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A GLIMPSE AT ISOALGEBRAIC SPACES

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Dedicated to the memory of Professor Gottfned Kothe

Isoaigebraic geometry is a rudiment of complex anaiysis which retains meaning over an aigebraic closed field C of characteristic zero instead of the filed **C** of complex numbers. In the classical casse C = **C** isoalgebraic spaces lie somewhat «in the middle» between aigebraic varieties and complex analytic spaces. In some sense, to be made precise below, isoalgebraic spaces over C form a smallest category containing the algebraic varieties over C, in which the inverse function theorem (hence the implicit function theorem) becomes right.

An introduction to isoalgebraic geometry (without proofs) from a somewhat naive viewpoint has been given by one of us already in 1981 [K]. A look at this introduction might still be helpful for the interested reader. We now intend to give a – slightly less naive – introduction to isoalgebraic spaces and to survey some results obtained since 1981. Proofs of most of the more difficult theorems are contained in [H], while a completely explicit systematic exposition is still lacking.

It is an honour for us to dedicate this survey article to the memory of Professor Gottfned Köthe, and perhaps this is also not quite inappropriate. Köthe has amplified our understanding of analysis enormously by penetrating this subject from the aigebraic side, more precisely, from the viewpoint of linear algebra (enriched by topology, but this is now very common in algebra). He certainly is one of the persons responsible for a drastic change of present day's feeling among the mathematicians, «where algebra stops and analysis begins» compared to the last century.

In due modesty we believe that from our endeavours in isoalgebraic geomeuy one also learns something about the border line between analysis and algebra, this time anaiysis and commutative algebra or, what is nearly the same, aigebraic geometry. Our main motivation for entering isoaigebraic geometry has been the need for results from this area for semiaigebraic topology. (If a locally semialgebraic space carries an isoaigebraic structure this has a lot of implications on its topology). But on the way we found much pleasure being forced to think about the relation between algebra and analysis in a somewhat new way. The isoalgebraic Kugelsatz in §5 below (theorem 5.2) may serve as an illustration that the border line between algebra and truly transcendentai analysis is not always where people usuaily think it is (at least where we have once believed it to be).

1. INTRODUCTION; THE SEMIALGEBRAIC SPACE $V(C)$

We shail work over a fixed aigebraic closed field C of characteristic zero. By an (algebraic) *variety* V we always mean a separated scheme V of finite type over Spec C, this being the most general reasonable choice for us. As usual we denote the set of geometric (= C-rational) points of V by $V(C)$.

If C is the field $\mathbf{C}$ of complex numbers then it is sometimes important, and often helpful, to apply to $V(\mathbf{C})$ complex analysis instead of just algebraic geometry. For example, if V is the affine line $\mathbf{A}^1$ over $\mathbf{C}$, hence $V(\mathbf{C}) = \mathbf{C}$, then already a disc D in $\mathbf{C}$ escapes the framework of all of agebraic geometry. To give still another example, the universal covering $V(\mathbf{C})^{\sim}$ of $V(\mathbf{C})$ as a complex analytic **space** is usually very different from the (profinite) universal covering $\hat{V}$ of V in the sense of algebraic geometry. Quite often $V(\mathbf{C})^{\sim}$ is the interesting space and not $\hat{V}$.

If $C \neq \mathbf{C}$ then it is impossible to deal with such spaces as D and $V(\mathbf{C})^{\sim}$ reasonably in a classical setting. Isoalgebraic geometry, to be described below, intends to fill this gap at least partially.

The basic ingredient leading to complex analysis on $V(\mathbf{C})$ in the case $C = \mathbf{C}$ is the field $\mathbf{R}$ of real numbers, the most important reason being that this field allows us to introduce a reasonable «strong» topology on $V(\mathbf{C})$ instead of the tembly coarse Zariski topology.

In general we choose, once and for all, a subfield R of C with $[C : R] = 2$. This is aiways possible, in fact in infinitely many different ways. We also choose a fixed square root i of -1 in C. Then $C = R(i)$ and the field R is real closed, hence has a unique total ordering compatible with addition and multiplication. The ordering makes R a topological field, the open intervals

$$]a, b[= \{x \in R | a < x < b\}$$

($a \in R, b \in R, a < b$) forming a basis of open sets. We use the element i to identify C with R^2. This makes C a topological space, in fact a topological field.

More generally we obtain from R a *strong topology* on $V(C)$ for every variety V as follows. Assume first that V is affine. Then we choose a (Zariski-) closed embedding $V \hookrightarrow \mathbf{A}^N$ into some affine standard space $\mathbf{A}^N$, and we equip $V(C)$ with the subspace topology in $\mathbf{A}^N(C) = C^N = R^{2N}$. It is easily seen that this strong topology on $V(C)$ does not depend on the choice of the embedding. If V is any variety we choose a covering $(V_i | i \in I)$ of V by affine Zariski-open subsets V_i with I finite. The intersections $V_i \cap V_j$ are again affine. Thus we have already established a strong topology on the sets $V_i(C)$ and $(V_i \cap V_j)(C) = V_i(C) \cap V_j(C)$. Every intersection $V_i(C) \cap V_j(C)$ is an open subspace of $V_i(C)$ and of $V_j(C)$ in their given strong topologies. This implies that we have a unique topology on $V(C)$ such that every $V_i(C)$ is an open subspace of $V(C)$. This is our strong topology on $V(C)$. It does not depend on the choice of the affine covering $(V_i | i \in I)$ of V. It is Hausdorff and is finer than the Zariski topology of $V(C)$, i.e. the subspace topology of $V(C)$ in V.

Unfortunately, whenever $R \not\cong \mathbf{R}$, the strong topology makes $V(C)$ a toially disconnected space. Our way out of this difficulty is to regard $V(C)$ as a *semialgebraic space* (over R).

We refer the reader to the paper [DK_1] for the basic theory of semiaigebraic spaces and the book [BCR] for background materiai. Later we shall also need parts of the theory of locally semialgebraic spaces, a slight generalization of semiaigebraic spaces. For this and some more advanced theorems on semiaigebraic spaces we refer to the book [DK]. A bnef survey on locally semialgebraic spaces has been given in [DK_2]. This paper also contains a section on covering maps which are not yet covered by [DK].

Here we just mention some formal ingredients of the definition of a semialgebraic space M. On the set M there is given axiomatically a set $\mathring{S}(M)$ of subsets which are called «open semialgebraic sets». The union and the intersection of finitely many open semialgebraic sets are again open semialgebraic. $\emptyset$ and M are open semialgebraic. Given some $U \in \mathring{S}(M)$ we call a family $(U_\lambda | \lambda \in \Lambda)$ in $\mathring{S}(M)$ an *admissible open covering of* U if $U_\lambda \subset U$ for every $\lambda \in \Lambda$ and U is aiready the union of finitey many U_λ. A *semialgebraic sheaf* F on M (of abelian groups, say) is an assignment $U \mapsto F(U)$ of an abelian group $F(U)$ to every $U \in \mathring{S}(M)$ and an assignment of a restriction homomorphism $s \mapsto s|V, F(U) \to F(V)$ for every pair $(U, V) \in \mathring{S}(M) \times \mathring{S}(M)$ with $V \subset U$, such that the usual sheaf axioms holds, but only with respect to admissible open coverings. As a final ingredient of a semialgebraic space there is given on M a semialgebraic sheaf $\mathcal{C}_M$ such that $\mathcal{C}_M(U)$ is a ring of R-valued functions on U for every $U \in \mathring{S}(M)$ and, of course, $h|V$ is the natural restriction of h to V for any open semialgebraic $V \subset U$ and $h \in \mathcal{C}_M(U)$ {i.e. $(h|V)(x) = h(x)$ for $x \in V$}. It is assumed that $\mathcal{C}_M(U)$ contains the constant functions, hence in an R-algebra. The $h \in \mathcal{C}_M(U)$ are called the *semialgebraic functions on* U.

If M is a semiaigebraic space then $S(M)$ denotes the boolean lattice of subsets of M generated by $\mathring{S}(M)$. The elements of $S(M)$ are cailed the *semiaigebraic subsets* of M. The *strong topology* on M is the topology on the set M, in the classical sense, with $\mathring{S}(M)$ a basis of open sets. Thus the open sets of M are the unions of arbitrary (not necessarily finite) families in $\mathring{S}(M)$. The axioms of a semialgebraic space [DK_1, §7] imply that every semialgebraic function $h : U \to R$ is continuous with respect to the strong topologies of M and R. Also, a semiaigebraic subset A of M is an element of $\mathring{S}(M)$ iff A is open in M in the strong topology.

In the following we always assume tacitly that a semiaigebraic space M is *separated*, i.e. that the strong topology is Hausdorff.

