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GENERALIZED SAMPLING APPROXIMATION OF MULTIVARIATE SIGNALS; THEORY AND SOME APPLICATIONS

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Dedicated to the memory of Professor Gottfried M. Köthe

1. INTRODUCTION

For a continuous and bounded kernel function $\varphi : \mathbb{R}^n \to \mathbb{C}$, and a continuous function f the multivariate sampling series is defined by

$$(1.1) \qquad (S_{\mathbf{W}}^{\varphi} f)(t) := \frac{1}{(\sqrt{2\pi})^n} \sum_{\mathbf{k}\in\mathbb{Z}^n} f\left(\frac{\mathbf{k}}{\mathbf{W}}\right) \varphi(\mathbf{Wt}-\mathbf{k}) \qquad (\mathbf{t}\in\mathbb{R}^n; \mathbf{W}\in\mathbb{R}_+^n).$$

In [6] the authors presented some qualitative and quantitative theorems on the approximation of $f \in C(\mathbb{R}^n)$ by $S_{\mathbf{W}}^{\varphi} f$, as well as a few first applications. Some theorems in this respect to be needed below are assertion (5.2) as well as Theorem 5.1 of Section 5. In the more theoretical part of this paper two further quantitative theorems are given. The first deals with the case of product kernels, namely Theorem 3.1, where univariate theory is used to yield convergence theorems with rates in an iterative way, together with an application. The second theorem, Theorem 4.1, is concerned with bandlimited kernels, in which case the convergence with rates of $S_{\mathbf{W}}^{\varphi} f$ is compared with and deduced from the approximation behaviour of the associated singular convolution integral of Fejér's type. The matter is applied to three concrete kernels.

The core of this paper is Section 5, devoted to the applications of the general theorems of [6] to box splines, especially to linear combinations of translates of box splines. Basic assumptions here are certain conditions upon the sum moments of the kernel φ. These applications are true multivariate results which cannot be deduced in any way from unvariate ones. This part of the paper can also be regarded as a contribution to the theory of multivariate spline approximation, dealt with from different sides by [3; 5; 12; 13; 15].

Let us finally note that this paper is concerned with direct (Jackson-type) approximation theorems; inverse (Bernstein-type) theorems are studied in [14].

2. NOTATIONS

As usual, $\mathbf{N}, \mathbf{N}_0, \mathbf{Z}$ denote the sets of all naturals, all non-negative integers, and all integers, respectively, $\mathbb{R}, \mathbb{R}_+, \mathbb{C}$ being the sets of all real, positive real, and complex numbers, respectively. Let $\mathbf{N}^n$ denote the sets of all n-tuples $\mathbf{k} = (k_1, \dots, k_n)$ of elements from $\mathbf{N}$; $\mathbf{N}_0^n, \mathbf{Z}^n, \mathbb{R}^n, \mathbb{R}_+^n$ are defined analogously. In particular, $\mathbb{R}^n$ is the Euclidean n-space endowed with the norm $||\mathbf{u}||_2 := (u_1^2 + \ldots + u_n^2)^{1/2}$, where $\mathbf{u} = (u_1, \ldots, u_n), u_\mu \in \mathbb{R}, \mu \in \{1, \ldots, n\}$. Thus vectors are given in bold-face. The unit coordinate vector $(\delta_{j\mu})_{\mu=1}^n, j = 1, \ldots, n$, is denoted by $\mathbf{e}^{(j)}$. Further, $\alpha\mathbf{u} := (\alpha u_1, \ldots, \alpha u_n)$ is the product of $\mathbf{u}$ with $\alpha \in \mathbb{R}$, $\mathbf{u} \cdot \mathbf{v} := \sum_{\mu=1}^{n} u_\mu v_\mu$ is the scalar product of $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$, but $\mathbf{uv} := (u_1 v_1, \ldots, u_n v_n)$; $\mathbf{u}/\mathbf{v}$ denotes the vector of fractions $(u_1/v_1, \ldots, u_n/v_n)$, and $\mathbf{u}^{-1}$ will be used for $\mathbf{1}/\mathbf{u}$. Also, $[\![\mathbf{u}]\!]$ is the vector $([\![u_1]\!], [\![u_2]\!], \ldots, [\![u_n]\!])$, where $[\![u_\mu]\!]$ is the largest integer not bigger than u_μ. For $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n, \mathbf{u} > \mathbf{v}$ if and only if $u_\mu > v_\mu$, and $\mathbf{u} > \zeta$ for $\zeta \in \mathbb{R}$ if and only if $u_\mu > \zeta$ for $1 \leq \mu \leq n$. By $[\mathbf{a}, \mathbf{b}]$ we understand the n-dimensional rectangle of all vectors $\mathbf{u} \in \mathbb{R}^n$ with $\mathbf{a} \leq \mathbf{u} \leq \mathbf{b}$. Further, standard multi-index notation is used, i.e., for $\mathbf{k} \in \mathbb{N}_0^n, \mathbf{u} \in \mathbb{R}^n$, let $|\mathbf{k}| := k_1 + \ldots + k_n$ and $\mathbf{u}^{\mathbf{k}} := u_1^{k_1} \cdot \ldots \cdot u_n^{k_n}$.

