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EXISTENCE OF STAR-PRODUCTS REVISITED

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Dedicated to the memory of Professor Gottfried Köthe

1. INTRODUCTION

Since the beginning of the theory of star-products, it was crucial to study the existence of such products. Vey [7] proved the existence when the third de Rham cohomology group of the manifold is vanishing (see also [5]).

Later, many papers proposed various other assumptions under which a symplectic manifold admits a star-product. It was finally shown [2], [3] that every symplectic manifold carries star-products.

The methods used in [2], [3] require a deep knowledge of the Hochshild or the Chevalley cohomology of the Poisson algebra to get rid of the possible obstructions encountered when building stepwise star-products or formal deformations of the Poisson algebra.

On the other hand, it is known from the beginning that every Darboux chart of a symplectic manifold has a preferred star-product, the Moyal star-product, and that two such Moyal products are always equivalent. It has thus been soon appealing to try to glue these local products to get a global one, by some Čech cohomology technique. But this direct approach did'nt succeed up to now because it quickly leads to an obstruction lying in the third de Rham cohomology space of the manifold which could not be avoided. It should be noted that this obstruction is due mainly to the fact that the Poisson algebra has a non vanishing center, namely $\mathbf{R}$.

Recently, H. Omori et al. gave an alternative proof of the existence of star-products by showing first that every symplectic manifold has a so-called Weyl manifold structure and then, that this implies that it also has a star-product [6]. In doing so, they have encountered a similar difficulty. Again, each Darboux domain of chart of the manifold has a local Weyl structure and the problem is to glue them by appropriately choosen transition maps. To avoid the de Rham obstruction, they have enlarged the structure in such a way that the disappointing Čech cocycle becomes a coboundary.

In fact, it is a trivial observation that the existence of a star-product is equivalent to the existence of a Weyl manifold structure. Thus, one can ask wether the heavy and technical formalism of [6], implied by the use of the Weyl structures, is really pertinent to solve the existence problem.

It is the purpose of the present paper to show that, on the contrary, one can easily achieve a proof of the existence of star-products staying within the standard framework of the theory. Using well known and simple tools, we directly show how to glue together the local Moyal products. Our method to get rid of the de Rham obstruction mentionned above is closely related to the key idea of [6]. Extending each local Moyal algebra by a non inner derivation, we obtain a family of algebras the centres of which are vanishing. Due to that fact, the gluing process still enconters a Čech cohomology obstruction, but in a case where the cohomology is vanishing. At the end of the paper, we show, whithin the same context, that each star-product of even order extends to a star-product, recovering in a simpler way one of our results of [2]. The fact that the existence of a star-product with a local and non formal derivation is equivalent to the exactness of the symplectic form is also obtained as a by-product of the proof.

2. MOYAL STAR-PRODUCTS

Notations 2.1. Let (M, F) be a symplectic manifold. A *Darboux chart* $(U, \varphi(x)) = (x^1, \ldots, x^{2n})$ is a chart such that $F = \sum_{i=1}^{n} dx^i \wedge dx^{i+n}$. We denote by $N(\Omega)$ (Ω open subset of M), the space of all smooth functions on Ω and by $N_\nu(\Omega)$ the space of formal series with coefficients in $N(\Omega)$. The k^{th} *-iterate of the Poisson bracket on* U is defined by

$$P^{(k)}(u, v) = \sum_{\substack{i_1 \ldots i_k \\ j_1 \ldots j_k}} F_{i_1 j_1} \ldots F_{i_k j_k} D_{x^{i_1}} \ldots D_{x^{i_k}} u D_{x^{j_1}} \ldots D_{x^{j_k}} v.$$

Definition 2.2. The *Moyal star-product on* U *related to the chart* (U, φ) is then the associative product on $N_\nu(U)$,

$$M_\nu = \sum_{k=0}^{\infty} \frac{\nu^k}{k!} P^{(k)},$$

where $P^{(0)}$ stands for the usual product $m : (u, v) \rightarrow uv$. The related *Moyal formal Lie algebra structure on* $N_\nu(U)$ is

$$P_\nu = \sum_{r=0}^{\infty} \frac{\nu^k}{(2k+1)!} P^{(2k+1)}.$$

Definition 2.3. Define now θ on $N_\nu(U)$ by

$$\theta = -2\, id + 4\nu D_\nu + L_\xi,$$

where ξ is the vector field on $U, \xi = \sum_{i=1}^{2n} x^i D_x i$. It is clear that it is a derivation of the Lie algebra $(N_\nu(U), P_\nu)$. It follows easily from [1] that if $H^1(U) = 0$, then the space of all derivations of $(N_\nu(U), P_\nu)$ is the direct sum of $\mathbb{C}_\nu \theta$ and the space of all inner derivations.

