

4.12. The following result can be proved by trace duality or by applying Maurey's extension theorem [27]. For details we refer the reader to ([3], Lemma 10.1), ([33], Theorem 5.5), ([37], Prop. 1) and ([48], Theorems 13.12 and 27.1).

Theorem. *Let* $T \in \mathfrak{L}(E, F)$. *Then*

$$G_n(T) = \inf c,$$

where the infimum ranges over all constants $c \ge 0$ *having the following property:*

For every subspace M *of* F *with* $\dim(M) \le n$ *there exists an operator* $T_M \in \mathfrak{L}(E, F)$ *such that*

$$\| T_M|\mathfrak{H} \| \le c, \quad T_M(E) \subseteq M \quad \text{and} \quad T_M x = Tx \quad \text{whenever } Tx \in M.$$

4.13. For $T \in \mathfrak{L}(E, F)$ and $n = 1, 2, \ldots,$ we define

$$z_n(T) := \sup \left\{ a_n(BTX) : \begin{array}{l} X \in \mathfrak{L}(l_2^n, E), \ \| X|\mathfrak{P}_2' \| \le 1 \\ B \in \mathfrak{L}(F, l_2^n), \ \| B|\mathfrak{P}_2 \| \le 1 \end{array} \right\}.$$

These quantities were comprensively studied in [33].

4.14 Proposition. *Let* $T \in \mathfrak{L}(E, F)$ *and* $n = 1, 2, \ldots$ *Then*

$$n z_n(T) \le G_n(T) \le \sum_{k=1}^{n} z_k(T).$$

Proof. By ([33], Lemma 2.3), we have

$$z_k(T) := \sup \left\{ a_k(BTX) : \begin{array}{l} X \in \mathfrak{L}(l_2^n, E), \ \| X|\mathfrak{P}_2' \| \le 1 \\ B \in \mathfrak{L}(F, l_2^n), \ \| B|\mathfrak{P}_2 \| \le 1 \end{array} \right\}$$

whenever $k \leq n$. The assertion now follows from

$$n a_n(BTX) \leq \| BTX | \mathfrak{S}_1 \| = \sum_{k=1}^{n} a_k(BTX)$$

by passing to the suprema over $X \in \mathfrak{L}(l_2^n, E)$ and $B \in \mathfrak{L}(F, l_2^n)$ with $\| X | \mathfrak{P}_2' \| \leq 1$ and $\| B | \mathfrak{P}_2 \| \leq 1$.

5. RELATIONSHIPS BETWEEN THE IDEAL NORMS G_n, H_n, AND K_n

5.1 Proposition. $K_n(T) \leq H_n(T)$ *for* $T \in \mathfrak{L}(E, F)$ *and* $n = 1, 2, \ldots$

Proof. The conclusion follows from

$$K_n(T) = \sup \left\{ K_n \left(T J_M^E \right) : M \subseteq E,\ \dim(M) \leq n \right\}$$

and

$$K_n \left(T J_M^E \right) \leq \| T J_M^E | \mathfrak{H} \| \leq H_n(T).$$

Remark. The identity map of the real Banach space l_1^n is an example of an operator T for which $K_n(T) < H_n(T)$ whenever n is an odd number; see 9.3. In the complex case, I do not know any operator T with $K_n(T) < H_n(T)$ for some n.

5.2 Proposition. $H_n(T) \leq G_n(T)$ *for* $T \in \mathfrak{L}(E, F)$ *and* $n = 1, 2, \ldots$

Proof. Let M be a subspace of E with $\dim(M) \leq n$. Then, by 4.10, we have

$$\| T J_M^E | \mathfrak{H} \| = G_n \left(T J_M^E \right) \leq G_n(T).$$

This implies that $H_n(T) \leq G_n(T)$.

Remark. Neither in the real case nor in the complex case, I know an operator T with $H_n(T) < G_n(T)$ for some n.

5.3. We now establish a fundamental inequality which goes back to G. Pisier ([43], Remark 13.4). He only considered the special case of identity maps. The generalization to arbitrary operators is, however, straightforward. Nevertheless, for the convenience of the reader, we present here a detailed proof.

Proposition. $G_n(T) \leq 2\,K_n(T)$ *for* $T \in \mathfrak{L}(E,F)$ *and* $n = 1,2,\ldots$

Proof. Let $X \in \mathfrak{L}(l_2^n, E)$ and $B \in \mathfrak{L}(F, l_2^n)$. Given $\varepsilon > 0$, in view of the famous Tomczak-Jägermann Theorem 5.4, there exist convex decompositions

$$X' = \sum_{h=1}^{p} \lambda_h A_h \quad \text{and} \quad B = \sum_{k=1}^{q} \mu_k B_k,$$

where

$$A_h : E' \xrightarrow{A_{1h}} l_2^n \xrightarrow{A_{2h}} l_2^n \quad \text{and} \quad B_k : F \xrightarrow{B_{1k}} l_2^n \xrightarrow{B_{2k}} l_2^n,$$

$$\left(\sum_{i=1}^{n} \| A'_{1h} e_i \|^2 \right)^{1/2} \leq \sqrt{2}(1+\varepsilon)\, \| X | \mathfrak{P}'_2 \| \,,\ \| A_{2h} \| \leq 1,$$

$$\left(\sum_{j=1}^{n} \| B'_{1k} e_j \|^2 \right)^{1/2} \leq \sqrt{2}(1+\varepsilon)\, \| B | \mathfrak{P}_2 \| \,,\ \| B_{2k} \| \leq 1.$$

Applying Lemma 5.5 to the dual operator T', we obtain

$$\begin{aligned}
&\| BTX | \mathfrak{S}_1 \| = \| X'T'B' | \mathfrak{S}_1 \| \leq \\
&\leq \sum_{h=1}^{p} \sum_{k=1}^{q} \lambda_h \mu_k \, \| A_{2h} A_{1h} T' B'_{1k} B'_{2k} | \mathfrak{S}_1 \| \leq \\
&\leq \sum_{h=1}^{p} \sum_{k=1}^{q} \lambda_h \mu_k \, \| A_{1h} T' B'_{1k} | \mathfrak{S}_1 \| \leq \\
&\leq K_n(T') \sum_{h=1}^{p} \sum_{k=1}^{q} \lambda_h \mu_k \left(\sum_{i=1}^{n} \| A'_{1h} e_i \|^2 \right)^{1/2} \left(\sum_{j=1}^{n} \| B'_{1k} e_j \|^2 \right)^{1/2} \leq \\
&\leq 2(1+\varepsilon)^2 K_n(T') \, \| B | \mathfrak{P}_2 \| \, \| X | \mathfrak{P}'_2 \| \,.
\end{aligned}$$

By the symmetry of K_n and property (G_4) in 4.5, we have

$$G_n(T) \leq 2(1+\varepsilon)^2 K_n(T).$$

Letting $\varepsilon \to 0$ yields the desired result.

