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A CHARACTERIZATION OF LAGRANGIAN DUAL PROBLEMS

J.-E. MARTINEZ-LEGAZ, I. SINGER

Dedicated to the memory of Professor Gottfried Köthe

Abstract. *For constrained primal infimization problems, we give a characterization of the Lagrangian dual objective function, regarded as a function of three variables (namely, of the target set, the primal objective function and the dual variables).*

1. INTRODUCTION

Let $(F \xrightarrow{u} X)$ be a (fixed) *system* (i.e. [5], [8], a triple consisting of two non-empty sets F and X and a mapping $u : F \to X$), Ω a non-empty subset of X, called *target set* and $h : F \to \overline{R} = [-\infty, +\infty]$ a function, called *the objective function.* We shall consider the *primal* infimization problem (with *constraint set* $u^{-1}(\Omega)$)

$$(1) \qquad (P) = (P_{\Omega,h}) \quad \alpha = \alpha_{\Omega,h} = \inf_{\substack{y\in F \\ u(y)\in\Omega}} h(y).$$

By a *dual* problem to (P) we shall mean any supremization problem of the form

$$(2) \qquad (Q) = (Q^{\Omega,h}) \quad \beta = \beta^{\Omega,h} = \sup \lambda(W),$$

where W is a fixed set (assumed non-empty, without loss of generality), called *dual constraint set* and $\lambda = \lambda^{\Omega,h} : W \to \overline{R}$ is a function, called *dual objective function.* We recall (see [1], [8]) that *the Lagrangian dual problem* to (P) is, by definition, the dual problem $\sup \lambda_{\text{Lagr}}(W)$ of (2), with (fixed) $\Omega \subseteq R^X$ (where R^X denotes the family of all functions $f : X \to R = (-\infty, +\infty)$) and with $\lambda_{\text{Lagr}} = \lambda^{\Omega,h}_{\text{Lagr}} : W \to \overline{R}$ defined by

$$(3) \qquad \lambda_{\text{Lagr}}(w) = \inf_{y\in F}\{h(y) - wu(y)\} \dot{+} \inf w(\Omega) \qquad (w \in W),$$

where $\dot{+}$ denotes the «lower addition» [3]on $\overline{R}$, defined by

$$(4) \qquad a \dot{+} b = a + b \quad \text{if } R \cap \{a, b\} \neq \emptyset \quad \text{or } a = b = \pm\infty,$$

$$(5) \qquad a \dot{+} b = -\infty \quad \text{if } a = -b = \pm\infty.$$

Key Words: primal infimization problem, Lagrangian dual problem, Fenchel conjugate.

The assumption that $W \subseteq R^X$ is fixed (i.e., it does not depend on F, u, Ω or h) is satisfied e.g. when $W = R^X$, or when X is a locally convex space and $W = X^*$, the family of all continuous linear functionals on X. On the other hand, we shall regard Ω and h as (variable) *parameters* of $(P), (Q)$; some of the advantages of this method have been shown in [8], [9] (for example, in [9], consideration of the case when $X = F, u = I_F$, the identity operator, and Ω is a singleton $\{y_0\}$ where $y_0 \in F$, has led to the concept of «the subdifferential of h at y_0, with respect to a primal-dual pair of optimization problems $\{(P), (Q)\}$», which encompasses, as particular cases, several known concepts of subdifferential). This also motivates our simplified notations $\beta^{\Omega,h}, \lambda^{\Omega,h}$, instead of $\beta^{u^{-1}(\Omega),h}, \lambda^{u^{-1}(\Omega),h}$ of [8].

The aim of the present Note is to give a characterization of the Lagrangian dual objective function $\lambda^{\Omega,h}_{\text{Lagr}}(w)$ as a function of three variables Ω, h and w (where $(F \xrightarrow{u} X)$ and $W \subseteq R^X$ are fixed). Namely, we shall show that four natural conditions on a function $\lambda^{\cdot,\cdot}(.)$ are necessary and sufficient in order to have

$$\lambda^{\Omega,h}(w) = \lambda^{\Omega,h}_{\text{Lagr}}(w) \qquad (\Omega \in 2^X \setminus \emptyset, h \in \overline{R}^F, w \in W), \tag{6}$$

where 2^X denotes the family of all subsets of X, and $\emptyset$ denotes the empty set; for simplicity, we write $2^X \setminus \emptyset$ instead of $2^X \setminus \{\emptyset\}$. Throughout this Note, we shall adopt the usual conventions $\inf \emptyset = +\infty, \sup \emptyset = -\infty$.