For a function $f : \mathbb{R}^n \to \mathbb{C}$,

$$D^{\mathbf{k}} f := \frac{\partial^{|\mathbf{k}|}}{\partial \mathbf{u}^{\mathbf{k}}} f := \frac{\partial^{|\mathbf{k}|}}{\partial u_1^{k_1} \ldots \partial u_n^{k_n}} f \qquad (|\mathbf{k}| = r)$$

is called an rth-order derivative of f. For $D^{k\mathbf{e}^{(j)}} f$ we simply write $D^{k_j} f$. Let $C(\mathbb{R}^n)$ be the space of all uniformly continuous and bounded functions $f : \mathbb{R}^n \to \mathbb{C}$, endowed with the usual supremum norm $||f||_{C(\mathbb{R}^n)}$; for $r \in \mathbb{N}_0, C^r(\mathbb{R}^n) := \{f \in C(\mathbb{R}^n); D^{\mathbf{k}} f \in C(\mathbb{R}^n), |\mathbf{k}| = r\}$ is the space of all r-fold continuously differentiable functions.

As a measure of smoothness of a function $f \in C(\mathbb{R}^n)$ the modulus of continuity with respecto to the rth-order difference is used, namely,

$$\omega_r(f; \boldsymbol{\delta}) := \sup\{|(\Delta_{\mathbf{h}}^r f)(\mathbf{t})|; \mathbf{t} \in \mathbb{R}^n, -\delta \leq \mathbf{h} \leq \boldsymbol{\delta}\},$$

where $\boldsymbol{\delta} \in \mathbb{R}_+^n, r \in \mathbb{N}$, and the difference operator $\Delta_{\mathbf{h}}^r$ is defined as

$$(\Delta_{\mathbf{h}}^r f)(\mathbf{t}) := \sum_{\mu=0}^{r} (-1)^{r-\mu} \binom{r}{\mu} f(\mathbf{t} + \mu\mathbf{h}) \qquad (\mathbf{t}, \mathbf{h} \in \mathbb{R}^n; r \in \mathbb{N}).$$

If $f \in L^1(\mathbb{R}^n)$, i.e., f is an absolutely integrable function over $\mathbb{R}^n$, then its Fourier transform $f^\wedge$ is defined by

$$f^\wedge(\mathbf{v}) := \frac{1}{(\sqrt{2\pi})^n} \int_{\mathbb{R}^n} f(\mathbf{u}) e^{-i\mathbf{u}\cdot\mathbf{v}} d\mathbf{u} \qquad (\mathbf{v} \in \mathbb{R}^n).$$

The operators (1.1) may be regarded as a discrete version of the singular convolution integral of Fejér's type with kernel φ, namely,

$$(2.1)\quad (I^{\varphi}_{\mathbf{W}} f)(\mathbf{t}) := \frac{1}{(\sqrt{2\pi})^n} \prod_{j=1}^{n} W_j \int_{\mathbb{R}^n} f(\mathbf{t}-\mathbf{u})\varphi(\mathbf{W}\mathbf{u})\,d\mathbf{u} \qquad (\mathbf{t} \in \mathbb{R}^n; \mathbf{W} \in \mathbb{R}^n_+).$$

If $\varphi \in L^1(\mathbf{R}^n)$ with $\varphi^\wedge(\mathbf{0}) = 1$, then $\{I^{\varphi}_{\mathbf{W}}\}_{\mathbf{W}\in\mathbb{R}^n_+}$ is a family of bounded linear operators from $C(\mathbf{R}^n)$ into itself, with operator norm $(\sqrt{2\pi})^{-n}\int_{-\infty}^{\infty} |\varphi(\mathbf{u})|\,d\mathbf{u}$, satisfying

$$\lim_{\mathbf{W}\to\infty} \|I^{\varphi}_{\mathbf{W}} f - f\|_{C(\mathbf{R}^n)} = 0 \qquad (f \in C(\mathbf{R}^n)),$$

i.e., the family $\{I^{\varphi}_{\mathbf{W}}\}$ defines a strong approximation process on $C(\mathbf{R}^n)$. Here $\mathbf{W} \to \infty$ means that each component of $\mathbf{W}$ tends to infinity.