5.4. Note that

$$\| A|\mathfrak{P}_2 \| \leq \left(\sum_{i=1}^{n} \| A'e_i \|^2 \right)^{1/2} \quad \text{for} \quad A \in \mathfrak{L}(E, l_2^n),$$

where $(e_1, \ldots, e_n)$ denotes the canonical basis of l_2^n. Hence, if $T \in \mathfrak{L}(E, F)$ is of the form

$$T : E \xrightarrow{A} l_2^n \xrightarrow{Y} F,$$

then we have

$$\| T|\mathfrak{P}_2 \| \leq \| Y \| \left(\sum_{i=1}^{n} \| A'e_i \|^2 \right)^{1/2}$$

We now formulate a converse of this result; see ([31], 19.1.8 and 19.2.9) and ([32], 1.7.14).

Factorization Theorem. *Let* $\varepsilon > 0$. *Every finite operator* $T \in \mathfrak{L}(E, F)$ *admits a factorization*

$$T : E \xrightarrow{A} l_2^n \xrightarrow{Y} F$$

such that

$$\| Y \| \left(\sum_{i=1}^{n} \| A'e_i \|^2 \right)^{1/2} \leq (1 + \varepsilon) \| T|\mathfrak{P}_2 \| .$$

In general, the dimension of the Hilbert space l_2^n must be chosen considerably larger than $\operatorname{rank}(T)$. However, on the cost of a more complicated representation, one may arrange that $n = \operatorname{rank}(T)$. For a proof of this result, we refer the reader to ([30], Cor. 18.1), ([46], Prop. 2) and ([48], Prop. 24.3).

Tomczak-Jägermann Theorem. *Let* $\varepsilon > 0$. *Every finite operator* $T \in \mathfrak{L}(E, F)$ *can be represented as a convex combination*

$$T = \sum_{h=1}^{p} \lambda_h T_h$$

of operators

$$T_h : E \xrightarrow{A_h} l_2^n \xrightarrow{Y_h} F$$

such that $n = \operatorname{rank}(T)$ *and*

$$\| Y_h \| \left(\sum_{h=1}^{n} \| A_h'e_i \|^2 \right)^{1/2} \leq \sqrt{2}(1 + \varepsilon) \| T|\mathfrak{P}_2 \| \quad \textit{for} \quad h = 1, \ldots, p.$$

Remark. The factor $\sqrt{2}$ can be dropped when we replace l_2^n by l_2^N with $N = \frac{1}{2}\, n(n+1)$ in the real case and $N = n^2$ in the complexe case.

5.5. In the proof of 5.3 we have used an auxiliary result.

Lemma. *Let* $T \in \mathfrak{L}(E,F)$ *and* $n = 1,2,\ldots$ *Then*

$$\| BTX|\mathfrak{S}_1 \| \leq K_n(T) \left(\sum_{i=1}^{n} \| Xe_i \|^2 \right)^{1/2} \left(\sum_{j=1}^{n} \| B'e_j \|^2 \right)^{1/2}$$

for $X \in \mathfrak{L}(l_2^n, E)$ *and* $B \in \mathfrak{L}(F, l_2^n)$.

Proof. By the polar decomposition theorem, there exists a unitary operator $S \in \mathfrak{L}(l_2^n)$ for which $SBTX \geq O$. Let (σ_{ij}) denote the representing matrix of S, and observe that BTX is represented by the matrix $(\langle Tx_i, b_j \rangle)$, where $x_i = Xe_i$ and $b_j = B'e_j$. We now obtain

$$\| BTX|\mathfrak{S}_1 \| = \| SBTX|\mathfrak{S}_1 \| = \text{trace}(SBTX) =$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{n} \sigma_{ij} \langle Tx_i, b_j \rangle \leq$$

$$\leq K_n(T) \left(\sum_{i=1}^{n} \| x_i \|^2 \right)^{1/2} \left(\sum_{j=1}^{n} \| b_j \|^2 \right)^{1/2} =$$

$$= K_n(T) \left(\sum_{i=1}^{n} \| Xe_i \|^2 \right)^{1/2} \left(\sum_{j=1}^{n} \| B'e_j \|^2 \right)^{1/2}.$$

5.6. Finally, we summarize the results from Propositions 5.1, 5.2 and 5.3.

Theorem. *Let* $T \in \mathfrak{L}(E,F)$ *and* $n = 1,2,\ldots$ *Then*

$$K_n(T) \leq H_n(T) \leq G_n(T) \leq 2K_n(T).$$

6. GROTHENDIECK NUMBERS

6.1. For $T \in \mathfrak{L}(E,F)$ and $n = 1,2,\ldots$, we define the *Grothendieck number*

$$\Gamma_n(T) := \sup \left\{ |\det(\langle Tx_i, b_j \rangle)|^{1/n} : \begin{array}{l} x_1,\ldots,x_n \in U_E \\ b_1,\ldots,b_n \in U_F^{\circ} \end{array} \right\}.$$

These quantities were comprehensively studied by S. Geiss ([4], [5] and [6]). Among others he proved the following statements:

$$(1) \qquad \Gamma_n(T) = \left(\prod_{k=1}^{n} a_k(T) \right)^{1/n} \quad \text{for} \quad T \in \mathfrak{L}(l_2^n).$$

(2) $$\Gamma_n(T') = \Gamma_n(T) \quad \text{for} \quad T \in \mathfrak{L}(E,F).$$

(3) $$\Gamma_n(ST) \leq \sqrt{\frac{e}{n}} \, \| S|\mathfrak{P}_2 \| \, \Gamma_n(T) \quad \text{for} \quad T \in \mathfrak{L}(E,F) \quad \text{and} \quad S \in \mathfrak{P}_2(F,G).$$

Note that Γ_n is not an ideal norm.

