ORTHONORMAL SETS IN REPRODUCING KERNEL SPACES
AND FUNCTIONAL COMPLETION
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Dedicated to the memory of Professor Gottfried Köthe

Let \( f_i(x) \) be a sequence of functions defined on a set \( S \). Suppose the function

\[
K(x, y) = \sum_i f_i(x) \overline{f_i(y)}
\]

makes sense for all points \((x, y)\) in \( S \times S \), i.e., that for every \( x \) the sequence \( \{f_i(x)\} \)
is summable square. Then in a known way the function \( K(x, y) \) is a positive matrix and corresponds to a reproducing kernel space \( \mathcal{H}_K(S) \) consisting of functions defined on \( S \). The question we ask here is whether the functions \( f_i(x) \) form a complete orthonormal system for that space.

It is easy to see that the answer is a negative one if the functions \( f_i \) are not linearly independent over \( S \), since every orthogonal system of functions is linearly independent. We suppose in the sequel, therefore, that the system of function \( f_i(x) \) is linearly independent.

In the special case that the system of functions is finite it turns out that it indeed is a complete orthonormal set; the problem is trickier in the infinite dimensional case.

If there are only \( N \) functions \( f_i(x), i = 1, 2, \ldots, N \) linearly independent over \( S \) we consider the space of all linear combinations

\[
g(x) = \sum_{i=1}^{N} a_i f_i(x)
\]

and introduce the quadratic norm defined by \( ||g||^2 = \sum_{i=1}^{N} |a_i|^2 \). For this space of functions

the point evaluations \( L_x(g) = g(x) \) are linear, hence continuous, and there exists a unique element \( K_x \) in the space such that \( g(x) = \langle g, K_x \rangle \). Thus our space is a reproducing kernel space, and by the definition of the norm, the \( f_i(x) \) are a complete orthonormal set. We can accordingly compute the associated kernel function using that orthonormal set and the kernel turns out to be our initial \( K(x, y) \). Hence the \( f_i(x) \) are indeed an orthonormal set in \( \mathcal{H}_K(S) \), as desired.

A more computational argument can also be given in the finite dimensional case. Since the \( f_i(x) \) are linearly independent, an elementary lemma in linear algebra guarantees the existence of \( N \) points \( x_1, x_2, \ldots, x_N \) in \( S \) such that the matrix

\[
F_{ij} = f_i(x_j)
\]
is non-singular. Let $H_{jk}$ be the inverse of $F$; since

$$K_j(x) = K(x, x_j) = \sum_k f_k(x) \overline{f_k(x_j)} = \sum_k H_{kj} f_k(x)$$

we have $\sum_j \overline{H_{jm}} K_j = f_m$ and by an easy calculation $(f_m, f_n) = \delta_{mn}$ i.e., the $f_k$ are orthonormal.

Before pursuing our argument for the general case it is worthwhile to recall some elementary facts concerning reproducing kernel spaces. Every such space is obtained from a mapping $\kappa$ of $S$ into a Hilbert space $\mathcal{H}$

$$\kappa : x \rightarrow k_x$$

which gives rise to a kernel function $K(x, y) = (k_y, k_x)$. There is a corresponding linear map $\kappa^*$ of $\mathcal{H}$ into $\mathcal{H}^*_K(S)$, a space of functions on $S$,

$$\kappa^* : f \rightarrow f(x) = (f, k_x).$$

The space $\mathcal{H}^*_K(S)$ is the reproducing kernel space associated with the kernel function $K(x, y)$. [1] The norm is the norm of the quotient $\mathcal{H}/\mathcal{N}$ where $\mathcal{N}$ is the null space of $\kappa^*$, a space necessarily closed. The mapping $\kappa^*$ is an isometry if and only if $\mathcal{N}$ is trivial.

A special case arises in the study of functional completion. Here we suppose that we are given a pre-Hilbert space $\mathcal{H}(S)$ of functions defined on $S$ such that the evaluation functionals $L_x$ are continuous. Here, as before, $L_x(f) = f(x)$, and of course these functionals admit a continuous extension to the (abstract) completion $\mathcal{H}^*$. On $\mathcal{H}^*$ $L_x$ is represented by an element $K_x$. Thus we have a map $\kappa$ of $S$ into $\mathcal{H}^*$ and an associated kernel function $K(x, y)$. Now the map $\kappa^*$ of $\mathcal{H}^*$ into $\mathcal{H}^*_K(S)$ is or is not an isometry.

If $\kappa^*$ is an isometry, it is clear that the initial $\mathcal{H}(S)$ was simply a dense subspace of $\mathcal{H}^*_K(S)$ and has the same norm as that space. The reproducing kernel space is the functional completion of $\mathcal{H}(S)$.