If M and N are semiaigebraic spaces then a morphism (f, ϑ) from the ringed space $(M, \mathcal{C}_M)$ to $(N, \mathcal{C}_N)$ is determined by its first component, a map f from the set M to the set N. These maps f are cailed the *semiaigebraic maps* from M to N. They are continuous in the strong topologies of M and N. It is well known (Tarski's projection theorem) that the image of a semialgebraic subset of M under a semialgebraic map $f : M \to N$ is a semialgebraic subset of N. Also the preimages of semialgebraic sets under semialgebraic maps are semialgebraic.

In the following words like «continuous», «open», «closed», «dense», ... will refer to the strong topology (except in the axiomatic part of §3).

The easiest examples of semialgebraic spaces are the affine standard spaces R^n with $n \in \mathbb{N}$. (For $n = 0$ we have the one-point-space). Recall that a subset A of R^n is classically called semialgebraic if A is a finite union of the sets $\{x \in R^n | P_1(x) > 0, \ldots, P_r(x) > 0, Q_1(x) \geq 0, \ldots, Q_s(x) \geq 0\}$ with P_i, Q_j polynomials in n variables with coefficients in R. Now $\mathring{S}(R^n)$ is defined as the set of classically semialgebraic sets in R^n in R^n which are open in the strong topology of R^n (coming from the topology of R). For $U \in \mathring{S}(R^n)$ the elements of $C_{R^n}(U)$ are the functions $h : U \to R$ which are continuous and have a (classically) semialgebraic graph $\Gamma(h) \subset U \times R \subset R^{n+1}$.

Notice that $S(R^n)$ is indeed the set of all classically semialgebraic subsets of R^n. It is known that the elements of $\mathring{S}(R^n)$ are the finite union of sets $\{x \in R^n | P_1(x) > 0, \ldots, P_r(x) > 0\}$ with $P_i \in R[T_1, \ldots, T_n]$, cf. [BCR, Chapter II].

If M is a semialgebraic space then it turns out that the elements of $C_M(M)$ are just the semialgebraic maps from M to R^1. Also, if $A \in S(M)$, then there exists a natural structure of semialgebraic space on the set A inherited from the semialgebraic space M. These spaces A are the *semialgebraic subspaces* of M. If $f : N \to M$ is a semialgebraic map and if $f(N) \subset A$, then f can be read as a semialgebraic map from N to A.

In the special case $M = R^n$ the semialgebraic space structure on A can be described as follows: $\mathring{S}(A)$ in the set of all subsets U of A which are classically semialgebraic in M and open in A. If $U \in \mathring{S}(A)$ then a function $h : U \to R$ is an element of $C_A(U)$ if h is continuous and the graph $\Gamma(h)$ of h is semialgebraic in $R^n \times R = R^{n+1}$.

The semialgebraic spaces which are isomorphic to a semialgebraic subspace of some R^n are called the affine semialgebraic *spaces* (over R). By definition [DK_1, §7] every semialgebraic space M has a covering by finitely many open semialgebraic subsets M_i of M such that every M_i (as a subspace of M) is affine.

A semialgebraic space M is called *connected* if M is not the disjoint union of two proper open semialgebraic subsets. It is known that then any two points of M can be joined by a semialgebraic path ([DK_1, §12], [BCR]; such a path is just a semialgebraic map from the unit interval [0, 1] in R to M). It is also known (loc. cit.) that every semialgebraic space M is the disjoint union of finitely many open semialgebraic subsets $M_1, \ldots, M_r$ which are connected. They are called the *connected components of* M.

If $U \in \mathring{S}(M)$ then we call a $\mathbb{C}$-valued function f on U semialgebraic if the real and the imaginary part of U are elements of $C_M(U)$. Of course, this just means that the map $f : U \to C = R^2$ is semiaigebraic.

That much about semiaigebraic spaces and maps. We shall obey the philosophy here that semialgebraic (or more generaily, locally semiaigebraic) spaces and maps **are** just the good substitute for topological spaces and continuous maps in the present setting.

We return to a variety V and now explain how the semialgebraic space structure on $V(C)$ is defined. Please look again at the introduction of the strong topology on $V(C)$ above. If V is affine and $V \hookrightarrow \mathbf{A}^N$ is a closed embedding, then $V(C)$ is a closed semialgebraic subset of $C^N = R^{2N}$. We equip $V(C)$ with the semiaigebraic subspace structure in C^N. This structure does not depend on the choice of the embedding. In generai, let again $(V_i | i \in I)$ be a finite covering of V by affine Zariski-open subsets. All the sets $V_i(C)$ and $(V_i \cap V_j)(C) = V_i(C) \cap V_j(C)$ carry a structure of an (affine) semialgebraic space and $V_i(C) \cap V_j(C)$ is an open subspace of $V_i(C)$ and of $V_j(C)$. Thus the semialgebraic space structures of the $V_i(C)$ glue together to a semiaigebraic space structure on $V(C)$ with $(V_i(C) | i \in I)$ an admissible open covering. This structure does not depend on the choice of the affine open covering $(V_i | i \in I)$ of V. The associated strong topology of this semialgebraic space is just the strong topology of $V(C)$ introduced above.

From now on we tacitly regard $V(C)$ not merely **as** a set but **as** a semialgebraic space. This space is obviously locally compiete [DK, Chap. I, §7], hence regular (in the semialgebraic sense), hence affine, cf. [Ro], [DK, p. 421. Notice that $V(C) = V_{red}(C)$ with V_{red} denoting the reduced variety associated with V.

If V is irreducible and has (aigebraic) dimension n then it turns out that $V(C)$ is connected and is pure of semiaigebraic dimension $2n$, i.e. every non empty open semialgebraic subset U of $V(C)$ has semialgebraic dimension $2n$ (cf. [DK_1, §8] for semiaigebraic dimension theory). U is also Zariski dense in V (cf. [H_1]; there the Zariski closure of arbitrary semialgebraic subsets of $V(C)$ has been computed).

Every morphism $\varphi : V \to W$ from V to a variety W restricts to a semiaigebraic map $\varphi_C : V(C) \to W(C)$. In particular, if U is a Zariski-open subset of V, then every $h \in \mathcal{O}_V(U)$ gives us a C-valued semialgebraic function h_C on $U(C)$. In the following we will call these maps φ_C (resp. functions h_C) *algebraic maps* (resp. *algebraic functions*). Notice that, for $x \in U(C)$, we have

$$h_C(x) = h(x) \in \mathcal{O}_{V,x}/m_{V,x} = C.$$

If $\varphi : V \to S, \psi : W \to S$ are morphisms between varieties then we can from the fibre product $V \times_S W$ with respect to φ and ψ. On the other hand we can form the semiaigebraic fibre product $V(C) \times_{S(C)} W(C)$ with respect to φ_C and ψ_C, cf. [DK_1, §7]. It is easily seen that

$$(V \times_S W)(C) = V(C) \times_{S(C)} W(C).$$

In particular $(S = Spec C)$ we have

$$(V \times W)(C) = V(C) \times W(C)$$

for any two varieties V, W.

Usually nice properties of a morphism $\varphi : V \to W$ give us nice properties of the semialgebraic map φ_C. For example, if φ is proper (in the algebraic sense) then φ_C is proper (in the semialgebraic sense, cf. [DK_1, §9]). The same holds for «finite» instead of «proper». If φ is etale then φ_C is a *local isomorphism*, i.e. every point $x \in V(C)$ has an open semialgebraic neighbourhood U such that $\varphi_C(U)$ is open (and, of course, semialgebraic) in $W(C)$ and φ_C restricts to a semialgebraic isomorphism from U to $\varphi_C(U)$. This last observation, which is based on the fact that the inverse function theorem is nght in semiaigebraic topology in constrast to Zariski topology, is the «prima movens» of isoalgebraic geometry. (cf. [K]).

If $f : M \to N$ is anly local isomorphism between semialgebraic spaces then it is known that there exists a finite covering $(U_i | i \in I)$ of M by open semialgebraic subsets such that f maps U_i isomorphically onto the open semialgebraic subset $f(U_i)$ of N [DK, p. 218].

2. ISOALGEBRAIC FUNCTIONS

Let again V be a variety. We start out to define a sheaf $\mathcal{A}_V$ of «isoalgebraic functions» on the semialgebraic space $V(C)$.

Given an open semialgebraic subset U of $V(C)$ we define a category $I(U)$ as follows. The objects of $I(U)$ are the iriples (V', U', f) with V' a variety, f an etale morphism from V' to V and U' an open semialgebraic subset of $V'(C)$ such that f_C restricts to a semialgebraic isomorphism from U' onto U. A morphism from an object (V', U', f) to an object (V'', U'', g) is a isomorphism of varieties $h : V'' \to V'$ (we reserve the arrows!) such that $f \circ h = g$ and $h_C(U'') \subset U'$, hence $h_C(U'') = U'$.