Remark. In the original version of the famous Geiss inequality (3) there appeared e instead of $\sqrt{e}$; see also ([33], Lemma 1.8). The best possible constant $c \geq 1$ such that $\Gamma_n(ST) \leq cn^{-1/2} \| B|\mathfrak{P}_2 \| \Gamma_n(T)$ for $n = 1, 2, \ldots$ is still unknown.

6.2 Proposition. $\Gamma_n(T) \leq K_n(T)$ *for* $T \in \mathfrak{L}(E,F)$ *and* $n = 1, 2, \ldots$

Proof. Given $x_1, \ldots, x_n \in U_E$ and $b_1, \ldots, b_n \in U_F^\circ$, we define $X \in \mathfrak{L}(l_2^n, E)$ and $B \in \mathfrak{L}(F, l_2^n)$ by

$$X := \sum_{i=1}^{n} e_i \otimes x_i \quad \text{and} \quad B := \sum_{j=1}^{n} b_j \otimes e_j.$$

Then $(\langle Tx_i, b_j \rangle)$ is the representing matrix of $BTX \in \mathfrak{L}(l_2^n)$. Hence, applying the inequality of means and Lemma 5.5, we obtain

$$|\det(\langle Tx_i, b_j \rangle)|^{1/n} \leq \Gamma_n(BTX) = \left(\prod_{k=1}^{n} a_k(BTX) \right)^{1/n} \leq$$

$$\leq \frac{1}{n} \left(\prod_{k=1}^{n} a_k(BTX) \right) = \frac{1}{n} \| BTX|\mathfrak{S}_1 \| \leq$$

$$\leq \frac{1}{n} K_n(T) \left(\sum_{i=1}^{n} \| x_i \|^2 \right)^{1/2} \left(\sum_{j=1}^{n} \| b_j \|^2 \right)^{1/2} \leq K_n(T).$$

This proves that $\Gamma_n(T) \leq K_n(T)$.

6.3 Proposition. $G_n(T) \leq e \sum_{k=1}^{n} \frac{1}{k} \Gamma_k(T)$ *for* $T \in \mathfrak{L}(E,F)$ *and* $n = 1, 2, \ldots$

Proof. Let $X \in \mathfrak{L}(l_2^n, E)$ and $B \in \mathfrak{L}(F, l_2^n)$. Assume that $\| X|\mathfrak{P}_2' \| \leq 1$ and $\| B|\mathfrak{P}_2 \| \leq 1$. Then

$$a_n(BTX) \leq \left(\prod_{k=1}^{n} a_k(BTX) \right)^{1/n} = \Gamma_n(BTX) \leq$$

$$\leq \sqrt{\frac{e}{n}} \, \| B|\mathfrak{P}_2 \| \, \Gamma_n(TX) \leq \sqrt{\frac{e}{n}} \Gamma_n(X'T') \leq$$

$$\leq \frac{e}{n} \, \| X'|\mathfrak{P}_2 \| \, \Gamma_n(T') \leq \frac{e}{n} \Gamma_n(T).$$

Hence

$$\| BTX|\mathfrak{S}_1 \| = \sum_{k=1}^{n} a_k(BTX) \le e \sum_{k=1}^{n} \frac{1}{k}\Gamma_k(T).$$

The desired result now follows from property (G_4) in 4.5.

6.4. The following example shows that, up to the factor e, the preceding inequality is the best possible.

Example. *Let* $S \in \mathfrak{L}(L_1[0,1], C[0,1])$ *be the operator of integration defined by*

$$S : x(t) \longrightarrow \int_0^s x(t)\,\mathrm{d}t.$$

Then

$$\Gamma_n(S) = 1 \quad \textit{and} \quad G_n(S) \asymp 1 + \log n.$$

Proof. Note that the extremal points of the closed unit ball of $C[0,1]'$ have the form $\beta\delta_t$, where δ_t denotes the Dirac measure at the point t and $|\beta| = 1$; see ([1], p. 441). Hence, in order to compute $\Gamma_n(S)$, it is enough to take the supremum over

$$|\det\left(\langle Sx_i, \delta_{t_j}\rangle\right)|^{1/n} \quad \text{with} \quad \| x_i|L_1 \| \le 1 \quad \text{and} \quad 0 \le t_1 < \ldots < t_n \le 1.$$

Substracting the $(j-1)$-th column from the j-th column and putting $t_o = 0$, we obtain

$$\det\left(\langle Sx_i, \delta_{t_j}\rangle\right) = \det\left(\int_{t_{j-1}}^{t_j} x_i(t)\,\mathrm{d}t\right).$$

Now it follows from Hadamard's inequality that

$$|\det\left(\langle Sx_i, \delta_{t_j}\rangle\right)| \le \prod_{i=1}^{n}\left[\sum_{j=1}^{n}\left|\int_{t_{j-1}}^{t_j} x_i(t)\,\mathrm{d}t\right|^2\right]^{1/2} \le$$

$$\le \prod_{i=1}^{n}\left[\sum_{j=1}^{n}\int_{t_{j-1}}^{t_j} |x_i(t)|\,\mathrm{d}t\right] \le 1.$$

This proves that $\Gamma_n(S) \le 1$. The lower estimate can be obtained by choosing non-negative functions x_i such that

$$\text{support}(x_i) = \left[\frac{i-1}{n}, \frac{i}{n}\right] \quad \text{and} \quad \int_0^1 x_i(t)\,\mathrm{d}t = 1$$

and letting $t_j = j/n$. Then

$$\langle Sx_i, \delta_{t_j} \rangle = \begin{cases} 1 & \text{if } i \leq j, \\ 0 & \text{if } i > j. \end{cases}$$