In some cases it is possible to compare the approximation behaviour of the discrete operators $S^{\varphi}_{\mathbf{W}} f$ of (1.1) with that of the integral $I^{\varphi}_{\mathbf{W}} f$ above (see Theorem 4.1). In Section 4 the following particular kernels will be considered, namely, the Fejér kernel

$$(2.2)\qquad F_n(\mathbf{t}) := \frac{1}{(\sqrt{2\pi})^n} \prod_{j=1}^{n} \left(\frac{\sin(t_j/2)}{t_j/2}\right)^2 \qquad (\mathbf{t} \in \mathbb{R}^n),$$

the kernel of de la Vallée Poussin

$$(2.3)\qquad \vartheta_n(\mathbf{t}) := \frac{4}{(\sqrt{2\pi})^n} \prod_{j=1}^{n} \frac{\sin(t_j/2)\sin(3t_j/2)}{t_j^2} \qquad (\mathbf{t} \in \mathbb{R}^n),$$

and the kernel of Bochner-Riesz

$$(2.4)\qquad b^{\gamma}_n(\mathbf{t}) := 2^{\gamma}\Gamma(\gamma+1)\|\mathbf{t}\|_2^{-(n/2+\gamma)} J_{n/2+\gamma}(\|\mathbf{t}\|_2) \qquad (\mathbf{t} \in \mathbb{R}^n)$$

for $\gamma > (n-1)/2$, J_λ being the Bessel function of order λ. Since these kernels will be used in the univariate case as well, the index n indicates the dimension of the variable $\mathbf{t}$.

Whereas Fejér's and de la Vallée Poussin's kernel are n fold products of the corresponding univariate versions, the Bochner Riesz kernel is of radial type, i.e., it depends only on $\|\mathbf{t}\|_2$. The essential property of all three is the fact that they belong to the class B_π, i.e., to the class of all functions $g \in L^1(\mathbb{R}^n)$ which are entire functions of exponential type π. This can be seen from their respective Fourier transforms (see [7, p. 516; 23; 24, p. 255; 18; 19, p. 109]),

$$F_n^\wedge(\mathbf{v}) = \prod_{j=1}^n F_1^\wedge(v_j), \qquad F_1^\wedge(\xi) = \begin{cases} 1-|\xi|, & |\xi| \le 1, \\ 0, & |\xi| > 1 \end{cases}$$

$$\vartheta_n^\wedge(\mathbf{v}) = \prod_{j=1}^n \vartheta_1^\wedge(v_j), \quad \vartheta_1^\wedge(\xi) = \begin{cases} 1, & |\xi| \le 1 \\ 2-|\xi|, & 1 < |\xi| \le 2, \\ 0, & |\xi| > 2 \end{cases}$$

$$(b_n^\gamma)^\wedge(\mathbf{v}) = \begin{cases} (1-\|\mathbf{v}\|_2^2)^\gamma, & \|\mathbf{v}\|_2 \le 1, \\ 0, & \|\mathbf{v}\|_2 > 1 \end{cases}$$

3. CONVERGENCE THEOREMS WITH RATES FOR PRODUCT KERNELS

In case that the kernel φ of the generalized sampling series (1.1) is of product type, i.e., $\varphi(\mathbf{t}) = \Pi_{j=1}^n \psi_j(t_j), \psi_j$ being univariate kernels satisfying the usual assumptions, assertions concerning the rate of approximation can be derived from univariate theory in an iterative way. The absolute (sum)-moment of $\varphi : \mathbb{R}^n \to \mathbb{C}$ of order $r \in \mathbb{N}_0$, needed below, is defined by

$$(3.1) \qquad m_r(\varphi) := \max_{|\mathbf{j}|=r} \sup_{\mathbf{t}\in\mathbb{R}^n} \frac{1}{(\sqrt{2\pi})^n} \sum_{\mathbf{k}\in\mathbb{Z}^n} |(\mathbf{t}-\mathbf{k})^{\mathbf{j}} \varphi(\mathbf{t}-\mathbf{k})|.$$

Theorem 3.1. *Let $\psi_1, \dots, \psi_n \in C(\mathbb{R}^1)$ be such that $m_0(\psi_j) < \infty$ and $\Sigma_{k\in\mathbb{Z}} \psi_j(\zeta - k) = \sqrt{2\pi}$ for all $\zeta \in \mathbb{R}, 1 \le j \le n$. Then for $\varphi(\mathbf{t}) := \Pi_{j=1}^n \psi_j(t_j) \in C(\mathbb{R}^n), t \in \mathbb{R}^n$, there holds*

$$\|S_{\mathbf{W}}^\varphi f - f\|_{C(\mathbb{R}^n)} \le \sum_{j=1}^n \left\{ \prod_{\substack{m=1 \\ m\neq j}}^n m_0(\psi_m) \right\} \|S_{W_j}^{\psi_j} f - f\|_{C(\mathbb{R}^n)}.$$

Here $S_{W_j}^{\psi_j} f$ denotes the univariate sampling operator with kernel ψ_j applied to f considered as a function of its j-th variable.