3. THE ALGEBRA $\mathbb{C}\theta \oplus N_\nu(U)$

Definition 3.1. Let again (U, φ) be a Darboux chart of (M, F). We shall denote by $A_\nu(U)$ the Lie algebra $\mathbb{C}\theta \oplus N_\nu(U)$, with the bracket

$$[\lambda\theta + u_\nu, \mu\theta + v_\nu] = \lambda\theta v_\nu - \mu\theta u_\nu + P_\nu(u_\nu, v_\nu).$$

It is clear that $\nu^k N_\nu(U)$ is an ideal in $A_\nu(U)$. We shall denote by $A_\nu^{(k)}(U)$ the quotient algebra $A_\nu(U)/\nu^k N_\nu(U)$.

Proposition 3.1. *The centre of $A_\nu(U)$ and of $A_\nu^{(k)}(U)$ is vanishing for each $k > 0$.*

Proof. Let $z = a\theta + \alpha_\nu$ belong to the centre. It follows from $[z, \nu] = 2a\nu$ that $a = 0$. As $z = \alpha_\nu$ belongs to the centre of $N_\nu(U)$, it is locally constant. Therefore

$$0 = [\theta, z] = \sum_{k \geq 0} \nu^k (4k - 2)\alpha_k$$

shows that $z = 0$. The same argument (mod ν^{k+1}) holds for $A_\nu^{(k)}(U)$.

Proposition 3.2. *If U is connected, the derivations of $A_\nu(U)$ and of $A_\nu^{(k)}$ $(k > 0)$ are inner.*

Proof. Let D be a derivation of $A_\nu(U)$. It is known that $(N_\nu(U), P_\nu)$ is equal to its derived ideal [1]. Therefore D induces a derivation $\mathscr{D}$ of $(N_\nu(U), P_\nu)$, which is local [1]. It makes thus sense to restrict D as a derivation D_ω of $A_\nu(\omega)$ for any open subset ω of U. We have

$$D_\omega(r\theta + u_\nu) = r(a\theta + \alpha_\nu) + \mathscr{D}|_\omega(u_\nu)(r \in \mathbb{C}, u_\nu \in N_\nu(\omega)),$$

for some $a \in \mathbb{C}$ and $\alpha_\nu \in N_\nu(\omega)$. If we choose ω connected and such that $H^1(\omega) = 0$, then $\mathscr{D}|_\omega$ can be written $ad(f_\nu^\omega) + b_\nu^\omega\theta$, as recalled after definition 2.3. Computing D_ω on $\theta u_\nu = [\theta, u_\nu]$ yields

$$(a + 4\nu D_\nu b_\nu^\omega)\theta u_\nu + ad(\theta f_\nu^\omega + \alpha_\nu)u_\nu = 0.$$

With $u_\nu = \nu$, we get $a + 4\nu D_\nu b_\nu^\omega = 0$. Thus $a = 0$, b_ν^ω reduces to its coefficient b^ω of degree 0 and the above equality reduces to $ad(\theta f_\nu^\omega + \alpha_\nu) = 0$. As ω is connected, it follows that the coefficients of $\gamma_\nu = \theta f_\nu^\omega + \alpha_\nu$ are constant. Therefore

$$D_\omega = ad_{A_\nu(\omega)}(b^\omega\theta + g_\nu^\omega),$$

where $g_k^\omega = f_k - \gamma_k/(4k-2)(k \in \mathbb{N})$. Since the centre of $A_\nu(\omega)$ is vanishing, b^ω and g_ν^ω are uniquely determined by D_ω.