In order to estimate $G_n(S)$, we note that the functions x_k and b_k defined by

$$x_k(t) := \sqrt{2} \cos \left(\frac{2k-1}{2} \pi t \right) \quad \text{and} \quad b_k(s) := \sqrt{2} \sin \left(\frac{2k-1}{2} \pi s \right)$$

constitute orthonormal systems in $L_2[0,1]$. Moreover,

$$Sx_k = \frac{2}{(2k-1)\pi} b_k \quad \text{for} \quad k = 1, 2, \ldots$$

See ([7], p. 120). Let I_x and I_b denote the canonical embeddings from $L_2[0,1]$ into $L_1[0,1]$ and from $C[0,1]$ into $L_2[0,1]$ respectively. Then $\| I_x | \mathfrak{P}'_2 \| = 1$. Define $X \in \mathfrak{L}(l_2^n, L_2)$ and $B \in \mathfrak{L}(L_2, l_2^n)$ by

$$X := \sum_{i=1}^{n} e_i \otimes x_i \quad \text{and} \quad B := \sum_{j=1}^{n} b_j \otimes e_j .$$

We now obtain that $\| I_x X | \mathfrak{P}'_2 \| \leq 1$, $\| B I_b | \mathfrak{P}_2 \| \leq 1$ and

$$\| B I_b S I_x X | \mathfrak{S}_1 \| = \frac{2}{\pi} \sum_{k=1}^{n} \frac{1}{2k-1} \asymp 1 + \log n.$$

Hence $G_n(S) \succ 1 + \log n$, by property (G_4) in 4.5. On the other hand, we have

$$G_n(S) \leq e \sum_{k=1}^{n} \frac{1}{k} \Gamma_k(S) = e \sum_{k=1}^{n} \frac{1}{k} \asymp 1 + \log n.$$

Remark. Operators of integration and summation were already used to produce various counerexamples; see ([20], p. 59), ([49], p. 571) and ([43], p. 177). The observation that $\Gamma_n(S) = 1$ is due to S. Geiss (unpublished).

7. OPERATOR IDEALS

7.1. Assume that with every operator T there is associated a scalar sequence

$$\| T \| = A_1(T) \leq A_2(T) \leq \ldots$$

such that all maps $A_n : T \to A_n(T)$ are ideal norms. Given $\rho \geq 0$, we denote by $\mathfrak{A}_\rho$ the collection of all operators T for which

$$\| T | \mathfrak{A}_\rho \| := \sup_n n^{-\rho} A_n(T)$$

is finite.

7.2. The following result can be proved by standard arguments.

Theorem. *$\mathfrak{A}_\rho$ is a Banach operator ideal.*

7.3. In view of 5.6, the Banach operator ideals $\mathfrak{G}_\rho$, $\mathfrak{H}_\rho$ and $\mathfrak{K}_\rho$ determined by the sequences (G_n), (H_n) and (K_n), respectively, coincide. It follows from

$$K_n(T) \leq H_n(T) \leq G_n(T) \leq n^{1/2} \| T \|$$

that $\mathfrak{G}_{1/2} = \mathfrak{H}_{1/2} = \mathfrak{K}_{1/2} = \mathfrak{L}$. Therefore, we can restrict our considerations to parameters ρ with $0 \leq \rho \leq 1/2$.

7.4. We can summarize the operator versions of some famous isomorphic characterizations of Hilbert spaces which are stated in Criterions 2.3, 3.3 and 4.8.

Theorem. $\mathfrak{G}_0 = \mathfrak{H}_0 = \mathfrak{K}_0 = \mathfrak{H}$.

7.5. Next, we establish an immediate consequence of Propositions 3.4 or 4.9

Proposition. *The operator ideal $\mathfrak{H}_\rho$ is symmetric, injective and surjective.*

Remark. It follows from 2.5 that the norm

$$\| T|\mathfrak{H}_\rho \| := \sup_n n^{-\rho} H_n(T)$$

is injective. As stated in 2.7, I do not know whether it is also surjective and symmetric. Those who are interested to have all of these properties must pass to the equivalent norms

$$\| T|\mathfrak{G}_\rho \| := \sup_n n^{-\rho} G_n(T) \quad \text{or} \quad \| T|\mathfrak{K}_\rho \| := \sup_n n^{-\rho} K_n(T).$$

7.6. Given $0 < r \leq 2$, we denote by $\mathfrak{L}^{(z)}_{r,\infty}$ the collection of all operators T for which

$$\| T|\mathfrak{L}^{(z)}_{r,\infty} \| := \sup_n n^{1/r} z_n(T)$$

is finite; see ([33], 2.7). This definition yields a 1-parameter scale of quasi-Banach operator ideals. In view of

$$n z_n(T) \leq G_n(T) \leq \sum_{k=1}^{n} z_k(T),$$

we have

$$\mathfrak{L}^{(z)}_{r,\infty} = \mathfrak{H}_\rho \quad \text{whenever} \quad 1/r + \rho = 1,\ 1 < r \leq 2 \quad \text{and} \quad 0 < \rho \leq 1/2.$$

However, in the limiting case $r = 1$ and $\rho = 0$ it turns out that $\mathfrak{H}^{\text{weak}} := \mathfrak{L}^{(z)}_{1,\infty}$ (the ideal of *weakly Hilbertian operators*) is strictly larger than $\mathfrak{H} = \mathfrak{H}_0$ (the ideal of *Hilbertian operators*). As already observed by G. Pisier ([42], p. 571) and ([43], p. 171), the operator of integration belongs to $\mathfrak{H}^{\text{weak}} \setminus \mathfrak{H}$; see Example 6.4 and ([33], Lemmas 1.11, 1.13 and 2.11).

8. GEOMETRIC PARAMETERS OF BANACH SPACES

8.1. Recall that $G_n(E)$, $H_n(E)$ and $K_n(E)$ denote the values of the ideal norms for the identity map of the Banach space E. The Grothendieck numbers $\Gamma_n(E)$ are defined analogously.

8.2. For every m-dimensional Banach space M the *Banach-Mazur distance* to l_2^m is given by

$$\mathrm{d}(M) := \inf \{\| T \| \| T^{-1} \| : T \in \mathfrak{L}(M, l_2^m), \text{ bijection}\}.$$

Remark. This concept goes back to the «Remarques» in Banach's monograph.