Proof. By definition of φ we may write

$$|S_{\mathbf{W}}^\varphi f(\mathbf{t}) - f(\mathbf{t})| = \left| \frac{1}{(\sqrt{2\pi})^n} \sum_{\mathbf{k}\in\mathbb{Z}^n} \left\{ f\left(\frac{\mathbf{k}}{\mathbf{W}}\right) - f(\mathbf{t}) \right\} \prod_{m=1}^n \psi_m(W_m t_m - k_m) \right|$$

$$= \left| \frac{1}{(\sqrt{2\pi})^n} \sum_{\mathbf{k}\in\mathbf{Z}^n} \sum_{j=1}^{n} \left\{ f\left(t_1,\ldots,t_{j-1},\frac{k_j}{W_j},\ldots,\frac{k_n}{W_n}\right) \right.\right.$$

$$\left.\left. - f\left(t_1,\ldots,t_j,\frac{k_{j+1}}{W_{j+1}},\ldots,\frac{k_n}{W_n}\right)\right\} \prod_{m=1}^{n} \psi_m(W_m t_m - k_m) \right|$$

$$\leq \sum_{j=1}^{n} \left\{ \frac{1}{(\sqrt{2\pi})^{n-1}} \prod_{\substack{m=1\\ m\neq j}}^{n} \sum_{k_m\in\mathbf{Z}} |\psi_m(W_m t_m - k_m)| \right\} \cdot$$

$$\sup_{\mathbf{u}_{[j]}\in\mathbf{R}^{n-1}} \left| \frac{1}{\sqrt{2\pi}} \sum_{k_j\in\mathbf{Z}} \left\{ f\left(u_1,\ldots,u_{j-1},\frac{k_j}{W_j},u_{j+1},\ldots,u_n\right) \right.\right.$$

$$\left.\left. - f(u_1,\ldots,u_{j-1},t_j,u_{j+1},\ldots,u_n)\right\} \psi_j(W_j t_j - k_j) \right|,$$

$\mathbf{u}_{[j]}$ being defined as $\mathbf{u}_{[j]} := (u_1,\ldots,u_{j-1},u_{j+1},\ldots,u_n)$. This gives the desired estimate.

It is known that the univariate Fejér-kernel F_1 of (2.2) and de la Vallée Poussin-kernel ϑ_1 of (2.3) satisfy the assumptions of Theorem 3.1 upon the ψ_j, and that there hold (cf. [8, Section 4.2; 9; 25]) for $f \in C(\mathbf{R}), 0 < \alpha < 1, 0 < \beta \leq 1, r \in \mathbf{N}_0$,

$$\|S_W^{F_1} f - f\|_{C(\mathbf{R})} = \mathcal{O}(W^{-\alpha}) \quad (W \to \infty) \iff \omega(f;\delta) = \mathcal{O}(\delta^\alpha) \quad (\delta \to 0+),$$

$$\|S_W^{F_1} f - f\|_{C(\mathbf{R})} = \mathcal{O}(W^{-1}) \quad (W \to \infty) \iff \omega(Hf;\delta) = \mathcal{O}(\delta) \quad (\delta \to 0+),$$

$$\|S_W^{\vartheta_1} f - f\|_{C(\mathbf{R})} = \mathcal{O}(W^{-r-\beta}) \quad (W \to \infty) \iff \omega_2(f^{(r)};\delta) = \mathcal{O}(\delta^\beta) \quad (\delta \to 0+),$$

where Hf denotes the Hilbert transform or conjugate function of f in the sense of [1, p. 128]. If $H_j f$ denotes the one dimensional Hilbert transform applied to $f \in C(\mathbf{R}^n)$ considered as a function of the j-th variable, then one has by Theorem 3.1,