2. PRELIMINARIES

In the sequel, we shall use the following tools:

a) Formula (3) can be also written in the form

$$\lambda^{\Omega,h}_{\text{Lagr}}(w) = -h^*(wu) \dotplus \inf w(\Omega) \qquad (w \in W), \tag{7}$$

where $h^* \in \overline{R}^{(R^F)}$ is the (generalized) Fenchel conjugate of $h \in \overline{R}^F$, defined (see e.g. [3], [2]) by

$$h^*(\Psi) = \sup_{y \in F}\{\Psi(y) - h(y)\} \qquad (\Psi \in R^F). \tag{8}$$

It is easy to check (and well known) that for every index set I we have

$$(\inf_{i \in I} h_i)^*(\Psi) = \sup_{i \in I} h_i^*(\Psi) \qquad (\{h_i\}_{i \in I} \subseteq \overline{R}^F, \Psi \in R^F). \tag{9}$$

b) We have (see [7], formula (3.7))

$$h = \inf_{(y,d)\in Epi\ h}\{\chi_{\{y\}} + d\} \qquad (h \in \overline{R}^F), \tag{10}$$

where $Epi\ h = \{(y,d) \in F \times R | h(y) \leq d\}$ is the epigraph of h and $\chi_{\{y\}}$ is the indicator function of the singleton $\{y\}$ (defined by $\chi_{\{y\}}(y') = 0$ if $y' = y$ and $\chi_{\{y\}}(y') = +\infty$ if $y' \in F \setminus \{y\}$).

c) We shall also use the following known formula (see [3], corollary 4.b) and [8], lemma 2.1): If E is a non-empy set, $f : E \to \overline{R}, b \in \overline{R}$ and $c \in R \cup \{-\infty\}$, then

$$\inf_{y\in E} f(y) \dot{+} c = \inf_{y\in E}\{f(y) \dot{+} c\}. \tag{11}$$

3. THE CHARACTERIZATION THEOREM

Theorem 1. *Let $(F \xrightarrow{u} X)$ be a system and $W \subseteq R^X$. For a function $\lambda = \lambda^{\cdot,\cdot}(.)$: $(2^X \setminus \emptyset) \times \overline{R}^F \times W \to \overline{R}$, the following statements are equivalent:*

1) λ is the function (3) (i.e., we have (6)).

2) For any index set I, any $\Omega \in 2^X \setminus \emptyset, \{h_i\}_{i\in I} \subseteq \overline{R}^F, w \in W, h \in \overline{R}^F, d \in R, \{\Omega_i\}_{i\in I} \subseteq 2^X \setminus \emptyset$ with $I \neq \emptyset$, and any $x \in X, y \in F$, we have

$$\lambda^{\Omega,\inf_{i\in I} h_i}(w) = \begin{cases} \inf_{i\in I} \lambda^{\Omega,h_i}(w), & \text{if } I \neq \emptyset \quad \text{or } \inf w(\Omega) > -\infty \\ -\infty, & \text{if } I = \emptyset \quad \text{and } \inf w(\Omega) = -\infty, \end{cases} \tag{12}$$

$$\lambda^{\Omega,h+d} = \lambda^{\Omega,h} + d, \tag{13}$$

$$\lambda^{\bigcup_{i\in I}\Omega_i,h} = \inf_{i\in I} \lambda^{\Omega_i,h}, \qquad \text{if} \quad h \not\equiv +\infty, \tag{14}$$

$$\lambda^{\{x\},\chi_{\{y\}}}(w) = w(x) - wu(y), \tag{15}$$

where the inf *in (14) is taken pointwise on W.*

Proof. Note that, by $W \subseteq R^X$, we have

$$\inf w(\Omega) < +\infty \qquad (\Omega \in 2^X \setminus \emptyset, w \in W). \tag{16}$$

1) $\Rightarrow$ 2). Assume 1), and let $I, \Omega, \dots$ be as in 2). If $I \neq \emptyset$ or $\inf w(\Omega) > -\infty$, then, by (6), (7), (9), and (11), we obtain

$$\lambda^{\Omega, \inf_{i\in I} h_i}(w) = -(\inf_{i\in I} h_i)^*(wu) \dot{+} \inf w(\Omega) = -\sup_{i\in I} h_i^*(wu) \dot{+} \inf w(\Omega) =$$

$$= \inf_{i\in I}(-h_i^*(wu)) \dot{+} \inf w(\Omega) = \inf_{i\in I}\{-h_i^*(wu) \dot{+} \inf w(\Omega)\} = \inf_{i\in I} \lambda^{\Omega, h_i}(w).$$

On the other hand, if $I = \emptyset$ and $\inf w(\Omega) = -\infty$, then, by (3),

$$\lambda^{\Omega, \inf_{i\in I} h_i}(w) = \lambda^{\Omega, +\infty}(w) = +\infty \dot{+} -\infty = -\infty,$$

which proves (12). Furthermore, by (3) and (11) we have (13). Now, if $h \not\equiv +\infty$, then $-h^*(wu) < +\infty$, whence, by (6), (7) and (11),

$$\lambda^{\cup_{i\in I}\Omega_i, h}(w) = -h^*(wu) \dot{+} \inf w\left(\cup_{i\in I}\Omega_i\right) = -h^*(wu) \dot{+} \inf_{i\in I} \inf w(\Omega_i) =$$

$$= \inf_{i\in I}\{-h^*(wu) \dot{+} \inf w(\Omega_i)\} = \inf_{i\in I} \lambda^{\Omega_i, h}(w),$$

which proves (14). Finally, by (6), (3) for $h = \chi_{\{y\}}$ and $\Omega = \{x\}$, we get

$$\lambda^{\{x\}, \chi_{\{y\}}}(w) = \inf_{y'\in F}\{\chi_{\{y\}}(y') - wu(y')\} + w(x) = -wu(y) + w(x),$$

which proves (15). Thus, 1) $\Rightarrow$ 2).