8.3. The following result is obvious, by ([31], B.4.6).

Theorem. *Let E be a Banach space and $n = 1, 2, \ldots$ Then*

$$H_n(E) = \sup \{\mathrm{d}(M) : M \subseteq E, \dim(M) \le n\}.$$

Remark. The right hand quantities were considered for the first time by M.I. Kadec/B.S. Mityagin ([13], Prop. 2.7) when they presented the Lindenstrauss-Tzafriri solution of the complemented subspace problem [25]. We also refer to ([48], p. 209).

8.4. Let M be any finite dimensional subspace of a Banach space E. Then the *relative Hilbertian projection constant* of M in E is given by

$$\gamma(M, E) := \inf \{\| P | \mathfrak{H} \| : P \in \mathfrak{L}(E), \text{ projection onto } M\}.$$

Remark. To my best knowledge, the first explicite definition of this quantity is given in ([48], p. 209). However, it has appeared implicitely in the work of D.R. Lewis [21], T. Figiel/N. Tomczak-Jägermann [3], H. König/J.R. Retherford/N. Tomczak-Jägermann [15] and G. Pisier [37] around 1979.

8.5. Next, we formulate an immediate consequence of Theorem 4.12.

Theorem. *Let E be a Banach space and $n = 1, 2, \ldots$ Then*

$$G_n(E) = \sup \{\gamma(M, E) : M \subseteq E, \dim(M) \le n\}.$$

Remark. The right hand quantities were considered for the first time by G. Pisier ([37], p. 4). For a comprehensive treatement we refer the reader to ([48], Chap. 6).

8.6. Let M be any finite dimensional subspace of a Banach space E. Then the *relative projection constant* of M in E is given by

$$\lambda(M, E) := \inf \{\| P \| : P \in \mathfrak{L}(E), \text{ projection onto } M\}.$$

Remark. This concept can be traced back to the early sixtieth when B. Gruenbaum [8] introduced a quantity which is now called the *absolute* projection constant of a finite dimensional Banach space.

8.7. For a Banach space E and $n = 1, 2, \ldots$, we define

$$P_n(E) := \sup\{\lambda(M, E) : M \subseteq E,\ \dim(M) \leq n\}.$$

Remark. Similar geometric parameters were considered for the first time by M.I.Kadec/B.S. Mityagin ([13], Prop. 27) when they presented the Lindenstrauss/Tzafriri solution of the complemented subspace problem [25]. We also refer to [14] and ([48], p. 209).

8.8. The following result is an immediate consequence of 5.6, 6.2, 8.5 and 8.7; see also ([48], p. 210).

Proposition. *Let E be any Banach space and $n = 1, 2, \ldots$ Then*

$$\Gamma_n(E) \leq K_n(E) \leq H_n(E) \leq G_n(E) \quad \text{and} \quad P_n(E) \leq G_n(E).$$

8.9. It would be extremely important to replace Definition 8.7 by a theorem analogous to the geometric interpretations given in 8.3 and 8.5.

Problem. *Does there exists an ideal norm P_n such that (in the sense of 8.1)*

$$P_n(E) = \sup\{\lambda(M, E) : M \subseteq E,\ \dim(M) \leq n\}?$$

8.10. We stress the fact that the proof of the following inequality works in the complex case only. I do not know whether the same result, possibly with a different constant, also holds in the real case.

Proposition. *Let E be a complex Banach space and $n = 1, 2, \ldots$ Then*

$$\Gamma_n(E) \leq \pi P_n(E).$$

Proof. Let $x_1, \ldots, x_n \in U_E$ and $a_1, \ldots, a_n \in U_E^0$. Additionally, we assume that the (n,n)-matrix $(\langle x_i, a_j \rangle)$ is non-singular. By the principle of related operators ([32], 3.3.4 and 3.3.5), the matrix $(\langle x_i, a_j \rangle)$ and the operator

$$T := \sum_{k=1}^{n} a_k \otimes x_k$$

have the same (non-zero) eigenvalues, which we denote by $\lambda_1, \dots, \lambda_n$. By a result from ([29], p. 331), we can find a subset K of $(1, \dots, n)$ with

$$\sum_{k=1}^{n} |\lambda_k| \leq \pi \, |\sum_K \lambda_k|$$

and, subsequently, a T-invariant subspace M of E such that the operator induced by T has precisely the eigenvalues λ_k with $k \in K$; ([32], 3.2.23). Since $\dim(M) \leq n$, there exists a projection P from E onto M with $\| P \| \leq P_n(E)$. Now it follows from

$$\sum_{k=1}^{n} |\lambda_k| \leq \pi |\sum_K \lambda_k| = \pi |\text{trace } (PT) =$$
$$= \pi |\sum_{k=1}^{n} \langle P x_k, a_k \rangle| \leq n\pi \, \| P \| \leq n\pi P_n(E)$$

and the inequality of means that

$$|\det(\langle x_i, a_j \rangle)|^{1/n} = \left(\prod_{k=1}^{n} |\lambda_k| \right)^{1/n} \leq \frac{1}{n} \sum_{k=1}^{n} |\lambda_k| \leq \pi P_n(E).$$

This proves the desired result.

8.11. Combining Propositions 6.3, 8.8 and 8.10, we obtain another important result of this paper.

Theorem. *Let E be a complex Banach space and $n = 1, 2, \dots$ Then*

$$P_n(E) \leq G_n(E) \leq e\pi \sum_{k=1}^{n} \frac{1}{k} P_k(E) \leq e\pi(1 + \log n) P_n(E).$$

8.12. It seems likely that the logarithmic factor in the inequality

$$(*) \qquad G_n(E) \leq e\pi(1 + \log n) P_n(E)$$

can be removed.

Problem. *Does there exist a constant* $c > 1$ *such that*

$$G_n(E) \leq cP_n(E)$$

for all Banach spaces E *and* $n = 1, 2, \ldots$?

Remark. Inequalities of the form

$$G_n(E) \leq cP_n(E)^\alpha$$

emerged in connection with the Lindestrauss-Tzafriri solution of the complemented subspace problem [25]. The exponent $\alpha = 32$ obtained in ([2], Theorem 6.7) was improved to $\alpha = 5$ in ([48], Theorem 29.4). Strangely enough, for infinite dimensional Banach spaces one can even take $\alpha = 2$; see ([13], Prop. 2.7) and ([48], Theorem 29.1).