Lemma 3.2. *a) If for $f \in C(\mathbf{R}^n)$ there exist constants $\alpha_j \in (0,1], 1 \leq j \leq n$, such that*

$$\|f(\mathbf{t} + h\mathbf{e}^{(j)}) - f(\mathbf{t})\|_{C(\mathbf{R}^n)} = \mathcal{O}(h^{-\alpha_j}) \quad (h \to 0)$$

in case $0 < \alpha_j < 1$, and

$$\|(\mathbf{H}_j f)(\mathbf{t} + h\mathbf{e}^{(j)}) - (H_j f)(\mathbf{t})\|_{C(\mathbf{R}^n)} = \mathcal{O}(h^{-1}) \quad (h \to 0)$$

in case $\alpha_j = 1$, *then there exists a constant* $C > 0$ *such that*

$$(3.2) \qquad \|S_{\mathbf{W}}^{F_n} f - f\|_{C(\mathbf{R}^n)} \leq C \sum_{j=1}^{n} W_j^{-\alpha_j}.$$

b) Let $g \in C(\mathbf{R}^n)$ *be such that its partial derivative* $D^{\mathbf{r}} g \in C(\mathbf{R}^n)$ *exists for some* $\mathbf{r} = (r_1, \dots, r_n) \in \mathbf{N}_0^n$, *and let there exist constants* $\beta_j \in (0,1], 0 < j \leq 1$ *such that* $\|D^{r_j} g(\mathbf{t} + h\mathbf{e}^{(j)}) - D^{r_j} g(\mathbf{t})\|_{C(\mathbf{R}^n)} = \mathcal{O}(h^{-\beta_j})$ *for all* $1 \leq j \leq n$. *Then there is a constant* $C > 0$ *with*

$$(3.3) \qquad \|S_{\mathbf{W}}^{\vartheta_n} g - g\|_{C(\mathbf{R}^n)} \leq C \sum_{j=1}^{n} W_j^{-r_j - \beta_j}.$$

A different application of Theorem 3.1 arises when using the following linear combinations of translates of univariate B-splines M_r, which can be defined in terms of their inverse Fourier transform as (see [8, Section 4.3])

$$M_r(\zeta) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left(\frac{\sin v/2}{v/2} \right)^r e^{iv\zeta} dv \qquad (\zeta \in \mathbf{R}).$$

For $r \in \mathbf{N}, r \geq 2$ let $b_{\mu r}, \mu = 0, 1, \dots, r-1$ be the unique solutions of the linear system (Vandermonde type)

$$(3.4) \qquad (-1)^j \sum_{\mu=0}^{[(r-1)/2]} b_{\mu r} \mu^{2j} = \left(\frac{1}{M_r^\wedge} \right)^{(2j)} (0) \quad \left(j = 0, 1, \dots, \left[\!\left[\frac{r-1}{2} \right]\!\right] \right).$$

Then

$$(3.5) \qquad \psi_r(\zeta) := b_{0r} M_r(\zeta) + \frac{1}{2} \sum_{\mu=1}^{[(r-1)/2]} b_{\mu r} \{ M_r(\zeta + \mu) + M_r(\zeta - \mu) \} \quad (\zeta \in \mathbf{R})$$

is a polynomial spline of degree $r-1$, having support in the compact interval $[-r/2 - (r-1)/2, r/2 + (r-1)/2]$, satisfying the assumptions of Theorem 3.1, as well as

$$(3.6) \qquad \|S_W^{\psi_r} f - f\|_{C(\mathbf{R})} \leq K_1 \omega_r(f; W^{-1}) \qquad (f \in C(\mathbf{R}); W > 0),$$

$$(3.7) \qquad \|S_W^{\psi_r} g - g\|_{C(\mathbf{R})} \leq K_2 W^{-r} \|g^{(r)}\|_{C(\mathbf{R})} \qquad (g \in C^r(\mathbf{R}); W > 0)$$

for some constants $K_1, K_2 > 0$, independent of f, g and W. Estimate (3.7) is also valid if only $g^{(r)}$ but not g itself is bounded.

Particular examples of kernels constructed according to (3.4)/(3.5) are given by

$$\psi_2(\zeta) = M_2(\zeta),$$
$$\psi_3(\zeta) = \frac{5}{4}M_3(\zeta) - \frac{1}{8}\{M_3(\zeta+1) + M_3(\zeta-1)\},$$
$$\psi_4(\zeta) = \frac{4}{3}M_4(\zeta) - \frac{1}{6}\{M_4(\zeta+1) + M_4(\zeta-1)\}.$$