If $\kappa^*$ is not an isometry it has a null space. Thus there exists a non-trivial element $g$ in $\mathcal{H}^*$ such that $(g, K_x) = 0$ for all $x$. This $g$ is the limit of a sequence $g_n$ in $\mathcal{H}(S)$ such that $g_n(x)$ converges to 0 for all $x$, although the norms $\|g_n\|$ are bounded away from 0. The space $\mathcal{H}(S)$ now appears as a dense subspace of $\mathcal{H}^*_K(S)$ but the initial norm on $\mathcal{H}(S)$ is not the norm induced on it by the reproducing kernel space; the norm is that of a quotient. In this case no functional completion of $\mathcal{H}(S)$ can exist.

These considerations make it fairly clear how we are to proceed in the general case of our problem. We form the space $\mathcal{P}(S)$ consisting of finite linear combinations of the functions
$f_i(x)$ and note that the representation of such a finite linear combination

$$g(x) = \sum_i a_i f_i(x)$$

is unique, owing to the linear independence of the $f_i(x)$. We again introduce the quadratic norm

$$\|g\|^2 = \sum_i \|a_i\|^2$$

and now $\mathcal{F}(S)$ appears as a pre-Hilbert space. The valuations $L_x$ are continuous linear functionals on $\mathcal{F}(S)$ because of the hypothesis that the sequence $\{f_i(x)\}$ is summable square. With the norm just introduced, the $f_i$ are a complete orthonormal set in the (abstract) completion $\mathcal{F}^*$ of $\mathcal{F}(S)$. Now, either $\mathcal{F}(S)$ has a functional completion or it does not.

If $\mathcal{F}(S)$ has a functional completion then the mapping $\kappa^*$ from $\mathcal{F}^*$ to $\mathcal{H}_K(S)$ is an isometry, and the orthonormal set $f_i$ maps into an orthonormal set in the reproducing kernel space.

If $\mathcal{F}(S)$ has no functional completion the mapping $\kappa^*$ is not an isometry, and so the image of the complete orthonormal set $f_i$ cannot be itself an orthonormal set. It follows that the functions $f_i(x)$ are not an orthonormal set in the space $\mathcal{H}_K(S)$.

We see that the $f_i(x)$ are an orthonormal set in the reproducing kernel space if and only if the map $\kappa^*$ has a trivial null-space. We are therefore able to state a final criterion.

**Theorem.** The functions $f_i(x)$ form a complete orthonormal set in $\mathcal{H}_K(S)$ if and only if, for every sequence $\{b_j\}$ summable square the function $B(x) = \sum b_j f_j(x)$ is identically zero on $S$ only when every coefficient $b_j$ vanishes.

Note that the criterion given in the theorem is a slight strengthening of the hypothesis of linear independence. Note also that our argument applies equally well in the finite-dimensional case.

It is still not clear as to whether or not the case when the initial functions are not an orthonormal set actually occurs. A moment's thought convinces us that it happens just as often as separable spaces $\mathcal{H}(S)$ occur which have no functional completion. For suppose that $\mathcal{H}(S)$ is a separable pre-Hilbert space with continuous evaluation functional $L_x$ which has no functional completion. By the Gram-Schmidt process we can construct an orthonormal set $f_i$ in $\mathcal{H}(S)$ which is complete in the abstract completion $\mathcal{H}^*$. Let $\mathcal{F}(S)$ be the subspace of finite linear combinations of the $f_i$; it is easy to see that this space has no functional completion either since it contains a Cauchy sequence converging pointwise to 0 not converging to 0 in norm. It follows that the $f_i(x)$ are not an orthonormal set in the corresponding reproducing kernel space, although the kernel function is indeed given by the formula

$$K(x, y) = \sum_i f_i(x) \overline{f_i(y)}.$$
The standard example of a functional pre-Hilbert space having no functional completion was given by Aronszajn. [2] For this purpose we consider the reproducing kernel space $\mathcal{H}^2(D)$ consisting of functions analytic in the unit disk $D = \{z : |z| < 1\}$ which are integrable square; the norm is of course the usual $L^2(D)$ norm. A convenient complete orthonormal set in the space is given by the functions

$$f_n(z) = \sqrt{\frac{n+1}{\pi}} z^n \quad n = 0, 1, 2, \ldots$$

and the corresponding kernel function is

$$K(z, w) = \frac{1}{\pi(1 - z\bar{w})^2}$$

For $S$ we select a sequence $\{z_n\}$ in $D$ with $|z_n|$ converging so rapidly to 1 that a non-trivial Blaschke product $B(z)$ vanishing on $S$ exists. Hence there exists a sequence of polynomials $p_n(z)$ converging to $B(z)$ in $\mathcal{H}^2(D)$ which converges pointwise on $S$ to 0. For $\mathcal{P}(S)$ we take the space of all polynomials restricted to $S$ in the norm of $\mathcal{H}^2(D)$. Manifestly $\mathcal{P}(S)$ has no functional completion and the associated reproducing kernel space $\mathcal{H}_K(S)$ has a different (and smaller) norm than that of $\mathcal{P}(S)$. The $f_n(z)$ are not an orthonormal set in $\mathcal{H}_K(S)$ although the kernel function for that space is the restriction of $K(z, w)$ to $S \times S$. 
REFERENCES