2) $\Rightarrow$ 1). Assume 2) and $h \not\equiv +\infty$, whence $Epi\ h \neq \emptyset$ and $\inf_{y\in F}\{h(y)) - wu(y)\} < +\infty$. Then, by (14), (10), (12), (13), (15) and (11) we obtain

$$\lambda^{\Omega, h}(w) = \inf_{x\in\Omega} \lambda^{\{x\}, h}(w) = \inf_{x\in\Omega} \lambda^{\{x\}, \inf_{(y,d)\in Epi\, h}\{\chi_{\{y\}}+d\}}(w) =$$

$$= \inf_{x\in\Omega} \inf_{(y,d)\in Epi\, h} \{\lambda^{\{x\}, \chi_{\{y\}}}(w) + d\} = \inf_{x\in\Omega} \inf_{y\in F}\{w(x) - wu(y) + h(y)\} =$$

$$= \inf_{x\in\Omega}\{w(x) + \inf_{y\in F}\{h(y) - wu(y)\}\} = \inf w(\Omega) \dot{+} \inf_{y\in F}\{h(y) - wu(y)\},$$

i.e., (6) for $h \not\equiv +\infty$. On the other hand, if 2) holds, then, by (12) for $I = \emptyset$, we have

$$\lambda^{\Omega, +\infty}(w) = \begin{cases} +\infty, & \text{if } \inf w(\Omega) > -\infty \\ -\infty, & \text{if } \inf w(\Omega) = -\infty, \end{cases} \tag{17}$$

i.e., (6) for $h \equiv +\infty$. Thus, 2) $\Rightarrow$ 1).

Remark 1. *a) Condition (14) can be replaced by the equivalent condition*

(18) $$\lambda^{\Omega,h} = \inf_{x\in\Omega} \lambda^{\{x\},h}, \qquad \text{if } h \not\equiv +\infty.$$

b) Condition (15) shows that, for any $y \in F$ and $w \in W$, $\lambda^{\{x\},\chi_{\{y\}}}(w) - w(x)$ does not depend on $x \in X$. Moreover, we also have the following property: for any $h \in \overline{R}^F$ and $w \in W$,

(19) $$\lambda^{\{x\},h}(w) - w(x) \qquad \textit{does not depend on } x \in X;$$

indeed, by (6), (3) for $\Omega = \{x\}$ and $W \subseteq R^X$, we obtain

(20) $$\lambda^{\{x\},h}(w) - w(x) = \inf_{y\in F}\{h(y) - wu(y)\} + w(x) - w(x) = -h^*(wu).$$

c) Note that the mapping $u : F \to X$ of (1) and (3) occurs only in condition (15). Also, (12)-(14) refer to properties for general sets Ω and functions h, and only (15) refers to elementary sets (singletons) and elementary functions (indicator functions of singletons).

4. CONCLUDING REMARKS

In the present Note, regarding the objective function $\lambda : W \to \overline{R}$ of a dual optimization problem (2) as a function not only of the dual variables $w \in W$, but also of the target set $\Omega \subseteq X$ and the primal objective function h, we have given a characterization of the Lagrangian dual objective function (3), by four natural conditions (12)-(15). Hence, one can obtain an axiomatic approach to the theory of Lagrangian dual optimization problems, by starting from these conditions.

Let us note that in the particular case of systems $(F \xrightarrow{I_F} F)$, i.e., when

(21) $$X = F, \qquad u = I_F, \qquad \Omega = G \subseteq F,$$

where I_F denotes the identity operator on F, (1) and (3) become, respectively, the primal problem

(22) $$(P) = (P_{G,h}) \qquad \alpha = \alpha_{G,h} = \inf h(G),$$

and the Lagrangian dual objective function (occurring in [6], [4], [8])

(23) $$\lambda_{\text{Lagr}}(w) = \inf_{y\in F}\{h(y) - w(y)\} \dot{+} \inf w(G) \qquad (w \in W),$$

where $W \subseteq R^F$; in this case, condition (15) becomes

(24) $$\lambda^{\{y'\},\chi_{\{y\}}}(w) = w(y') - w(y) \qquad (y', y \in F).$$

We shall show elsewhere that some other dual optimization problems to (1) can be also studied with similar methods.

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Juan-Enrique Martínez-Legaz
Department of Applied Mathematics and Analysis
University of Barcelona
08071 Barcelona, Spain

Ivan Singer
Institute of Mathematics of the Romanian Academy
P.O. Box 1-764
70700 Bucharest, Romania