The category $I(U)$ is filtered [Mi, p. 3051. We assign to every object (V', U', f) the C-algebra $\mathcal{O}_{V'}(V')$ and obtain a direct system of C-algebras. We define

$$P_V(U) := \varinjlim_{I(U)} \mathcal{O}_{V'}(V').$$

Varying U we obtain a presheaf of C-algebras P_V on $V(C)$. It is separated [Mi, p. 49]. We define $\mathcal{A}_V$ as the semiaigebraic sheaf associated to P_V.

An element of $\mathcal{A}_V(U)$ may be viewed as a family $(h_i | i \in I)$ arising as follows. There is given a finite covering $(U_i | i \in I)$ of U by open semialgebraic subsets. For each $i \in I$ there is given an object (V_i, U_i', f_i) of $I(U_i)$, and h_i is an element of $\mathcal{O}_{V_i}(V_i)$. For any two indices $i \neq j$ in I the elements h_i and h_j are «compatible». This means the following. Let

$U'_{ij} := U'_i \cap f_i^{-1}(U_i \cap U_j)$ and $U'_{ji} := U'_j \cap f_j^{-1}(U_i \cap U_j)$. Then (V_i, U'_{ij}, f_i) and (V_j, U'_{ij}, f_i) are both objects of $I(U_i \cap U_j)$. h_i and h_j are compatible iff there exist morphisms from these two objects into a third object (V_{ij}, U_{ij}, f_{ij}) of $I(U_i \cap U_j)$ which send h_i and h_j to the same element of $\mathcal{O}_{V_{ij}}(V_{ij})$.

This looks complicated. It is much easier to visualize $\mathcal{A}_V$ if the variety V is *reduced*, as we explain now. If (V', U', f) is an object of $I(U)$ and h is an element of $\mathcal{O}_{V'}(V')$ then V' is again reduced and h may be identified with the algebraic function h_C on $V'(C)$. In the triple (V', U', f) we may replace V' by the Zariski-open subset which is the complement of the union of all irreducible components which do not meet U'. Thus we may assume that U' is Zariski-dense in V'. Now h is determined by the semialgebraic function $h_C|U'$. On the other hand, U' defines a semialgebraic section $s : U \xrightarrow{\sim} U' \hookrightarrow V'(C)$ of f_C over U. Altogether we obtain a semialgebraic function $h_C \circ s$ on U. It turns out that $P_V(U)$ may be identified with the set of functions on U obtained in this way. Now P_V becomes a sub-presheaf of the sheaf $\mathcal{C}_{V(C)} \otimes_R C$ of C-valued semialgebraic functions on $V(C)$, hence $\mathcal{A}_V$ becomes a subsheaf of $\mathcal{C}_{V(C)} \otimes_R C$.

We call the elements of $\mathcal{A}_V(U)$ the *isoalgebraic functions* on U with respect to V (even if V is not reduced). In the reduced case a semialgebraic function $h : U \to C$ is isoalgebraic iff there exists a finite covering $(U_i | i \in I)$ of U by open semialgebraic subsets such that each restriction $h|U_i$ admits an *etale factorization*, i.e. there exists an etale morphism $f_i|V_i \to V$, a semialgebraic section $s_i : U_i \to V_i(C)$ of $(f_i)_C$ over U_i and some $h_i \in \mathcal{O}_{V_i}(V_i)$ such that $h|U_i = (h_i)_C \circ s_i$.

$$
(2.1) \qquad \begin{array}{ccccc}
 & & V_i(C) & & \\
 & {\scriptstyle s_i} \nearrow & \downarrow (f_i)_C & \searrow^{(h_1)_C} & \\
U_i & \hookrightarrow & V(C) & & C \\
 & & h|U_i & &
\end{array}
$$

Things are even better if the variety V is *normal*. Then every isoalgebraic function on U has a *global etale factorization* (cf. [K, §2] for the explicit description of a canonical such factorization). If, in addition, U is connected then the isoalgebraic functions on U also obey an *identity principle*. An isoalgebraic function f on U which vanishes on some non empty open subset of U vanishes everywhere on U. In particular, $\mathcal{A}_V(U)$ is an integral domain.

We write down the proto-typical example of an isoalgebraic function. Let V be reduced and U be a *simply connected* open semialgebraic subset of $V(C)$. This means that U is connected and that the semialgebraic fundamental group of U (cf. [DK_2], [DK, Chap. III]) vanishes. Notice that such sets U abound in $V(C)$. Every semialgebraic space M is known to be a union of finitely many simply connected semialgebraic subsets, since M can be triangulated ([DK, ,(BCR, Cap. 9]). Let

$$p(z,T) = T^n + a_1(z)T^{n-1} + \ldots + a_n(z)$$

be a polynomial whose coefficients are algebraic functions $a_1, \ldots, a_n$ on $V(\mathbb{C})$. Assume that the discriminant of $p(z,T)$ vanishes nowhere on U. Let

$$M := \{(z,t) \in U \times C \mid p(z,t) = 0\}.$$

This is a semiaigebraic subset of $V(\mathbb{C}) \times \mathbb{C}$, hence a semialgebraic space. The natural projection $\pi : M \to U, (z,t) \mapsto z$, is a semiaigebraic covering [DK, ,§5] of degree n. Since U is simply connected this covering is trivial (loc. cit.). This means that M consists of n connected components $M_1, M_2, \ldots, M_n$ and that π restricts to semialgebraic isomorphisms $\pi_i : M_i \xrightarrow{\sim} U$. The C-valued functions $h_1, \ldots, h_n$ on U defined by $\pi_i^{-1}(z) = (z, h_i(z))$ are isoalgebraic. In the polynomiai nng $\mathcal{A}_V(U)[T]$ we have the factorization

$$p(z,T) = (T - h_1(z)) \ldots (T - h_n(z))$$

Thus $h_1, \ldots, h_n$ are the «semialgebraic roots» of $p(z,T)$.

Isoaigebraic functions are amenable to truly locai considerations, at least if V is reduced, by the following theorem [H, Satz 10.61.

Theorem 2.2. *Assume that V is reduced and that $f : U \to \mathbb{C}$ is a semialgebraicfunction on some open semialgebraic subset U of $V(C)$. Assumefurther that every point p of U has some open semialgebraic neighbourhood such that the restriction of f to this neighbourhood is isoalgebraic. Then f is isoalgebraic.*

Let us look at the stalk A,, of $\mathcal{A}_V$ at some point p of $V(\mathbb{C})$ for any variety V ! It is a local ring. O,, embeds into **A,,** by a local injection and thus will be regarded a local subring of $\mathcal{A}_{V,p}$. It follows immediately from the definition of $\mathcal{A}_{V,p}$ (cf. [K, p. 134]), that $\mathcal{A}_{V,p}$ in the *henselization* $\mathcal{O}^h_{V,p}$ (cf. [R]) of $\mathcal{O}_{V,p}$.

We have a natural local injection of A,, into the $m_{V,p}$-adic completion $\hat{\mathcal{O}}_{V,p}$ of $\mathcal{O}_{V,p}$, and thus regard A, a local subring of $\hat{\mathcal{O}}_{V,p}$. If V is normal at p then $\hat{\mathcal{O}}_{V,p}$ is an integra1 domain. In this case, by a iheorem of Nagata [N, Th. 44.11, A,, is the set of all elements of $\hat{\mathcal{O}}_{V,p}$ which are algebraic over the quotient field of $\mathcal{O}_{V,p}$.

Using already some genuine isoalgebraic geometry, one can prove the following global version of Nagata's theorem [H, Satz 10.11.

Theorem 2.3. *Assume that V is affine and normal. A C-valued semialgebraic function f on some connected open semialgebraic subset U of $V(C)$ is isoalgebraic iff there exist algebraic functions $a_0, a_1, \ldots, a_n$ on V such that a_0 does not vanish everywhere on U and, for every $z \in U$,*

$$a_0(z)f(z)^n + a_1(z)f(z)^{n-1} + \ldots + a_n(z) = 0$$

In the case $R = \mathbf{R}$ it suffices to assume that f is continuous instead of semialgebraic.

This theorem reminds us of the classical definition of ***Nash functions*** as «algebraic real analytic functions» [AM, p. 88]. Indeed, the modern theory of Nash functions over a real closed field – or even on a real spectrum – also starts with semialgebraic sections of etale maps ([Roy]; already Artin and Mazur have been well aware of this approach [loc. cit.]). In some sense we are just doing the analogue of this theory over C. But this analogue has its own «complex analytic flavour». Moreover there is a closer connection than just analogy: The real and the imaginary part of an isoalgebraic function – say, for V reduced – both are Nash functions.

In the case $V = \mathbf{A}^n$ it can be seen particularly well that isoalgebraic functions show a local behaviour similar to complex analytic functions, as it should be.

Let $z_1, \ldots, z_n$ be the standard coordinate functions on C^n, and let x_j and y_j denote the real and the imaginary part of z_j. Let f be a non constant isoalgebraic function on some connected open semialgebraic subset U of C^n.

f is an *open* map. (This remains true if V is any reduced irreducible variety, cf. [H, Prop. 7.16]). Thus neither the real part nor the imaginary part nor the modulus $|f|$ of f attains a local extremum at any point in U.