Given two overlapping sets ω_1 and ω_2, we have then $b^{\omega_1} = b^{\omega_2}$ and $g_\nu^{\omega_1} = g_\nu^{\omega_2}$ in $\omega_1 \cap \omega_2$. Thus, there exists unique $b \in \mathbb{C}$ and $g_\nu \in N_\nu(U)$ such that

$$D = ad_{A_\nu(U)}(b\theta + g_\nu).$$

Remark. Observe that the connectedness of U is required only to fix b. Thus the derivations with range in $N_\nu(U)$ are always inner.

4. PRINCIPAL AUTOMORPHISMS

Consider two Darboux charts (U, φ_α) and (U, φ_β). We index by α the objects related to $(U, \varphi_\alpha) : M_{\alpha,\nu}, P_{\alpha,\nu}, \theta_\alpha = \theta, A_{\alpha,\nu}(U), \ldots$

Definition 4.1. A *principal isomorphism (of order $k > 0$)* of N_ν or $N_\nu^{(k')} = N_\nu / \nu^{k'} N_\nu$ $(k' > k)$ is an isomorphism

$$T : (N_\nu(U), P_{\alpha,\nu}) \to (N_\nu(U), P_{\beta,\nu}) \text{ or } T : N_\nu^{(k')}(U) \to N_\nu^{(k')}(U)$$

of the type

$$T = id + \nu^k T'_\nu$$

where T'_ν is formal, local and vanishing on the constants. If $\alpha = \beta$ and if $H^1(U) = 0$, it is known that such a T has the form exp $\nu^k ad(u_\nu)$ for some $u_\nu \in N_\nu(U)$ or $N_\nu^{(k')}(U)$.

Definition 4.2. *A principal isomorphism (of order $k > 0$) of A or $A^{(k')}(k' > k)$* is an isomorphism

$$Q : A_{\alpha,\nu}(U) \to A_{\beta,\nu}(U) \quad or \quad Q : A_{\alpha,\nu}^{(k')}(U) \to A_{\beta,\nu}^{(k')}(U)$$

such that $Q(\theta_\alpha) = \theta_\beta \bmod N_\nu$ and which restricts to a principal isomorphism of order k of N_ν or $N_\nu^{(k')}$.

Proposition 4.1. *If $H^1(U) = 0$, every principal isomorphism T of N_ν (resp. $N_\nu^{k'}, k' > k$) extends to a principal isomorphism Q defined by*

$$Q : \begin{cases} \theta_\alpha \to \theta_\beta + q_\nu, \\ u_\nu \to T u_\nu, \end{cases}$$

where q_ν is characterized by

$$ad_\beta(q_\nu) = -\theta_\beta + T \circ \theta_\alpha \circ T^{-1} \tag{1}$$

and is unique mod $\mathbb{C}_\nu$ (resp. mod $\mathbb{C}_\nu + \nu^k N_\nu$*).*

Proof. As defined, Q extends T if and only if it satisfies (1). The right hand side of (1) is a derivation of $(N_\nu(U), P_\beta)$ and it is formal. Since $H^1(U) = 0$, it can be written $ad_\beta(q_\nu)$ and q_ν is unique mod $\mathbb{C}_\nu$.

Proposition 4.2. *If U is connected, the principal automorphisms of order k of $A_\nu(U)$ are the maps* $\exp\ ad_{A_\nu(U)}(f_\nu)$ $(f_\nu \in N_\nu(U), f_i \in \mathbb{C}$ *for* $i < k$). *If* $\exp\ ad_{A_\nu(U)}(f_\nu)$ *and* $\exp\ ad_{A_\nu(U)}(g_\nu)$ *are principal automorphisms of order k and*

$$\exp\ ad_{A_\nu(U)}(f_\nu) \circ \exp\ ad_{A_\nu(U)}(g_\nu) = \exp\ ad_{A_\nu(U)}(h_\nu), \tag{2}$$

then

$$h_\nu = f_\nu + g_\nu \mod \nu^{k+1}.$$

Proof. Let Q be a principal automorphism of order k, of the form

$$Q : \begin{cases} \theta \to \theta + q_\nu \\ u_\nu \to u_\nu + \nu^k T_\nu u_\nu. \end{cases}$$