We stress the fact that the estimate $G_n(E) \leq cP_n(E)^\alpha$ with $\alpha > 1$ is better than (*) only in the rare case when the sequence $(P_n(E))$ grows very slowly.

The most important step towards inequality (*) was already done in the Thesis of S. Geiss ([4], Satz 2.3.1) who proved that

$$G_n(E) \leq e(1 + \log n) \max_{1 \leq k \leq n} \Gamma_k(E).$$

8.13. For $0 \leq \rho \leq 1/2$, we let

$$\mathbf{H}_\rho := \left\{E : (n^{-\rho} H_n(E)) \in l_\infty\right\}.$$

Replacing $H_n(E)$ by $G_n(E)$, $K_n(E)$, $P_n(E)$ and $\Gamma_n(E)$, the classes $\mathbf{G}_\rho$, $\mathbf{K}_\rho$, $\mathbf{P}_\rho$ and $\mathbf{\Gamma}_\rho$ can be defined in the same way.

8.14. We denote by $\mathbf{L}$ the class of arbitrary Banach spaces.

Theorem. $\mathbf{G}_{1/2} = \mathbf{H}_{1/2} = \mathbf{K}_{1/2} = \mathbf{P}_{1/2} = \mathbf{\Gamma}_{1/2} = \mathbf{L}$.

Beweis. This follows from 4.7 and 8.8.

8.15. We denote by $\mathbf{H}$ the class of all Banach spaces which are isomorphic to Hilbert spaces. The following theorem summarizes various characterizations of this class due to J. I. Joichi [12], S. Kwapień ([17], [18], Cor. 1, and [19], Prop. 3.1) and J. Lindenstrauss/L. Tzafriri [25].

Theorem. $\mathbf{G}_0 = \mathbf{H}_0 = \mathbf{K}_0 = \mathbf{P}_0 = \mathbf{H}$.

Remark. Using the 2-convexified Tsirelson space, G. Pisier ([43], Chap. 13) showed that $\mathbf{\Gamma}_0$ (the class of *weak Hilbert spaces*) is strictly larger than $\mathbf{H}$.

8.16. Next we treat the general (complex!) case.

Theorem. $\mathbf{G}_\rho = \mathbf{H}_\rho = \mathbf{K}_\rho = \mathbf{P}_\rho = \Gamma_\rho$ *for* $0 < \rho < 1/2$.

Proof. This follows from the inequalities stated in 6.3, 8.8 and 8.10.

8.17. The class $\mathbf{H}_\rho$ enjoys very nice permanence properties.

Theorem. $\mathbf{H}_\rho$ *is stable under passing to subspaces, quotients, duals and finite direct sums.*

Proof. Note that $\mathbf{H}_\rho = \{E : I_E \in \mathfrak{H}_\rho\}$ and apply Proposition 7.5.

8.18. Furthermore, we see from 3.6 that the classes $\mathbf{H}_\rho$ behave very well under complex interpolation.

Proposition. *Let* $\rho = (1-\Theta)\rho_0 + \Theta\rho_1$ *for* $0 \le \rho_0, \rho_1 \le 1/2$ *and* $0 < \Theta < 1$. *Then* $E_0 \in \mathbf{H}_{\rho_0}$ *and* $E_1 \in \mathbf{H}_{\rho_1}$ *imply* $[E_0, E_1]_\Theta \in \mathbf{H}_\rho$.

8.19. It is unknown whether a similar result holds for real interpolation.

Problem. *Let* $\rho = (1-\Theta)\rho_0 + \Theta\rho_1$ *for* $0 \le \rho_0, \rho_1 \le 1/2$ *and* $0 < \Theta < 1$. *Assume that* $|1/q - 1/2| \le \rho$. *Do* $E_0 \in \mathbf{H}_{\rho_0}$ *and* $E_1 \in \mathbf{H}_{\rho_1}$ *imply* $(E_0, E_1)_{\Theta,q} \in \mathbf{H}_\rho$?

Remark. We see from ([49], p. 128) that the answer is affirmative for $q = 2$.

8.20. For the definition of the concepts «*weak type* p» and «*weak cotype* q», we refer to [26], [34] and ([43], p. 168). The following result is taken from ([26], p. 106) and ([34], 3.14).

Proposition. *Let* $1/p = 1/2 + \rho$ *and* $1/q = 1/2 - \rho$ *for* $0 \le \rho < 1 : 2$. *Then every Banach space* $E \in \mathbf{H}_\rho$ *has weak type* p *and weak cotype* q.

8.21. For $0 < \rho \le 1/2$, we let

$$\mathbf{H}_\rho^0 := \{E : (n^{-\rho} H_n(E)) \in c_0\}.$$

Replacing $H_n(E)$ by $G_n(E)$, $K_n(E)$, $P_n(E)$ and $\Gamma_n(E)$, the classes $\mathbf{G}_\rho^0, \mathbf{K}_\rho^0, \mathbf{P}_\rho^0$ and Γ_ρ^0 can be defined in the same way.

8.22. We denote by **B** the class of all B-*convex Banach spaces*, which are characterized by the property that they do not contain the spaces l_1^n uniformly. A classical result of V.D. Milman/H. Wolfson [28] now reads as follows. See also [35] and [37].

Theorem. $\mathbf{G}_{1/2}^0 = \mathbf{H}_{1/2}^0 = \mathbf{K}_{1/2}^0 = \mathbf{P}_{1/2}^0 = \Gamma_{1/2}^0 = \mathbf{B}$.

8.23. We conclude this section by formulating an open problem which goes back to G. Pisier ([38], [39] and [40]).

Problem. *Is it true that*

$$\mathbf{H}_\rho^0 = \bigcup_{0<\sigma<\rho} \mathbf{H}_\sigma \quad \text{for} \quad 0 < \rho \leq 1/2?$$

Remark. As shows by G. Pisier ([38], [39]), for $\rho = 1/2$, the answer is affirmative in the setting of Banach lattices.