The partial derivatives $\frac{\partial f}{\partial x_j}$ and $\frac{\partial f}{\partial y_j}$ exist at every point in U although, of course, the topological field R is not complete in general. The real and the imaginary part of f obey the Cauchy-Rimann equations. Conversely if u and v are R-valued Nash functions on U obeying the Cauchy-Riemann equations then $u + iv$ is isoalgebraic on U.

The partial derivatives $\frac{\partial f}{\partial z_j}$ are again isoalgebraic functions. Thus also all higher derivatives $\frac{\partial^\alpha f}{\partial z^\alpha}$ exist and are isoalgebraic on U. {We may use such a notation since indeed $\frac{\partial^2 f}{\partial z_j \partial z_k} = \frac{\partial^2 f}{\partial z_k \partial z_j}$}. We can form the Taylor series of f at any point $a = (a_1, \ldots, a_n)$ of U and this series is the image of f in $\hat{O}_{V,a} = C[[z_1 - a_1, \ldots, z_n - a_n]]$. {Recall that $\mathcal{A}_{V,a} \subset \hat{O}_{V,a}$.}

The field R is called *microbial* if R contains some element $\vartheta > 0$ with $(\vartheta^n | n \in \mathbf{N})$ converging to zero (a «microbe»). Microbial fields abound among real closed fields. For example, if C is the algebraic closure of some field which is finitely generated over $\mathbf{Q}$, then

R is automatically microbial. Now, if R is microbial, then the Taylor series of f at a point a E U converges to f in some neighbourhood of a (cf. [K] fora more explicit statement). Conversely,if f is a C-valued semialgebraic function on U which locally everywhere is the iimit of some power series, then f is isoalgebraic.

Proofs of these facts are contained in [H, $2 and §10]. More generaily all this remains valid on a smooth n-dimensional variety V instead of $\mathbf{A}^n$ since then every point of $V(C)$ has an open semialgebraic neighbourhood which is *isoalgebraically* isomorphic (see below) to the open unit ball in C".

We hope that the reader has gained some confidence that isoalgebraic functions occur «in nature» and thus is willing to accept a reasonable definition of isoalgebraic spaces based on these functions. We did not say much about isoaigebraic «functions» in the non reduced case. But also non reduced isoalgebraic spaces will be needed for functonal (and other) reasons, as in algebraic and in analytic geometry.

3. DEFINITION OF LOCALLY ISOALGEBRAIC SPACES

We shall work in the category of *ringed spaces* over a field F with either $F = R$ or $F = C$, as defined in [DK, Chap. I, §1]. Such a ringed space is a pair $(X, \mathcal{O}_X)$ consisting of a *generalized topological space* X [loc. cit.] and a sheaf of F-algebras. On a generalized topological space there is axiomatically given a set $\mathring{\mathcal{T}}(X)$ of subsets which are called *«open subsets»* and, for every U E $\mathring{\mathcal{T}}(X)$, a set (or better class) $\mathrm{Cov}_X(U)$ of families in $\mathring{\mathcal{T}}(X)$ which are called *«admissible open coverings»* of U.

Usually only the union of *finitely many* open subsets of X is again open. By one of the axioms [loc. cit.] then this finite family is an admissible covering of the union., A generalized topological space is a site in the sense of Grothendieck. Thus sheaf theory makes sense on it.

A *morphism* from $(X, O,)$ to a second ringed space $(Y\mathcal{O},)$ over F is a pair $(f, 19)$ consisting of a continuous map $f: X \to Y$ between generalized topological spaces and a homomorphism 6 : O, $\to f_*\mathcal{O}_X$. Here «continuous» has the obvious meaning: If V E $\mathring{\mathcal{I}}(Y)$ then $f^{-1}(V)$ E $\mathring{\mathcal{T}}(X)$, and if $(V_\lambda|\lambda$ E $\mathbf{A})$ E $\mathrm{Cov}_Y(V)$ then $(f^{-1}(V_\lambda)|\lambda$ E $\mathbf{A})$ E $\mathrm{Cov}_X(f^{-1}(V))$. The homomorphism ϑ may be thought of as a family of F-algebra homomorphisms $\vartheta_{U,V} : \mathcal{O}_Y(V) \to \mathcal{O}_X(U)$ with U E $\mathring{\mathcal{T}}(X)$, V E $\mathring{\mathcal{T}}(Y)$, $f(U)$ C V, and the obvious compatibility conditions.

If no confusion is likely then we shall denote a ringed space $(X$, O,) by the single letter X and a morphism $(f, 6)$ by the single letter f. All the maps $\vartheta_{U,V}$ then will be denoted by F^*.

If X is a ringed space over F then every open subset of X may again be regarded as a ringed space over F. These are the *open subspaces* of X.

Every semialgebraic space M is a ringed space over R, with stnicture sheaf C_M. By definition, a *locally semialgebraic space* (over R) is a ringed space M over R which has some admissible open covering $(M_i | i \in I) \in \mathrm{Cov}_M(M)$ such that every M_i, as an open subspace of M, is isomorphic to a semialgebraic space. We usually denote the structure shcaf of a locall semialgebraic space M again by C_M.

A morphism (f,δ) between locally semialgebraic spaces (M, C_M) and (N, C_N) is determined by its first component f. These maps $f : M \to N$ are called the *locally semialgebraic maps from M to N*.

Of course, a variety V = (V, O,) is a ringed space over C, this time with first component V a genuine topological space, albeit almost never Hausdorff. We associate with V a ncw ringed space V^h over C as follows. The underlying generalized topological space of V^h is the same as that of the semialgebraic space $V(C)$. The stnicture sheaf $\mathcal{O}_{V^h}$ is the sheaf A, introduced in §2. Slightly abusively we may write $V^h = (V(C), A_V)$. Notice that the choice of the field R in C is essential for the definition of V^h.

Definition 1. *A* locally isoalgebraic space over (C,R) *is a ringed space* $(X, \mathcal{O}_X)$ *over* C *which has an admissible open covering* $(X_i | i \in I)$ *such that every open subspace* X_i *is isomorphic to an open subspace of some* V^h, *where* V *is a variety. If one can choose* I *jinite, then* $(X, \mathcal{O}_X)$ *is called* isoalgebraic.

Usually we shall suppress the words «over (C,R)», since always the fields C and R will remain fixed.

As a first example, if V is a variety, then V^h is an isoalgebraic space. Every open subspace of an isoalgebraic (resp. locally isoalgebraic) space is again an isoalgebraic (resp. locally isoalgebraic) space. In §6 we shall discuss some natural examples of truly locally isoalgebraic spaces.

Let $(X, \mathcal{O}_X)$ be a locally isoalgebraic space. There lives a canonical sheaf C_X on the generalized topological space X such that (X, C_X) is a locally semialgebraic space. (X, C_X) is called the locally semialgebraic space associated to the locally isoalgebraic space $(X, \mathcal{O}_X)$ and is denoted by $|X|$. C_X is constructed in the following way. Let $(X_i | i \in I)$ be an admissible open covering of X such that every X_i is isomorphic to an open subspace of V_i^h for some variety V_i, $\varphi_i : X_i \xrightarrow{\sim} U_i \subseteq V_i^h$. Via φ_i every X_i becomes a semialgebraic space, hence we have a sheaf C_{X_i} on X_i. The C_{X_i} glue together to the sheaf C_X.

Let $(X, \mathcal{O}_X)$ be a locally isoalgebraic space. All stalks $\mathcal{O}_{X,x}$ are local rings with residue field C. X is called reduced (rcsp. normal, resp. smooth) if all local rings $\mathcal{O}_{X,x}$ are reduced (resp, normal, resp. regular). Let U be an open subsct of X and $f \in \mathcal{O}_X(U)$. For every $x \in U$, $f(x)$ denotes the image of f in the residue field $\mathcal{O}_{X,x}/m_x \simeq C$. The C-valued function $\overline{f} : U \to C, x \mapsto f(x)$ is locally semialgebraic, i.e. f is a section of $C_X \otimes_R C$ over

U. If X is reduced then the sheaf homomorphism $\mathcal{O}_X \to \mathcal{C}_X \otimes_R \mathrm{C}, f \mapsto \bar{f}$ is injective. This follows from the description of the sheaf $\mathcal{A}_V$ for a reduced variety V (§2). The elements of $\mathcal{O}_X(\mathrm{V})$ *are* called the *isoalgebraic functions* on U (even if X is not reduced).