As

$$ad(q_\nu) = (1 + \nu^k T_\nu) \circ \theta \circ (1 + \nu^k T_\nu)^{-1} - \theta$$

we have $ad_\nu(q_\nu) = 0 \mod \nu^{k+1}$, hence $q_i \in \mathbb{C}$ for $i < k$. If T_0 is the coefficient of ν^k in T_ν,

$$D : \begin{cases} \theta \to \nu^k q_k \\ u_\nu \to \nu^k T_0 u_\nu \end{cases}$$

induces a derivation of $A_\nu^{(k+1)}(U)$. As easily seen, D is of the form $ad_{A_\nu(U)}(\nu^k q'_k)$ mod ν^{k+1} and

$$Q = \exp\ ad_{A_\nu(U)} \left(\sum_{i=0}^{k-1} \nu^i c_i + \nu^k q'_k \right) \mod \nu^{k+1}$$

where $c_i = -(4i-2)^{-1} q_i$. Now $Q \circ \exp -ad_{A_\nu(U)} \left(\sum_{i=0}^{k-1} \nu^i c_i + \nu^k q'_k \right)$ is a principal automorphism of order $k+1$. Applying the lemma below, it follows by induction that we can find q'_j's such that

$$Q = \exp\ ad_{A_\nu(U)} \left(\sum_{i=0}^{k-1} \nu^i c_i + \sum_{j=k}^{N} \nu^j q'_j \right) \mod \nu^{N+1}, \forall N \geq k,$$

and finally

$$Q = \exp\ ad_{A_\nu(U)} \left(\sum_{i=0}^{k-1} \nu^i c_i + \sum_{j \geq k} \nu^i q'_j \right).$$

It follows easily from (2) that $ad_{A_\nu}(f_\nu + g_\nu) = ad_{A_\nu}(h_\nu)$ mod ν^{k+1}. Therefore, $h_\nu = f_\nu + g_\nu$ mod ν^{k+1}.

Lemma 4.3. *Let* $S = \exp\ ad_{A_\nu(U)}(f_\nu)$ *and* $T = \exp\ ad_{A_\nu(U)}(g_\nu)$, *where* $f_\nu, g_\nu \in N_\nu(U)$, $f_0, g_0 \in \mathbb{C}$. *Then* $S \circ T = \exp\ ad_{A_\nu(U)}(h_\nu)$ *for some* $h_\nu \in N_\nu(U)$.

Proof. Assume that $S \circ T$ and $R = \exp\ ad_{A_\nu(U)}(h_\nu)$ induce the same automorphism of $A_\nu^{(k)}(U)$ for some $k \geq 1$, i.e. that $S \circ T \circ R^{-1}$ is the identity on $A_\nu^{(k)}(U)$. Then

$$S \circ T \circ R^{-1}(\alpha) = \alpha + \varphi(\alpha), \forall \alpha \in A_\nu(U),$$

where $\varphi : A_\nu(U) \to \nu^k N_\nu(U)$. As easily seen, φ induces a derivation on $A_{\nu,x_0}^{(k+1)}(U)$. Thus $\varphi = ad_{A_\nu(U)}(\nu^k h_k')$ mod ν^{k+1} and

$$S \circ T \circ R^{-1} = \exp\ ad_{A_\nu(U)}(\nu^k h_k') \text{ mod } \nu^{k+1}.$$

This may be rewritten

$$S \circ T = \exp\ ad_{A_\nu(U)}(h_\nu + \nu^k h_k') \text{ mod } \nu^{k+1}$$

since, clearly, $\exp\ ad_{A_\nu(U)}(h_\nu) \circ \exp\ ad_{A_\nu(U)}(\nu^k h_k')$ and $\exp\ ad_{A_\nu(U)}(h_\nu + \nu^k h_k')$ only differ by terms in $\nu^{k+1} N_\nu(U)$.