9. EXAMPLES

9.1. The following result goes back to V.I. Gurarii/M.I. Kadec/V.I. Macaev ([9], Lemma 2).

Example. *Let* $1 \leq p \leq \infty$. *Then, in the real and complex case, we have*

$$G_n(l_p^n) = H_n(l_p^n) = n^{|1/p-1/2|} \textit{ for } n = 1,2,\ldots$$

Proof. It follows from $G_n(l_2^n) = 1$ that

$$H_n(l_p^n) \leq G_n(l_p^n) \leq \| I : l_p^n \to l_2^n \| \, G_n(l_2^n) \, \| I : l_2^n \to l_p^n \| \leq n^{|1/p-1/2|}.$$

On the other hand,

$$G_n(l_p^n) \geq H_n(l_p^n) \geq \mathrm{d}(l_p^n) = n^{|1/p-1/2|};$$

see ([48], p. 280).

9.2. We call $n = 1,2,\ldots$ an *Hadamard number* if there exists an (n,n)-matrix $A_n = (\alpha_{ij})$ such that

$$\alpha_{ij} = \pm 1 \quad \text{and} \quad A_n A_n' = n I_n,$$

where I_n denotes the unit (n,n)-matrix.

All Hadamard number are necessarily multiplies of 4. However, it is a long-standing open problem whether this property also suffices. In any case, all powers of 2 are Hadamard. The required matrices can be obtained by induction:

$$A_{2n} := \begin{pmatrix} A_n & A_n \\ A_n & -A_n \end{pmatrix} \quad \text{and} \quad A_1 := (1).$$

For more information we refer the reader to [50].

9.3 Example. *Let* $1 \le p \le \infty$. *Then, in the complex case, we have*

$$\Gamma_n(l_p^n) = K_n(l_p^n) = n^{|1/p-1/2|} \quad for \quad n = 1, 2, \ldots$$

The same holds in the real case $(p \ne 2)$ *if and only if* n *is an Hadamard number.*

Proof. By 8.8 and 9.1,

$$\Gamma_n(l_p^n) \le K_n(l_p^n) \le H_n(l_p^n) \le n^{|1/p-1/2|}.$$

To obtain the lower estimate for $1 \le p \le 2$ and in the complex case, we put

$$a_j = \left(\alpha_{1j}, \ldots, \alpha_{nj}\right) \quad \text{with} \quad \alpha_{ij} := \exp\left(\frac{2\pi\sqrt{-1}}{n}\, ij\right).$$

Note that $\| a_j | l_{p'}^n \| = n^{1/p'}$. Hence

$$K_n(l_p^n) \ge \Gamma_n(l_p^n) \ge |\det\left(\langle e_i, n^{-1/p'} a_j\rangle\right)|^{1/n} =$$

$$= n^{-1/p'} |\det\left(\alpha_{ij}\right)|^{1/n} = n^{1/p-1/2};$$

see also [5]. In the real case, the same argument works if there exists an Hadamard matrix $a = (\alpha_{ij})$.

We now assume that $K_n(l_p^n) = n^{1/p-1/2}$, where $n = 1, 2, \ldots$ and $1 \le p < 2$ are fixed. Then there exist $x_i = (\xi_{i1}, \ldots, \xi_{1n}) \in l_p^n$ and a unitary (n,n)-matrix $S = (\sigma_{ij})$ such that

$$\left(\sum_{j=1}^{n} \| \sum_{i=1}^{n} \sigma_{ij} x_i | l_p^n \|^2\right)^{1/2} = n^{1/p-1/2} \left(\sum_{i=1}^{n} \| x_i | l_p^n \|^2\right)^{1/2}$$

This implies that in the following chain of inequality we even have equalities:

$$\left(\sum_{j=1}^{n} \left[\sum_{k=1}^{n} |\sum_{i=1}^{n} \sigma_{ij}\xi_{ik}|^p\right]^{2/p}\right)^{1/2} \le$$

$$\le n^{1/p-1/2} \left(\sum_{j=1}^{n}\sum_{k=1}^{n} |\sum_{i=1}^{n} \sigma_{ij}\xi_{ik}|^2\right)^{1/2} =$$

$$= n^{1/p-1/2} \left(\sum_{i=1}^{n}\sum_{k=1}^{n} |\xi_{ik}|^2\right)^{1/2} \le$$

$$\le n^{1/p-1/2} \left(\sum_{i=1}^{n} \left[\sum_{k=1}^{n} |\xi_{ik}|^p\right]^{2/p}\right)^{1/2}.$$

Hence

$$\left[\sum_{k=1}^{n}|\sum_{i=1}^{n}\sigma_{ij}\xi_{ik}|^{p}\right]^{1/p} = n^{1/p-1/2}\left[\sum_{k=1}^{n}|\sum_{i=1}^{n}\sigma_{ij}\xi_{ik}|^{2}\right]^{1/2}$$

for $j = 1,\ldots,n$ and

$$\left[\sum_{k=1}^{n}|\xi_{ik}|^{2}\right]^{1/2} = \left[\sum_{k=1}^{n}|\xi_{ik}|^{p}\right]^{1/p}$$

for $j = 1,\ldots,n$. By ([10], Theorem 19), we conclude from the latter set of equations that x_i must be a multiple of a unit vector. In view of

$$K_n(l_p^m) \leq m^{1/p-1/2} \quad \text{for} \quad m = 1,\ldots,n,$$

the vectors $x_1,\ldots,x_n$ do not belong to any proper subspace of l_p^n. Thus, by a permutation, we can arrange that $x_i = \xi_i e_i$. Now the first set of equations reads as follows:

$$\left[\sum_{k=1}^{n}|\sigma_{kj}\xi_{k}|^{p}\right]^{1/p} = n^{1/p-1/2}\left[\sum_{k=1}^{n}|\sigma_{kj}\xi_{k}|^{2}\right]^{1/2}$$

for $j = 1,\ldots,n$. Therefore, ([10], Theorem 18) implies that, for fixed j, all numbers $|\sigma_{1j}\xi_1|,\ldots,|\sigma_{nj}\xi_n|$ are equal. We denote the common value by α_j.