Lemma 3.3. *Let $\mathbf{r} \in \mathbb{N}^n$ with $r_j \geq 2$. Suppose that $\varphi(\mathbf{t}) := \Pi_{j=1}^n \psi_{r_j}(t_j)$ for $\mathbf{t} \in \mathbb{R}^n$, the ψ_{r_j} defined as in (3.4)/(3.5). Then there exist constants $K_1, K_2 > 0$ such that for all $\mathbf{W} \in \mathbb{R}_+^n, f \in C(\mathbb{R}^n)$ and all $g \in C(\mathbb{R}^n)$ with $D^{\mathbf{r}} g \in C(\mathbb{R}^n)$,*

$$(3.8)\quad ||S_{\mathbf{W}}^{\varphi} f - f||_{C(\mathbb{R}^n)} \leq K_1 \sum_{j=1}^{n} \sup_{\mathbf{u}_{[j]} \in \mathbb{R}^{n-1}} \omega_{r_j}(f(u_1, \ldots, u_{j-1}, \cdot, u_{j+1}, \ldots, u_n); W_j^{-1}),$$

$$(3.9)\quad ||S_{\mathbf{W}}^{\varphi} g - g||_{C(\mathbb{R}^n)} \leq K_2 \sum_{j=1}^{n} W_j^{-r_j} ||D^{r_j} g||_{C(\mathbb{R}^n)}.$$

The modulus of continuity above is the one-dimensional modulus as applied fo f as a function of the j-th variable. The proof follows by Theorem 3.1 and (3.6)/(3.7).

4. CONVERGENCE THEOREMS WITH RATES FOR BANDLIMITED KERNELS

It is known in the univariate case that the approximation error of the generalized sampling series (1.1) can be estimated from above and below by the error of the associated singular convolution integral (2.1) (see [8, Section 4.2; 20; 21; 25]). Let us state the multivariate version. Its proof will not be given as it can be taken over almost verbatim, using standard multiindex notation.

Theorem 4.1. *Let $\varphi \in B_\pi^1$ with $\varphi^\wedge(\mathbf{0}) = 1$. There exist contants $c_1, c_2 > 0$, depending only on φ, such that*

$$c_1 ||I_{\mathbf{W}}^{\varphi} f - f||_{C(\mathbb{R}^n)} \leq ||S_{\mathbf{W}}^{\varphi} f - f||_{C(\mathbb{R}^n)} \leq c_2 ||I_{\mathbf{W}}^{\varphi} f - f||_{C(\mathbb{R}^n)} \quad (f \in C(\mathbb{R}^n); \mathbf{W} \in \mathbb{R}_+^n).$$

This theorem allows one to transfer practically all known results concernend with the approximation by convolution integrals to general sampling series.

Let us apply it in particular to the kernels φ equal to F_n, ϑ_n and b_n^γ defined in (2.2), (2.3), (2.4), respectively. In case of Corollary 4.2 below we choose the vectors $\mathbf{W}$ and $\boldsymbol{\delta}$ such that all of their components are equal; so these vectors will be identified with their components W and δ, respectively.

Corollary 4.2. *a) For $f \in C(\mathbb{R}^n), \varphi := F_n$ and $0 < \alpha < 1$ the following two assertions are equivalent:*

(i) $$\|S_W^\varphi f - f\|_{C(\mathbb{R}^n)} = \mathcal{O}(W^{-\alpha}) \qquad (W \to \infty),$$

(ii) $$\omega_2(f;\delta) = \mathcal{O}(\delta^\alpha) \qquad (\delta \to 0+).$$

b) For $f \in C(\mathbb{R}^n), \varphi := \vartheta_n, r \in \mathbb{N}_0$ and $0 < \alpha \leq 1$ the following two statements are equivalent:

(i) $$\|S_W^\varphi f - f\|_{C(\mathbb{R}^n)} = \mathcal{O}(W^{-r-\alpha}) \qquad (W \to \infty),$$

(ii) $f \in C^r(\mathbb{R}^n)$ and for all $\mathbf{j} \in \mathbb{N}_0^n, |\mathbf{j}| = r$ there holds

$$\omega_2(D^{\mathbf{j}} f;\delta) = \mathcal{O}(\delta^\alpha) \qquad (\delta \to 0+).$$

The proof follows by Theorem 4.1 and the fact that the corresponding results are true for the n-dimensional singular convolution integrals $I_W^\varphi f$ with Fejér's and de la Vallée Poussin's kernel, respectively. The implications (i) $\Rightarrow$ (ii) for the two convolution integrals can be shown iteratively using one dimensional results (cf. [22; 7, p. 149; 23]). The converse implications are proved as in the univariate case noting that F_n and ϑ_n are entire functions of exponential type π, (cf. [7, p. 148] for the one dimensional proof, and [19, Chapter 3] for the properties of entire functions of exponential type).