To conclude by induction, it remains to observe that, since $f_0, g_0 \in \mathbb{C}$,

$$\exp\ ad_{A_\nu(U)}(f_\nu) \circ \exp\ ad_{A_\nu(U)}(g_\nu) = \exp\ ad_{A_\nu(U)}(f_\nu + g_\nu) \text{ mod } \nu.$$

5. GLUING MOYAL STAR-PRODUCTS

Proposition 5.1. *Let* $U_\alpha (\alpha \in A)$ *be a contractible covering of* M *such that the* U_α*'s are the domains of Darboux charts* $(U_\alpha, \varphi_\alpha)$. *Let* M_α *be the corresponding Moyal star-products. There exist a collection of maps* $T_\alpha : N_\nu(U_\alpha) \to N_\nu(U_\alpha)$, *such that* $T_\alpha -$ *id are formal, differential and vanishing on the constants and that*

$$T_\alpha^* M_\alpha = T_\beta^* M_\beta$$

on each nonempty $U_{\alpha\beta} = U_\alpha \cap U_\beta$, *where* $T_\alpha^* M_\alpha$ *is the pull-back of* M_α *by* T_α. *Thus, if we define* M_ν *by*

$$M_\nu = T_\alpha^* M_\alpha$$

on U_α, M_ν is a star-product on (M,F).

Proof. If $U_{\alpha\beta} \neq \phi$, M_α and M_β are equivalent on $U_{\alpha\beta}$: there exists $R_{\alpha\beta} = id + \nu^2 T'_{\alpha\beta,\nu^2}$ where $T'_{\alpha\beta,\nu^2}$ is formal, differential and vanishing on the constants, such that

$$R_{\alpha\beta} : (N_\nu(U_{\alpha\beta}), M_\beta) \to (N_\nu(U_{\alpha\beta}), M_\alpha)$$

is an isomorphism [4]. Then $T_{\alpha\beta} = id + \nu T'_{\alpha\beta,\nu}$ is an isomorphism between $(N_\nu(U_{\alpha\beta}), P_\beta)$ and $(N_\nu(U_{\alpha\beta}), P_\alpha)$.

Fix some total order on A. Choose such a $T_{\alpha\beta}$ for $\alpha < \beta$ and define $T_{\beta\alpha} = T_{\alpha\beta}^{-1}$. Since $T_{\alpha\beta}$ is a principal isomorphism of order 1, we can extend it to a principal isomorphism of order 1, $Q_{\alpha\beta} : A_\beta(U_{\alpha\beta}) \to A_\alpha(U_{\alpha\beta})$. Set again $Q_{\beta\alpha} = Q_{\alpha\beta}^{-1}$.

If $U_{\alpha\beta\gamma} \neq \phi$,

$$Q_{\alpha\beta\gamma} = Q_{\alpha\beta} Q_{\beta\gamma} Q_{\gamma\alpha}$$

is still a principal isomorphism of order 1 of $A_\alpha(U_{\alpha\beta\gamma})$. Moreover,

$$Q_{\alpha\beta\gamma} = Q_{\alpha\beta} Q_{\beta\gamma\delta} Q_{\beta\alpha} Q_{\alpha\beta\delta} Q_{\alpha\gamma\delta}^{-1}. \tag{3}$$

By proposition 4.2,

$$Q_{\alpha\beta\gamma} = \exp\ ad_{A_\alpha}(q_{\alpha\beta\gamma}),$$

where

$$q_{\alpha\beta\gamma} = \sum \nu^i q^i_{\alpha\beta\gamma},\ q^0_{\alpha\beta\gamma} \in \mathbb{C}$$

It is straightforward that

$$Q_{\alpha\beta} Q_{\beta\gamma\delta} Q_{\beta\alpha} = \exp\ ad_{A_\alpha}(Q_{\alpha\beta} q_{\beta\gamma\delta}).$$

Using again proposition 4.2, (3) yields

$$q_{\alpha\beta\gamma} = Q_{\alpha\beta} q_{\beta\gamma\delta} + q_{\alpha\beta\delta} - q_{\alpha\beta\delta} \bmod \nu^2.$$