Similar to a semialgebraic space we equip every locally semialgebraic space $(X,\mathcal{C}_X)$ (and hence also every locally isoalgebraic space) with a strong topology. This is the topology generated by $\mathring{\mathcal{T}}(X)$. Henceforth words like open, closed, continuous, ... all refer to the strong topology of X. A subset U of X is called *open semialgebraic* if $U \in \mathring{\mathcal{T}}(X)$ and the open subspace $(U, \mathcal{C}_X|U)$ is a semialgebraic space. Since «open» revers to the strong topology we cali the elements of $\mathring{\mathcal{T}}(X)$ now *open locally semialgebraic* subsets of X. (This notation is justified by the following observation: A subset S of X is an element of $\mathring{\mathcal{T}}(X)$ iff $S \cap U$ is an open semialgebraic subset of U for every open semialgebraic subspace $(U, \mathcal{C}_X|U)$ of $(X,\mathcal{C}_X)$.)

Definition 2. *By a morphism between locally isoalgebraic spaces over* (C,R), *or* isoalgebraic morphism *for short, we mean a morphism between such spaces in the category of ringed spaces over* C.

Let $f = (g,\vartheta) : (X,\mathcal{O}_X) \to (Y,\mathcal{O}_Y)$ be a morphism between locally isoalgebraic spaces. Then $g : |X| \to |Y|$ is locally semialgebraic and, for every $x \in X$, the ring homomorphism $\vartheta_x : \mathcal{O}_{Y,g(x)} \to \mathcal{O}_{X,x}$ is local, since ϑ_x is a C-algebra homomorphism and $C \xrightarrow{\sim} \mathcal{O}_{X,x}/m_x$. If X is reduced then f is determined by g, since $\mathcal{O}_X$ is a subsheaf of the sheaf $\mathcal{C}_X \otimes_R \mathrm{C}$ of C-vaiued locally semiaigebraic functions on X.

Let $(X,\mathcal{O}_X)$ be a localiy isoalgebraic space and z the coordinate function of $\mathbf{A}^1$. As in algebraic geometry, the map $f \mapsto f^*(z)$ from the set $\mathrm{Hom}(X,(\mathbf{A}^1)^h)$ of isoalgebraic morphisms $X \to (\mathbf{A}^1)^h$ to the set $\mathcal{O}_X(X)$ is bijective. Since $(\mathbf{A}^n)^h$ is the product $(\mathbf{A}^1)^h \times \dots \times (\mathbf{A}^1)^h$ in the category of locaily isoalgebraic spaces, the map $\mathrm{Hom}(X,(\mathbf{A}^n)^h) \to \mathcal{O}_X(X)^n, f \mapsto (f^*(z_1),\dots,f^*(z_n))$ is also bijective ($z_1,\dots,z_n$ denote the coordinate functions of $\mathbf{A}^n$).

Let $f : X \to Y$ be an algebraic morphism of varieties. f induces an isoalgebraic morphism $f^h = (g,\vartheta) : X^h \to Y^h$ in the follwoing way. We put $g = f_C$. We define a presheaf morphism $\eta : P_Y \to g_* P_X$ whose associated sheaf morphism is ϑ. Let U be an open semialgebraic subset of $Y(\mathrm{C})$ and s an element of $P_Y(\mathrm{V})$ represented by an object (Y', U', p) of $I(U)$ and an element t of $\mathcal{O}_{Y'}(Y')$. Let X' be the fibre product $X \times_Y Y'$ and $p' : X' \to X$ and $f' : X' \to Y'$ the projections. $(X', (f'_C)^{-1}(\mathrm{V}'), p')$ is an object of $I(g^{-1}(\mathrm{V}))$. We define $\eta(s)$ to be the element of $P_X(g^{-1}(\mathrm{V}))$ represented by $(X', (f'_C)^{-1}(U'), p')$ and $(f')^*(t) \in \mathcal{O}_{X'}(X')$.

Thus we have established a functor $X \mapsto X^h, f \mapsto f^h$ from the category of varieties to the category of isoaigebraic spaces. Using some comparison theorems between coherent sheaves on aigebraic varieties X and coherent sheaves on their associated isoalgebraic spaces X^h (cf. §4), one can prove [H, **12.111.**

Theorem 3.1. *Thefunctor* $X \mapsto X^h, f \mapsto f^h$ *from the category of varieties to the category of isoalgebriac spaces isfullyfaithful.*

Let $f : X \to Y$ be an etale morphism of varieities and U an open semialgebraic subset of $X(C)$ such taht $f|U$ is injective. Then $V := f(U)$ is an open semiaigebraic subset of $Y(C)$ and $f^h|U : (U, \mathcal{A}_X|U) \to (V, \mathcal{A}_Y|V)$ is an isoalgebraic isomorphism. This is an immediate consequence of the definition of the sheaves $\mathcal{A}_X$ and $\mathcal{A}_Y$.

Let X and Y be varieties and $f : U \to Y^h$ be an isoaigebraic morphism, where U is an open subspace of X^h. Then locally f is the composition of the inverse of an isoalgebraic isomorphism just described and an aigebraic morphism. There exists a finite open covering $(U_i | i \in I)$ of U such that each restriction $f|U_i$ admits an etale factorization,i.e. there exists an etale morphism $g_i : X_i \to X$,an open semialgebraic subset U_i' of $X_i(C)$ and an algebraic morphism $f_i : X_i \to Y$ such that $g_i|U_i' : U_i' \to U_i$ is bijectiveand $f|U_i = (f_i)^h \circ s_i$, where $s_i : U_i \to U_i'$ is the inverse of the isoalgebraic isomorphism $(g_i)^h|U_i' : U_i' \to U_i$.

$$\begin{array}{ccccc} & & X_i^h & & \\ s_i & \nearrow & \downarrow (g_i)_h & \searrow & f_i^h \\ U_i & \hookrightarrow & X^h & & Y^l \\ & & f|U_i & & \end{array}$$

In the special case X reduced and $Y = \mathbf{A}^1$ this etale factorization was aiready described in **52** (cf. diagram **2.1).**

4. COHERENT SHEAVES

Let $(X, \mathcal{O}_X)$ be a ringed space over C and $\mathcal{F}$ of a sheaf of $\mathcal{O}_X$ -modules on X. $\mathcal{F}$ is called of *finite type on X* if there exists an admissible open covering $(U_i | i \in I)$ of X such that, for every $i \in I, \mathcal{F}|U_i$ is finitely generated (i.e. there exists a surjective $\mathcal{O}_X|U_i$-morphism $(\mathcal{O}_X|U_i)^n \to \mathcal{F}|U_i)$. $\mathcal{F}$ is called *coherent* if $\mathcal{F}$ is of finite type on X and if for every $U \in \mathring{\mathcal{T}}(X)$ and every $(\mathcal{O}_X|U$-morphism $\mathcal{O}_X|U)^n \to \mathcal{F}|U$ the kernel is of finite type on U.

Coherent sheaves play an important role in many parts of isoalgebraic geometry.

First we consider the fundamental coherence theorems of complex analysis. They remain true in isoalgebraic geometry:

1. The structure sheaf $\mathcal{O}_X$ of a locally isoalgebraic space $(X, \mathcal{O}_X)$ is coherent.

2. A subset A of a locally isoalgebraic space $(X, \mathcal{O}_X)$ is cailed *locally isoalgebraic* if there exists an admissible open covering $(U_i | i \in I)$ of X such that, for every $i \in I$, $A \cap U_i$ is the zero set of finitely many isoaigebraic functions on U_i. To a locally isoaigebraic subset A of X we define a sheaf J_A of ideais on X by $J_A(U) = \{f \in \mathcal{O}_X(U) | f(x) = 0$ for every $x \in A \cap U\}$. J_A is coherent.

Let X be a locaily isoaigebraic space. The support $\mathrm{supp}(\mathcal{F})$ of a coherent sheaf $\mathcal{F}$ on X is defined to be $\{x \in X | \mathcal{F}_x \neq 0\}$. Notice that we define the support of a sheaf here only for coherent sheaves. For an arbitrary sheaf on X this definition would not be good. The reason for this is that the family of functors $(\mathcal{F} \mapsto \mathcal{F}_x | x \in X)$ does not share the good proprieties of the sheaf theory on a topological space. For instance, the family of functors $(\mathcal{F} \mapsto \mathcal{F}_x | x \in X)$ is neither faithful nor conservative. (The resmction of this family of functors to the category of coherent sheaves on X is faithful and conservative).

The support of a coherent sheaf on X is a locally isoalgebraic subset of X. Let $\mathcal{I}$ be a coherent sheaf of ideals on X and let A be the support of $\mathcal{O}_X/\mathcal{I}$. We have Hilbert's Nullstellensatz: J_A is the radical ideal sheaf of $\mathcal{I}$.