Corollary 4.3. *Let $f \in C(\mathbb{R}^n), \varphi := b_n^\gamma, \gamma > (n+3)/2$ and $0 < \alpha \leq 2$. The following statements are equivalent:*

(i) $$\|S_{\mathbf{W}}^\varphi f - f\|_{C(\mathbb{R}^n)} = \mathcal{O}(\|\mathbf{W}^{-1}\|_2^\alpha) \qquad (\mathbf{W} \to \infty),$$

(ii) $$\omega_2(f;\boldsymbol{\delta}) = \mathcal{O}(\|\boldsymbol{\delta}\|_2^\alpha) \qquad (\|\boldsymbol{\delta}\|_2 \to 0).$$

For the proof one makes use of Theorem 4.1 and the following lemma.

Lemma 4.4. *For $f \in C(\mathbb{R}^n), \varphi := b_n^\gamma, \gamma > (n+3)/2$ and $0 < \alpha \leq 2$ the following statements are equivalent:*

(i) $$\|I_{\mathbf{W}}^\varphi f - f\|_{C(\mathbb{R}^n)} = \mathcal{O}(\|\mathbf{W}^{-1}\|_2^\alpha) \qquad (\mathbf{W} \to \infty),$$

$$(ii) \qquad \omega_2(f;\boldsymbol{\delta}) = \mathcal{O}(\|\boldsymbol{\delta}\|_2^\alpha) \qquad (\|\boldsymbol{\delta}\|_2 \to 0).$$

Proof. Since b_n^γ is symmetric with respect to the origin, i.e., $b_n^\gamma(\mathbf{u}) = b_n^\gamma(-\mathbf{u})$ for $\mathbf{u} \in \mathbb{R}^n$, one has for arbitrary $\mathbf{t} \in \mathbb{R}^n$ (cf. [7 p. 142]),

$$\begin{aligned}
|(I_{\mathbf{W}}^{\varphi} f)(\mathbf{t}) - f(\mathbf{t})| &\le \frac{1}{2(\sqrt{2\pi})^n} \prod_{j=1}^{n} W_j \int_{\mathbb{R}^n} |f(\mathbf{t}-\mathbf{u}) + f(\mathbf{t}+\mathbf{u}) - 2f(\mathbf{t})|\varphi(\mathbf{W}\mathbf{u})\,d\mathbf{u} \\
&= \frac{1}{2(\sqrt{2\pi})^n} \int_{\mathbb{R}^n} \left|f\left(\mathbf{t} - \frac{\mathbf{u}}{\mathbf{W}}\right) + f\left(\mathbf{t} + \frac{\mathbf{u}}{\mathbf{W}}\right) - 2f(\mathbf{t})\right|\varphi(\mathbf{u})\,d\mathbf{u} \\
&\le \frac{1}{2(\sqrt{2\pi})^n} \int_{\mathbb{R}^n} \omega_2\left(f;\frac{\mathbf{u}}{\mathbf{W}}\right)\varphi(\mathbf{u})\,d\mathbf{u} \le M\|\mathbf{W}^{-1}\|_2^\alpha \int_{\mathbb{R}^n} \|\mathbf{u}\|_2^\alpha \varphi(\mathbf{u})\,d\mathbf{u}.
\end{aligned}$$

This gives the implication (ii) $\Rightarrow$ (i), since the latter integral is finite in view of $\gamma > (n+3)/2$. For the converse one may again proceed as in the univariate case, b_n^γ being an entire function of exponential type.

Observe that the investigations of the approximation theoretical behaviour of multivariate convolution *integrals* having radial kernels, such as b_n^γ, are not truly multidimensional; they can be reduced to the univariate case using polar coordinates. However, this remark does not apply to multivariate convolutions *sums.*

5. CONVERGENCE THEOREMS FOR BOX SPLINES

5.1. Properties of box splines; convergence theorems

In [6] it was shown that if for a function $\varphi \in C(\mathbb{R}^n)$ the moment $m_0(\varphi)$ (cf. (3.1)) is finite, where the convergence of the infinite series is uniform on compact sets, and

$$(5.1) \qquad \frac{1}{(\sqrt{2\pi})^n} \sum_{k\in\mathbb{Z}^n} \varphi(\mathbf{t}-\mathbf{k}) = 1 \qquad (\mathbf{t} \in \mathbb{R}^n),$$

then $S_{\mathbf{W}}^{\varphi}$ is a bounded linear operator from $C(\mathbb{R}^n)$ into itself with operator norm $m_0(\varphi)$, and

$$(5.2) \qquad \lim_{W\to\infty} \|S_{\mathbf{W}}^{\varphi} f - f\|_C = 0 \qquad (f \in C(\mathbb{R}^n)).$$