Since

$$Q_{\alpha\beta} q_{\beta\gamma\delta} = q_{\beta\gamma\delta} \bmod \nu^2,$$

we get

$$q^0_{\alpha\beta\gamma} = q^0_{\beta\gamma\delta} + q^0_{\alpha\beta\delta} - q^0_{\alpha\gamma\delta} \tag{4}$$

and

$$q^1_{\alpha\beta\gamma} = q^1_{\beta\gamma\delta} + q^1_{\alpha\beta\delta} - q^1_{\alpha\gamma\delta}. \tag{5}$$

As $Q_{\alpha\beta\alpha} = id$, one has $q^1_{\alpha\beta\alpha} = 0$. Setting $\delta = \alpha$ in (5), it thus follows that $q^1_{\alpha\beta\gamma} = q^1_{\beta\gamma\alpha}$. But $Q^{-1}_{\alpha\beta\gamma} = Q_{\alpha\gamma\beta}$. Thus $q^1_{\alpha\beta\gamma} = -q^1_{\alpha\gamma\beta}$. We have also $Q_{\alpha\beta\gamma} = Q_{\alpha\beta}Q_{\beta\gamma\alpha}Q_{\beta\alpha}$, hence $q^1_{\alpha\beta\gamma} = q^1_{\beta\gamma\alpha}$.

Hence $q^1_{\alpha\beta\gamma}$ is skewsymmetric in the indices α, β, γ.

The same holds for $q^0_{\alpha,\beta,\gamma}$ and the relation (4) states that $q^0_{\alpha,\beta,\gamma}$ defines a 2-cocycle in the Čech cohomology of M. We will discuss its interpretation later on.

Let $\varphi_\alpha (\alpha \in A)$ be a locally finite partition of the unity subordinate to $U_\alpha (\alpha \in A)$. Then

$$q^1_{\alpha\beta\gamma} = s^1_{\beta\gamma} + s^1_{\alpha\beta} - s^1_{\alpha\gamma}$$

where

$$s^1_{\alpha\beta} = \sum_{\mu \in A} \varphi_\mu q^1_{\alpha\beta\mu}.$$

Observe that $s^1_{\alpha\beta}$ is skewsymmetric in α, β.

Define

$$Q^1_{\alpha\beta} = \exp ad_{A_\alpha}\left(-\frac{\nu}{2}s^1_{\alpha\beta}\right) \circ Q_{\alpha\beta} \circ \exp ad_{A_\beta}\left(\frac{\nu}{2}s^1_{\beta\alpha}\right)$$

and set

$$Q^1_{\alpha\beta\gamma} = Q^1_{\alpha\beta}Q^1_{\beta\gamma}Q^1_{\gamma\alpha}.$$

It is easily seen that $Q^1_{\beta\alpha} = (Q^1_{\alpha\beta})^{-1}$ and that

$$Q^1_{\alpha\beta\gamma} = \exp ad_{A_\alpha}(q'_{\alpha\beta\gamma})$$

where

$$q'_{\alpha\beta\gamma} = q_{\alpha\beta\gamma} + \nu(s^1_{\beta\alpha} + s^1_{\alpha\gamma} + s^1_{\gamma\beta}) \bmod \nu^2$$

Thus $q'_{\alpha\beta\gamma} \in \mathbb{C}_\nu \bmod \nu^2$. Therefore $Q^1_{\alpha\beta\gamma}$ is a principal automorphism of order 2.

By induction on k, we can then define isomorphisms $Q^k_{\alpha\beta}$ such that

$$Q^k_{\alpha\beta} = Q^{k+1}_{\alpha\beta} \bmod \nu^{k+1}.$$

Define $Q'_{\alpha\beta}$ by

$$Q'_{\alpha\beta} = Q^k_{\alpha\beta} \bmod \nu^{k+1}, \forall k,$$

and by $T'_{\alpha\beta}$ their restrictions to N_ν. Then then $Q'_{\alpha\beta}$'s are still isomorphisms such that $Q'_{\alpha\beta}Q'_{\beta\alpha} = id$ and $T'_{\alpha\beta}T'_{\beta\gamma}T'_{\gamma\alpha} = id$. In other words, the new $T'_{\alpha\beta}$'s satisfy the Čech cocycle condition.