Let $(X, \mathcal{O}_X)$ be a locally isoalgebraic space. We want to define locally isoaigebraic subspaces of X. Let $\mathcal{I}$ be a coherent sheaf of ideals on X. We equip $Y := \mathrm{supp}(\mathcal{O}_X/\mathcal{I})$ with the weakest structure of a generalized topologicai space such that the inclusion map $i : Y \hookrightarrow X$ is continuous. We have the sheaf $i^{-1}(\mathcal{O}_X/\mathcal{I})$ on Y. $(Y, i^{-1}(\mathcal{O}_X/\mathcal{I}))$ is a ringed space over C and we have a canonical morphism $i : (Y, i^{-1}(\mathcal{O}_X/\mathcal{I})) \to (X, \mathcal{O}_X)$ of ringed spaces over C. All ringed spaces over C arising in this way are called *locally isoalgebraic subspaces* of X. This notation is justified, since one can prove that locally isoalgebraic subspaces are locally isoalgebraic spaces in the sense of Definition 1 of §3.

A first example of a locally isoalgebraic subspace of X is the reduction of X: J_X is a coherent sheaf of ideals on X. The locally isoalgebraic subspace of X defined by J_X is called the reduction of X and is denoted by X_{red}.

According to Definition 1 of §3 the open subspaces of the ringed spaces V^h, V a variety, are the local models for the locally isoalgebraic spaces. In complex analytic geometry the locally closed subspaces of $\mathbf{C}^n$ are the local models for the complex analytic spaces. The same is true in isoalgebraic geometry: Every locally isoalgebraic space $(X, \mathcal{O}_X)$ has an admissible open covenng $(U_i | i \in I)$ such that every $(U_i, \mathcal{O}_X|U_i)$ is isomoprhic to an isoalgebraic subspace of an open subspace of some $(\mathbf{A}^n)^h$.

The finite coherence theorem holds in isoalgebraic geometry:

Theorem 4.1. *Let $f : X \to Y$ be a finite morphism of locally isoalgebraic spaces and let $\mathcal{F}$ be a coherent sheaf on X. Then $f_*(\mathcal{F})$ is a coherent sheaf on Y.*

We do not know whether the direct image sheaf $f_*(\mathcal{F})$ of a coherent sheaf $\mathcal{F}$ under a proper isoalgebraic maps f is a coherent. The proper coherence theorem (all direct image sheaves $R^n f_*(\mathcal{F})$ of a coherent sheaf $\mathcal{F}$ under a proper isoalgebraic map f are coherent) does not hold. For example, $H^1(\mathbf{P}^1(\mathbb{C}), \mathcal{A}_{\mathbf{P}^1})$ is not finite dimensional. But nevertheless, using some special aspects of isoalgebraic geometry, one can prove Remmert's proper mapping theorem.

Theorem 4.2. *Let $f : X \to Y$ be a proper morphism of locally isoalgebraic spaces. Then $f(X)$ is a locally isoalgebraic subset of Y.*

The connection between isoalgebraic geometry and algebraic geometry is rather strong. This is reflected, for example, by the following theorem.

Theorem 44.3. *Let $\mathcal{F}$ be a coherent sheaf on an isoalgebraic space X. Let f be an isoalgebraicfunction on X and set $U = \{x \in X \mid f(x) \neq 0\}$. Then thefollowing holds:*
i) For every $s \in \mathcal{F}(X)$ with $s|U = 0$ there exists a natural number n with $f^n . s = 0$.
ii) For every $s \in \mathcal{F}(U)$ there exists a natural number n such that $f^n .s$ can be extended to a section of $\mathcal{F}$ over X.

As mentioned aiready above Serre's beautiful **GAGA** Principles do not remain true in isoalgebraic geometry. For instance, $H^1(\mathbf{P}^1(C), \mathcal{A}_{\mathbf{P}^1}) \neq H^1(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1})$. But is turns out that, whenever there is a comparison theorem between varieties and their associated isoalgebraic spaces, the comparison theorem holds for every variety and not only for projective varieties. We give some examples.

Let X be a variety and let $\varphi : X^h \to X$ be the canonical morphism of ringed spaces over C.

«Chow's Theorem» 4.4. *Every isoalgebraic subset A of X^h is algebraic, i.e. there exists a subvariety Y of X with $A = Y(C)$.*

Let $\mathcal{G}$ be a coherent siieaf on X^h. We deduce from Theorem **4.3** that $\varphi_*(\mathcal{G})$ is a quasi-coherent sheaf on X. But even more, one can prove that $\varphi_*(\mathcal{G})$ is a coherent sheaf on X.

Now let $\mathcal{F}$ be a coherent sheaf on X. Then $\varphi^*(\mathcal{F})$ is a coherent sheaf on X^h. One can show that the canonical morphism $\Gamma(X, \mathcal{F}) \to \Gamma(X(C), \varphi^*(\mathcal{F}))$ is bijective. In particular, $\mathcal{A}_X(X(C)) = \mathcal{O}_X(X)$.

Not every coherent sheaf $\mathcal{G}$ on X^h is algebraic (i.e. isomorphic to a sheaf $\varphi^*(\mathcal{F})$ with $\mathcal{F}$ a coherent sheaf on X). For example, if X is affine reduced and $\dim X \geq 1$, then there exists an invertible sheaf $\mathcal{G}$ on X^h with no nontrivial global section. But if a coherent sheaf $\mathcal{G}$ on X^h has enough globai sections, then $\mathcal{G}$ is algebraic. More precisely

Theorem 4.5. *φ_* and φ^* are quasi-inverse functors between the category of coherent sheaves on X and the category of coeherent sheaves $\mathcal{F}$ on X^h which satisfy the following property: There exists a covering $(U_i | i \in I)$ of X by Zariski-open subsets such that, for every $i \in I$, the* ***sheaf*** *$\mathcal{F}|U_i(C)$ is generated by its global sections.*

5. EXTENSION OF ISOALGEBRAIC FUNCTIONS AND GLOBAL IRREDUCIBLE COMPONENTS

We first state the classical theorem conceming the extension of isoalgebraaic functions. The Riemann removable singularity theorem is one of the basic resulis in the elementary isoalgebraic geometry.

Let $(X, \mathcal{O})$ be a normal connected locally isoalgebraic space and A a closed locally semialgebraic subset of X (i.e. $X - A \in \mathring{\mathcal{T}}(X)$). An isoalgebraic function $f \in \mathcal{O}(X - A)$ on $X - A$ is called ***weakly bounded near*** A, if for every point $x \in A$ there exisis a $t \in R$ such that for every neighbourhood U of x in X there exists some $y \in U - A$ with $|f(y)| < t$. Here $\dim^{sa} X$ and $\dim^{sa} A$ denote the semialgebraic dimensions of X and A.

Theorem 5.1. *Let $f \in \mathcal{O}_X(X - A)$ be an isoalgebraic function on $X - A$.*
i) If $\dim^{sa} A \leq \dim^{sa} X - 1$ (i.e. $X - A$ is dense in X) and f has a continuous extension to X, then this extension is isoalgebraic.
ii) If $\dim^{sa} A \leq \dim^{sa} X - 2$ and f is weakly bounded near A, then f has an isoalgebraic extension to X.
iii) If $\dim^{sa} A \leq \dim^{sa} X - 3$, then f has an isoalgebraic extension to X.

ii) and iii) are the first and second Riemann removable singularity theorems. Since we work only in the semialgebraic category and have the semialgebraic dimension theory at our disposal, the isoalgebraic version of the Riemann extension theorem is a little bit sronger than the complex analytic version: Let $(C, R) = (\mathbf{C}, \mathbf{R})$. To the locally isoalgebraic space X there is associated in a canonical way a complex analytic space X^{an}. It is connected and normal. f is a holomorphic function on X^{an}. Obviously, if f is locally bounded near A, then f is weakly bounded near A. If A is thin in X^{an}, then $\dim^{sa} A \leq \dim^{sa} X - 2$ and if A is thin of order ≥ 2 in X^{an}, then even $\dim^{sa} A \leq \dim^{sa} X - 4$.

Another classical extension theorem in complex analysis in the Kugelsatz of Hartogs.

Theorem 5.2. *Let X be a connected normal affine variety of dimension ≥ 2 and let K be a complete semialgebraic subset of $X(C)$ such that $X(C) - K$ is connected. Then the restriction $\mathcal{A}_X(U) \to \mathcal{A}_X(U - K)$ is bijective for each open semialgebraic neighbourhood U of K in $X(C)$.*

For $X = \mathbf{A}^n$ one can prove a slightly stronger version of Hartogs' theorem.

Theorem 5.2'. *Let U be an open semialgebraic subset of C^n and K a closed semialgebraic subset of U. Let p be the projection $C^n = C^{n-1} \times \mathbf{C} \to C^{n-1}$. We assume*
i) U and $U - K$ are connected.
ii) $p|K : K \to p(U)$ is proper and $p(K) \neq p(U)$.
Then every isoalgebraic function on $U - K$ can be continued to an isoalgebraic function on U.

Here are two classical examples to which one can apply Theorems 5.2 and 5.2' respectively.