Using the fact that (σ_{ij}) is unitary, we conclude that

$$|\xi_i|^2 = \sum_{j=1}^{n}|\sigma_{ij}\xi_i|^2 = \sum_{j=1}^{n}\alpha_j^2$$

for $i = 1,\ldots,n$. Thus all numbers $|\xi_1|,\ldots,|\xi_n|$ are equal. We denote the common value by ξ. Using again the fact that (σ_{ij}) is unitary, we get

$$\xi^2 = \sum_{i=1}^{n}|\sigma_{ij}\xi_i|^2 = n\alpha_j^2.$$

Next, $|\sigma_{ij}|\xi = |\sigma_{ij}\xi_i| = \alpha_j$ yields $|\sigma_{ij}| = n^{-1/2}$. Since σ_{ij} was assumed to be real, we finally see that $n^{1/2}\sigma_{ij} = \pm 1$. This completes the proof for $1 \leq p < 2$. The case $2 < p \leq \infty$ can be treated by passing to the dual space.

Remark. In the real case, sophisticated computations show that

$$\Gamma_3\left(l_1^3\right) = 4^{1/3} \approx 1,58, \quad K_3\left(l_1^3\right) = 5/3 \approx 1,66 \quad \text{and} \quad H_3\left(l_1^3\right) = 3^{1/2} \approx 1,73.$$

9.4. A recent result of H. König/N. Tomczak-Jägermann [16] states that, for all real and complex Banach spaces E,

$$P_n(E) \leq n^{1/2} - cn^{-1/2},$$

where $c > 0$ is a constant. Hence

$$P_n(l_1^n) < \Gamma_n(l_1^n) = n^{1/2}$$

for all Hadamard numbers n. This shows that $P_n(E)$ can be strictly smaller than $\Gamma_n(E) \leq K_n(E) \leq H_n(E) \leq G_n(E)$.

It seems to be extremely difficult to compute the quantity $P_n(E)$ for concrete Banach spaces. In particular, we have the

Problem. *Which is the value of $P_n(l_1^n)$ for $p \neq 2$?*

9.5. Following G. Pisier [39], a complex Banach space E is said to be $\Theta-$ *Hilbertian* $(0 < \Theta < 1)$ if it is isomorphic to a space $[E_0, H]_\Theta$ obtained by complex interpolation between a suitable Banach space E_0 and a Hilbert space H.

9.6. We now formulate an immediate consequence of 8.18.

Proposition. *Let $0 < \Theta < 1$ and $\rho = (1-\Theta)/2$. Then every Θ-Hilbertian Banach space belongs to $\mathbf{H}_\rho$.*

9.7. We denote by L_p the Banach space of all p-integrable scalar functions living on an arbitrary measure space (Ω, μ). The next result is due to D.R. Lewis ([21], Cor. 4).

Example. *Let $1 \leq p \leq \infty$ and $\rho = |1/p - 1/2|$. Then $L_p \in \mathbf{H}_\rho$.*

Proof. We treat the case when $1 < p < 2$. Then $L_p = [L_1, L_2]_\Theta$, where Θ is defined by $1/p = (1-\Theta)/1 + \Theta/2$.

9.8. We denote by $L_p(E)$ the Banach space of all Bochner p-integrable E-valued functions living on an arbitrary measure space (Ω, μ).

Example. *Let $1 \leq p \leq \infty$, $\rho = |1/p - 1/2|$ and $\Theta = 1 - 2\rho$. If E is Θ-Hilbertian, then $L_p(E)$ belongs to $\mathbf{H}_\rho$.*

Proof. We treat the case when $1 < p < 2$. Since $E = [E_0, H]_\Theta$ and $1/p = (1-\Theta)/1 + \Theta/2$, by ([49], p. 121), we have

$$L_p(E) = L_p\left([E_0, H]_\Theta\right) = \left[L_1\left(E_0\right), L_2(H)\right]_\Theta.$$

9.9. It is unknown whether the preceding result extends as follows.

Problem. *Let* $1 \leq p \leq \infty$ *and* $\rho = |1/p - 1/2|$. *Does* $E \in \mathbf{H}_\rho$ *imply* $L_p(E) \in \mathbf{H}_\rho$*?*

9.10. The Schatten-von Neumann classes $\mathfrak{S}_p(H) := \mathfrak{L}_p^{(a)}(H)$ can be viewed as non-commutative analoga of the function spaces L_p. Therefore, the next result is closely related to 9.7. Its original proof is due to N. Tomczak-Jägermann ([47], Cor. 2.10) see also ([22], Cor. 5.3).

Example. *Let* $1 \leq p \leq \infty$ *and* $\rho = |1/p - 1/2|$. *Then* $\mathfrak{S}_p(H) \in \mathbf{H}_\rho$.

Proof. Use the interpolation argument from 9.7.

9.11. Note that the following result is asymmetric.

Example. *Let* $1 \leq p \leq \infty$. *If* $1 \leq p, q \leq 2$, *then* $\mathfrak{P}_2(l_p, l_q) \in \mathbf{H}_\rho$, *where* $\rho = \max(1/p - 1/2, 1/q - 1/2)$. *However, in the remaining cases* $\mathfrak{P}_2(l_p, l_q)$ *even fails to be B-convex.*

Proof. Very recently, G. Pisier [44] showed that $\mathfrak{P}_2(l_p, l_q)$ is Θ-Hilbertian, where Θ is the minimum of α and β defined by $1/p = (1-\alpha)/1 + \alpha/2$ and $1/q = (1-\beta)/1 + \beta/2$ provided that $1 \leq p, q \leq 2$. The remaining case follows from ([23], Theorem 3 and Cor. 1).

9.12. Finally, we present an example which shows that the class $\mathbf{H}_\rho$ with $\rho > 0$ contains non-reflexive Banach spaces. The following construction is taken from G. Pisier/Quanhua Xu [45].

Let v_1^0 denote the Banach space of all zero sequences $x = (\xi_k)$ for which

$$\| x | v_1^0 \| := \sum_{k=1}^{\infty} |\xi_k - \xi_{k+1}|$$

is finite. Then (v_1^0, c_0) constitutes an interpolation couple.

Example. $H_n((v_1^0, c_0)_{1/2,2}) \asymp 1 + \log n$.

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