Setting

$$S_{\alpha,\nu} = \sum_{\mu\in A} \varphi_\mu T'_{\mu\alpha},$$

we easily see that $S_{\alpha,\nu} = id + \nu S'_\alpha$ (S'_ν formal, differential, vanishing on the constants) and that

$$T'_{\alpha\beta} = S^{-1}_{\alpha,\nu} S_{\beta,\nu}.$$

Thus

$$S^{-1*}_{\beta,\nu} P_\beta = S^{-1*}_{\alpha,\nu} P_\alpha,$$

wherefrom

$$S^{-1*}_{\beta,\nu^2} M_\beta = S^{-1}_{\alpha,\nu^2} M_\alpha$$

when $U_{\alpha\beta} \neq \phi$.

We now turn to explain the meaning of the constants $q^0_{\alpha,\beta,\gamma}$. First observe that the $Q'_{\alpha,\beta}$'s constructed in the above proof are such that

$$Q'_{\alpha\beta\gamma} = Q'_{\alpha\beta}Q'_{\beta\gamma}Q'_{\gamma\alpha} = \exp ad_{A_\alpha} q^0_{\alpha\beta\gamma}. \tag{6}$$

With the notation of that proof, it follows from prop. 4.1 that there exist $q_{\alpha\beta} \in N_\nu(U_{\alpha\beta})$ such that $Q'_{\alpha\beta}(\theta_\beta) = \theta_\alpha + q_{\alpha\beta}$. These are uniquely determined mod $\mathbb{C}_\nu$. Now, applying both members of (6) to θ_α and identifying the coefficients of ν^0 leads to

$$2\, q^0_{\alpha\beta\gamma} = q^0_{\alpha\beta} + q^0_{\beta\gamma} + q^0_{\gamma\alpha}.$$

Since $Q'_{\alpha\beta}Q'_{\beta\alpha} = id$ and $Q'_{\alpha\alpha} = id$, we have also $q^0_{\alpha\beta} + q^0_{\beta\alpha} = 0$ and $q^0_{\alpha\alpha} = 0$. On the other hand (Prop. 4.1), it follows from $ad_\alpha q_{\alpha\beta} = T'^{*}_{\beta\alpha}\theta_\beta - \theta_\alpha$ that $\xi_\beta - \xi_\alpha = H_{q^0_{\alpha\beta}}$, where H_u denotes the Hamiltonian vector field associated to u. This may be rewritten as $i(\xi_\alpha)F - i(\xi_\beta)F = dq^0_{\alpha\beta}$. Since, on U_α, $2F = L_{\xi_\alpha}F = di(\xi_\alpha)F$, we have shown

Proposition 5.2. *The constants $q^0_{\alpha\beta\gamma}$ define a cocycle of the Čech cohomology of M, the class of which coincides with the de Rham cohomology class of F under the canonical isomorphism between the Čech and de Rham cohomology.*

This allows us to state

Proposition 5.3. *Let P_ν be the formal deformation of (N, P) obtained in the proof of prop. 5.1. Then P_ν admits a non inner local derivation if and only if F is exact.*

Proof. We use freely the notations of the proof of prop. 5.1. Assume first that F is exact. Then there exist constants $c_{\alpha\beta}$ such that $2q^0_{\alpha\beta\gamma} = c_{\alpha\beta} + c_{\beta\gamma} + c_{\gamma\alpha}$. We choose the principal automorphisms $Q_{\alpha\beta}$ so that $q^0_{\alpha\beta\gamma} = 0$ (if $Q_{\alpha\beta}\theta_\beta = \theta_\alpha + f_{\alpha\beta}$, change it by $Q_{\alpha\beta}\theta_\beta = \theta_\alpha + f_{\alpha\beta} - c_{\alpha\beta}$). Then the $Q'_{\alpha\beta}$'s constructed in the proof verify $Q'_{\alpha\beta\gamma} = id$ whenever $U_{\alpha\beta\gamma} \neq \emptyset$. It follows easily that if $Q'_{\alpha\beta}\theta_\beta = \theta_\alpha + q_{\alpha\beta}$,

$$q_{\alpha\beta} + T'_{\alpha\beta}q_{\beta\gamma} + T'_{\alpha\gamma}q_{\gamma\alpha} = 0.$$