1) Let $P_s := \{(z_1,\ldots,z_n) \in C^n \mid |z_i| < s, i = 1,\ldots,n\}$ denote the polycylinder of radius s. Let $n \geq 2, s > r > 0$. Then every isoalgebraic function on $P_s - \overline{P}_r$ can be extended to an isoalgebraic function on P_s.

2) Let $q_1,\ldots,q_n, s \in R$ with $0 < q_i < s, i = 1,\ldots,n$. Then every isoalgebraic function on $\{(z_1,\ldots,z_n) \in C^n \mid |z_i| < s, i = 1,\ldots,n-1 \text{ and } q_n < |z_n| < s\} \cup \{(z_1,\ldots,z_n) \in C^n \mid |z_i| < q_i, i = 1,\ldots,n-1 \text{ and } |z_n| < s\}$ can be continued to an isoalgebraic function on the polycylinder P_s.

In complex analysis the essential tool in the proof of Riemann's theorem or Hartogs' theorem is the Cauchy integral formula. Clearly, in isoalgebraic geometry this method is not possible. Here the methods are purely algebraic or semialgebraic.

Let $(X, \mathcal{O}_X)$ be a reduced locally isoalgebraic space. The non-normal locus $\{x \in X \mid \mathcal{O}_{X,x}$ is not normal$\}$ is denoted by $N(X)$. It is a locally isoalgebraic subset of X. An isoalgebraic morphism $f : \hat{X} \to X$ is called a *normalization* of X if the following conditions are satisfied:

i) $\hat{X}$ is a normal locally isoalgebraic space.

ii) f is finite.

iii) $\hat{X} - f^{-1}(N(X))$ is dense in $\hat{X}$ and the restriction $\hat{X} - f^{-1}(N(X)) \to X - N(X)$ of f is an isoalgebraic isomorphism.

Theorem 5.3. *Every reduced locally isoalgebraic space X has a normalization. The normalization is uniquely determined up to an isomorphism.*

The uniqueness of the normalization follows from the Riemann removable singularity theorem. Let $(X_i | i \in I)$ be an admissible open covering of X such that every X_i is isomorphic to an open subspace of some V^h, where V is a reduced variety. Let $g : \hat{V} \to V$ be the algebraic normalization of V. Then $g^h : \hat{V}^h \to V^h$ is a normalization of the isoalgebraic space V^h. Hence every X_i has a normalization $f_i : \hat{X}_i \to X_i$. Since the normalization is unique, the f_i glue together to a normalization $f : \hat{X} \to X$.

Finally we state the fundamental global decomposition theorem. The essential ingredients of its proof are the Riemann externsion theorem and the finite mapping theorem 4.1.

Let $(X, \mathcal{O}_X)$ be a locally isoalgebraic space. A locally isoalgebraic subset A of X is called *irreducible* if there are no proper locally isoalgebraic subsets A_1 and A_2 of A with $A = A_1 \cup A_2$. The maximal irreducible locally isoalgebraic subsets of X are called the *irreducible* components of X. For example, if V is a variety with irreducible components $V_1,\ldots,V_n$, then Theorem 4.4 implies that $V_1(C),\ldots,V_n(C)$ are the irreducible components of V^h.

Theorem 5.4. *Let $(Z_i | i \in I)$ be the family of connected components of the locally semialgebraic space $X - N(X_{red})$ and let $\overline{Z}_i$ be the closure of Z_i in X. Then $(\overline{Z}_i | i \in I)$ is the family of irreducible components of X. The family $(\overline{Z}_i | i \in I)$ is locally finite (i.e. for every open semialgebraic subset U of X there are only finitely many $i \in I$ with $\overline{Z}_i \cap U \neq \emptyset$).*

Let Z be a connected component of $X - N(X_{red})$. The delicate point in the proof of Theorem **5.4** is to show that $\overline{Z}$ is a locally isoalgebraic subset of X. The difficulties arise, because we have to show that there exists an *admissibble* open covering $(U_i | i \in I)$ of X such that $\overline{Z} \cap U_i$ is the zero set of finitely many isoalgebraic functions on U_i, whereas $\overline{Z}$ is the closure of Z in the *strong* topoiogy of X. Let $f : \hat{X} \to X_{red}$ be the normalization of X_{red}. Let W be the connected component of $\hat{X}$ containing $f^{-1}(Z)$. The set $f^{-1}(Z)$ is dense in W. It follows from the finite mapping theorem **4.1** that $\overline{Z} = f(W)$ is a locally isoalgebraic subset of X.

6. COVERINGS

Let $f : X \to Y$ be a morphism of locally semialgebraic spaces. f is cailed a *covering* if f is locally trivial with discrete fibres, i.e. there exists an admissible open covering $(U_i | i \in I)$ of Y having the property that every $f^{-1}(U_i)$ is a direct sum of open subspaces, $f^{-1}(U_i) = \coprod_{j \in J} V_j$, such that $f|V_j : V_j \to U_i$ is an isomorphism for every $j \in J$. If, moreover, all fibres of f are finite then f is called a *finite covering.*

As a first example, if $f : X \to Y$ is an algebraic covering of varieties then $f_C : X(C) \to Y(C)$ is a finite semialgebraic covering. Let X be a connected locally semialgebraic space and let x be a point of X. The relation between the semialgebraic fundamental group $\pi_1(X, x)$ of X and the coverings of X is the same as in topology ([DK, §5]):

1) $p \mapsto p_*(\pi_1(Y, y))$ is a bijection from the set of isomorphism classes of coverings $p : (Y, y) \to (X, x)$ with Y connected to the set of subgroups of $\pi_1(X, x)$. p is finite iff $p_*(\pi_1(Y, y))$ has finite index in $\pi_1(X, x)$.

2) There exists a *universal covering* $p : (\tilde{X}, \tilde{x}) \to (X, x)$ of X which is determined by the following property: $\tilde{X}$ is connected and for every covering $q : (Y, y) \to (X, x)$ there exists a unique lovally semialgebraic map $f : (\tilde{X}, \tilde{x}) \to (Y, y)$ with $p = q \circ f$. $\tilde{X}$ is simply connected and there is a (canonical) bijection from $\pi_1(X, x)$ to the fibre $p^{-1}(x)$.

If $f : Y \to X$ is a covering with Y semialgebraic then f is finite. But in generai, $\pi_1(X, x)$ is not finite. In this case the universal covering $\tilde{X}$ is not semialgebraic.

The Riemann existence theorem states that the algebraic fundamental group of a connected variety X over C is isomorphic to the profinite completion of the fundamental group of the topological space $X(\mathbf{C})$ [Mi, p. 40]. This remains true if we replace $\mathbf{R}$ by an arbitrary real closed field.

Theorem 6.1. *Let X be a variety. The functor, which associates with any algebraic covering $Y \to X$ the semialgebraic covering $Y(C) \to X(C)$, is an equivalence from the category of algebraic coverings of X to the category of finite semialgebraic coverings of $X(C)$.*

In a first step one proves Theorem **6.1** for Zariski-open subsets of $\mathbf{A}^1$. This special case follows easily from the description of coverings by subgroups of the fundamental group and the fact that Theorem **6.1** is true for $R = \mathbf{R}$. Then the general result of Theorem **6.1** is obtained by the same proof as in the topological case $R = \mathbf{R}$ using etale cohomology **[SGA 4, XVI.4.1]**.

A *locally isoalgebraic structure* on a locally semialgebraic space (X,C) is a sheaf $\mathcal{O}$ on the generalized topological space X such that $(X,\mathcal{O})$ is a locally isoalgebraic space with associated locally semialgebraic space (X,C). Two locally isoalgebraic structures $\mathcal{O}_1$ and $\mathcal{O}_2$ on (X,C) are called isomorphic if $(X,\mathcal{O}_1)$ and $(X,\mathcal{O}_2)$ are isoalgebrically isomorphic.

Let $(X,\mathcal{O}_X)$ be a locally isoalgebraic space and let $f : Y \to X$ be a locally semialgebraic covering. Then there exist a locally isoalgebraic structure $\mathcal{O}_Y$ on Y and a sheaf homomorphism $\vartheta : \mathcal{O}_X \to f_*\mathcal{O}_Y$ such that $(f,\vartheta) : (Y,\mathcal{O}_Y) \to (X,\mathcal{O}_X)$ is an isoalgebraic covering (isoalgebraic coverings are defined analogously to semialgebric coverings). $\mathcal{O}_Y$ and ϑ are uniquely determined.