In other words,

$$q_{\alpha\beta} - q_{\alpha\gamma} = -T'_{\alpha\beta}q_{\beta\gamma}$$

as $q_{\beta\alpha} = -T'_{\beta\alpha}q_{\alpha\beta}$. Thus

$$\sum_\mu \varphi_\mu q_{\mu\beta} - \sum_\mu \varphi_\mu q_{\mu\gamma} = -\left(\sum_\mu \varphi_\mu T'_{\mu\beta}\right) q_{\beta\gamma} = -S_\beta q_{\beta\gamma}.$$

Therefore

$$q_{\beta\gamma} = g_\beta - T'_{\beta\gamma}g_\gamma$$

where $g_\beta = -S_\beta^{-1}\sum_\mu \varphi_\mu q_{\mu\beta}$. Hence

$$S^*_\beta S^{-1*}_\gamma(\theta_\gamma + adg_\gamma) = T^*_{\gamma\beta}(\theta_\gamma + adg_\gamma)$$

$$= \theta_\beta + ad(q_{\beta\gamma} + T'_{\beta\gamma}g_\gamma) = \theta_\beta + adg_\beta.$$

This shows that there exists a derivation θ of P_ν which is local and whose restrictions to the different U_α's are given by $S^{-1*}_\alpha(\theta_\alpha + adg_\alpha)$. As these are not formal, θ is not inner.

Conversely, assume that P_ν has a local derivation θ which is not inner. There exist then $c_\alpha \in \mathbb{C}_\nu$ and series $f_\alpha \in N_\nu(U_\alpha)$ such that $S^*_\alpha\theta = c_\alpha\theta_\alpha + adf_\alpha$. One has $S^*_\beta\theta = T'^*_{\alpha\beta}S^*_\alpha\theta$ and this reads

$$c_\beta\theta_\beta + adf_\beta = c_\alpha(\theta_\beta + adq_{\beta\alpha}) + adT'_{\beta\alpha}f_\alpha. \tag{7}$$

In particular, the c_α's coincide. Since θ is not inner, their common value is non vanishing. It may be assumed to take the form $\nu^t b$ with $b \in \mathbb{C}_\nu$ and $b_0 = 1$. Equating the coefficients of ν^t in (7) shows that for some constants $c_{\alpha\beta}$

$$q^0_{\beta\alpha} = f^t_\beta - f^t_\alpha + c_{\beta\alpha},$$

where f_α^t is the coefficient of ν^t in f_α.

This implies that $2\,q^0_{\alpha\beta\gamma} = c_{\alpha\beta} + c_{\beta\gamma} + c_{\gamma\alpha}$. Therefore, the Čech cohomology class of the $q^0_{\alpha\beta\gamma}$'s is vanishing and the symplectic form F is exact. Hence the result.

Proposition 5.4. *Each star-product of even order on (M,F) extends to a star-product.*

Proof. Let

$$M_\nu = \sum_{k=0}^{2n-1} \nu^k C_k$$

be a star-product of order $2\,n$. With the notation of prop. 5.1, choose D_{α,ν^2} such that

$$M_\nu = D^*_{\alpha,\nu^2} M_\alpha \bmod \nu^{2n},$$

In the proof of prop. 5.1, we can choose $R_{\beta\alpha} = D_\beta \circ D_\alpha^{-1} \bmod \nu^{2n}$. It then turns out that $T'_{\beta\alpha} = D_\beta \circ D_\alpha^{-1} \bmod \nu^n$ and that

$$S_{\alpha,\nu} = \sum_\mu \varphi_\mu D_\mu \circ D_\alpha^{-1} \bmod \nu^n.$$

Define $D = \sum_\mu \varphi_\mu D_\mu$. Then

$$D^*_{\nu^2} S^{-1*}_{\alpha,\nu^2} M_\alpha = D^*_\alpha M_\alpha \bmod \nu^{2n}.$$

Thus, if M'_ν is the star-product obtained in prop. 5.1,

$$D^*_{\nu^2} M'_\nu = M_\nu \bmod \nu^{2n}.$$

Hence $D^*_{\nu^2} M'_\nu$ extends M_ν to a star-product.

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