Theorem 6.2. *Let X and Y be the normal connected varieties. Let V be a locally isoalgebraic space and let $f : V \to X^h$ and $g : V \to Y^h$ be isoalgebraic coverings. Then X and Y have a common algebraic covering, i.e. there exist algebraic coverings $W \to X$ and $W \to Y$.*

We may assume that V is connected. $(X \times Y)^h$ is the product of X^h and Y^h in the category of locally isoalgebraic spaces. Therefore f and g define an isoalgebraic morphism $s : V \to (X \times Y)^h$. The crucial (but not difficult) point in the proof of Theorem **6.2** is that $s(V)$ is contained in an n-dimensional isoalgebraic subset of $(X \times Y)^h$, where n is the dimension of V. Indeed, the Zariski-closure Z of $s(V)$ in $X \times Y$ has dimension n. Let $p : Z \to X$ and $q : Z \to Y$ be the projections. Using the fact that an isoalgebraic space has only finitely many irreducible components one can show that $s(V)$ is an n-dimensional isoalgebraic subset of Z^h and that $p|s(V) : s(V) \to X(C)$ and $q|s(V) : s(V) \to Y(C)$ are finite. Since Z is irreducible, we have $s(V) = Z(C)$. Thus p and q are finite. Let $t : W \to Z$ be the normalization of Z. Then $p \circ t : W \to X$ and $q \circ t : W \to Y$ are the coverings we are looking for [(1)].

Corollary 6.3. *Let X and Y be connected normal varieties. If X^h and Y^h have a common isoalgebraic covering then X^h and Y^h have also a common finite isoalgebraic covering.*

Now we give a detailed example of an isoalgebraic covering.

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(1) **We thank Mikahel Gromov for a discussion which led to this proof.**

For every natural number n we put $V_n = \{z \mathbf{E} C||z| < n\}$. Let V be the union of all V_n. V is a valuation ring of C with maximal ideal $m_V = \{z \mathbf{E} C||z| < \frac{1}{n}$ for every n E $\mathbf{N}\}$. If R is archimedean then $V = C$, and if R is not archimedean then V is not a semialgebraic subset of C. Let O, **be** the sheaf of isoalgebraic functions on V,. We consider the inductive limit iim $\rightarrow$ $(V,, O,)$ of the open subspaces $(V,, O,)$ of $(\mathbf{A}^1)^h$ in the category of ringed spaces over C. This space can be described as follows. The underlying sets is V. **A** subset U of V is an eiement of $\mathring{\mathcal{T}}(V)$ iff $U \cap V_n$ E $\mathring{\mathcal{T}}(V_n)$ for every n E **N** . Let U be an eiement of $\mathring{\mathcal{T}}(V)$. **A** family $(U_i|i$ E $I)$ of elements of $\mathring{\mathcal{T}}(V)$ is an admissible open covering of U iff, for every $n \in$ **N** ,$(U_i \cap V_n|i$ E $I)$ is an admissible open covering of $U \cap V_n$.$\mathcal{O}_V(U)$ is the ring of all functions f $: U \rightarrow$ C such that $f|U \cap V_n$ is an isoalgebraic function on $U \cap V_n$ for every $n \in$ **N**.

This space $(V$,O,) **is** locally isoalgebraic. Let C, be the associated sheaf of locally semialgebraic functions. Then C, is the ring of all functions $f : U$ -+ R such that $f|U \cap V_n$ is semialgebraic for every n E **N** .

The residue field $K := V/m_V$ is the algebraic closure of the real closed field $S := Y \cap R/m_V \cap R$. Let $\pi : V \rightarrow K$ be the projection. By a *lattice* of V we mean a subgroup L of V which is generated by two elements ω_1 and ω_2 such that $\pi(\omega_1)$ and $\pi(\omega_2)$ **are** linearly independent over S.

Let $L = \mathbf{Z}\omega_1 +$ Zw, **be** a lattice of V and let $p_L : V \rightarrow V/L$ be the projection. We equip V/L with the quotient structure, i.e. $\mathring{\mathcal{T}}(V/L)$ is the set of all subsets U of V/L with $p_L^{-1}(U)$ E $\mathring{\mathcal{T}}(V)$,Cov$_{V/L}(U)$ is the set of all families$(U_i|i$ E $\mathbf{i})$ with $(p_L^{-1}(U_i)|i$ E $\mathbf{I})$ E Cov$_V(p_L^{-1}$(U)) and the structure sheaf $\mathcal{O}_L$ is defined by $\mathcal{O}_L(U) = \{f \mathbf{E} \mathcal{O}_V(p_L^{-1}(U))|f$ is constant on $p_L^{-1}(x)$ for every x E U}. Then p_L is a morphism of ringed spaces over C. We set $W = \{s\omega_1 + t\omega_2|s,t$ E R and $-\frac{1}{2} < s,t < \frac{1}{2}\}$ and E $= \{0, \frac{1}{2}\omega_1, \frac{1}{2}\omega_2, \frac{1}{2}\omega_1 + \frac{1}{2}\omega_2\}$. For everye $\in E$ and $l \in L$, $p_L(e+W)$ is an open subset of V/L and $p_L|e+l+W : e+l+W \rightarrow p_L($e $+ W)$ is an isomorphism of ringed spaces over C. Furthermore, $(p_L($e $+ W)|e$ E $E)$ is an admissible open covering of V/L. Hence $(V/L$, O,) is an isoalgebraic space and p_L is an isoalgebraic covering. Let C_L be the associated sheaf of semialgebraic functions on V/L. Then $C_L(U) = \{f$ E $C_V(p_L^{-1}(U))|f$ is constant on $p_L^{-1}(x)$ foreverys E $U\}$. Since (V, C_V) is simply connected,p, $: (V$,C $\rightarrow (V/L$, C,) is the universal covenng of $(V/L$, C,).

Let U be a connected open locally semialgebraic subset of V and let f be an isoalgebraic function on U. We assume that $x_0 + L \subseteq$ Uforsomepoints, $\in V$ and that $f(x_0) = f(x_0 + l)$ for every 1$\in L$. Then f is constant. This is a consequence of the global etale factorization of f (cf. §2). This shows that the isoalgebraic space $(V/L, \mathcal{O}_L)$ is not algebraic.

Next we want to parametrizc the isomorphism classes of spaces $(V/L$, O,). Let L : ***M*** be two lattices of V and let f $: V/L \rightarrow V/M$ be an isoalgebraic isomorphism. f lifts to an isoalgebraic isomorphism $\tilde{f} : V \longrightarrow V$. It is of the form $z \mapsto az + b$ with b in V and a a unit of V. Hence V/L and V/M are isornorphic if and only if the lattices L and M are linearly equivalent, i.e. there exists a unit a of V with $aL = M$.

We denote by HV the set $\{x \in V | \operatorname{Im}(x) > \frac{1}{n}$ for some $n \in \mathbf{N}\}$. $SL(2,\mathbf{Z})$ acts on HV in the usual way, $A\tau = \frac{a\tau+b}{c\tau+d}$ with $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $\tau \in HV$. Every lattice of V is linearly equivalent to a lattice of the form $\mathbf{Z} + \mathbf{Z}\tau$ with $\tau \in HV$ and two lattices $\mathbf{Z} + \mathbf{Z}\tau_1$ and $\mathbf{Z} + \mathbf{Z}\tau_2$ with $\tau_1, \tau_2 \in$ HV are linearly equivalent if and only if $\tau_1 = A\tau_2$ with some $A \in SL(2,\mathbf{Z})$. Hence the orbit space of the action of $SL(2,\mathbf{Z})$ on HV parametrizes the isomorphism classes of spaces $(V/L, O,)$.

S^1 denotes the one dimensional sphere over the real closed field R. For every lattice L of V the semialgebraic space $(V/L$, C,) is isomorphic to $S^1 \times S^1$. Hence we get

Theorem 6.4. On *the torus $S^1 \times S^1$ there are injinitely many non isomorphic smooth isoalgebraic structures $\mathcal{O}$ such that $(S' \times S', O)$ is not algebraic.*

Corollary **6.3** is not true for arbitrary normal isoalgebraic spaces. There exist lattices L **and** M of V such that V/L and V/M have no common finite isoalgebraic covering. Let $s : V \rightarrow V/L \times V/M$ **be** the isoalgebraic morphism induced by p_L and p_M. We conclude from the proof of Theorem 6.2 that $s(V)$ is not contained in an one dimensional isoalgebraic subset of $V/L \times V/M$.

Two projective srnooth algebraic curves X and Y are called isogenous if they have a common algebraic covering. The universal covering of the semialgebraic space $X(\mathbf{C})$, where X is aprojective smooth algebraic curve, is isomorphic to $(V$,C,) . Hence every isogeny class of curves induces, up to isomorphism, a locally isoalgebraic structure on $(V$,C,) . We conclude from Theorem 6.2 that non isogeneous curves induce non isomorphic locally isoalgebraic structures on $(V$,C,) . Thus we obtain

Theorem 6.5. *On the locally semialgebraic space $(V$,C,) there exist infinitely many non isomorphic smooth locally isoalgebraic structures.*

The last theorem should be compared with the complex analytic situation. If (C,R) = **(C,R)** then $V = \mathbf{C}$. On the complex plane there are (up to isomorphism) only two smooth holomorphic structures.

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