In this section it will be shown that this result can be applied to certain box splines, first introduced in [3] (for their properties see [4; 5; 15; 16; 17; 10, Chapter 2; 2, §13]). The following definition is appropriate: Let A be an $n \times m$-matrix with column vectors $A_\mu \in \mathbf{Z}^n \setminus \{\mathbf{0}\}, \mu = 1, 2, \ldots, m$, and $rk(A) = n$. The box spline M_A is then defined via

$$\int_{\mathbf{R}^n} M_A(\mathbf{t}) g(\mathbf{t}) d\mathbf{t} = \int_{Q^m} g(A\mathbf{x}) d\mathbf{x} \qquad (g \in C(\mathbf{R}^n)), \tag{5.3}$$

$Q^m := \left[-\frac{1}{2}, \frac{1}{2}\right]^m$ being the m-dimensional unit cube. Since M_A is defined only a.e. by (5.3), it is assumed that M_A is continuous whenever possible. It follows that

$$M_A(\mathbf{t}) \geq 0 \quad (\mathbf{t} \in \mathbf{R}^n), \quad \operatorname{supp}(M_A) = AQ^m, \tag{5.4}$$

in particular, M_A has compact support. If $\rho = \rho(A)$ is the largest integer for which all submatrices generated from A by deleting ρ columns have rank n, then $M_A \in C^{\rho-1}(\mathbf{R}^n)$. Further, the M_A are piecewise polynomials, i.e., polynomial splines of (total) degree $m - n$.

Basic here is that the Fourier transform of M_A is given by

$$\begin{aligned} M_A^\wedge(\mathbf{v}) &= \frac{1}{(\sqrt{2\pi})^n} \prod_{\nu=1}^{m} \operatorname{sinc}\left(\frac{1}{2\pi} \sum_{\mu=1}^{n} v_\mu a_{\mu\nu}\right) \\ &\equiv \frac{1}{(\sqrt{2\pi})^n} \prod_{\nu=1}^{m} \operatorname{sinc}\left(\frac{\mathbf{v} \cdot A_\nu}{2\pi}\right) \qquad (\mathbf{v} \in \mathbf{R}^n), \end{aligned} \tag{5.5}$$

$a_{\mu\nu}$ denoting the entries of A. For further properties of the M_A see the linerature cited above. Thus, if $\rho(A) \geq 1$ and $\varphi(\mathbf{t}) := (\sqrt{2\pi})^n M_A(\mathbf{t})$, then $\varphi \in C(\mathbf{R}^n)$ has compact support, and $\varphi^\wedge(\mathbf{0}) = 1, \varphi^\wedge(2\pi\mathbf{k}) = 0$ for $\mathbf{k} \in \mathbf{Z}^n \setminus \{\mathbf{0}\}$; the latter hold since the entries of A are integers and $rk(A) = n$. By [6, La. 3.2] this is equivalent to (5.1); the conditions upon $m_0(\varphi)$ are satisfied automatically as φ has compact support. So (5.2) holds for this φ, i.e.,

Corollary 5.1. *For $\varphi(\mathbf{t}) := (\sqrt{2\pi})^n M_A(\mathbf{t})$ one has for each $f \in C(\mathbf{R}^n)$,*

$$\lim_{\mathbf{W} \to \infty} \left\| \frac{1}{(\sqrt{2\pi})^n} \sum_{\mathbf{Wt} - \mathbf{k} \in AQ^m} f\left(\frac{\mathbf{k}}{\mathbf{W}}\right) M_A(\mathbf{Wt} - \mathbf{k}) - f(\mathbf{t}) \right\|_{C(\mathbf{R}^n)} = 0. \tag{5.6}$$

Example 1. Choose integers $m_j \geq 2$ for $j = 1, 2, \ldots, n$. Define an $n \times m$-matrix $A = (a_{\mu\nu})$ by $a_{\mu\nu} := 1$ provided $\Sigma_{j=1}^{\mu-1} m_j < \nu \leq \Sigma_{j=1}^{\mu} m_j$, and $a_{\mu\nu} := 0$ otherwise, i.e.,

$$A = \begin{pmatrix} 1 & \ldots & 1 & 0 & \ldots & 0 & \ldots & 0 & \ldots & 0 \\ 0 & \ldots & 0 & 1 & \ldots & 1 & \ldots & 0 & \ldots & 0 \\ \vdots & & \vdots & \vdots & & \vdots & & \vdots & & \vdots \\ 0 & \ldots & 0 & & \ldots & & \ldots & 1 & \ldots & 1 \end{pmatrix}; \tag{5.7}$$

then $[(\sqrt{2\pi})^n M_A]^\wedge(\mathbf{v}) = \Pi_{j=1}^n (sinc(v_j/2\pi))^{m_j}$. Now $(sinc(\cdot/2\pi))^r$ is just the Fourier transform of the univariate B-spline M_r of degree $r-1$. Hence by the uniqueness theorem for Fourier transforms, $M_A(\mathbf{t})$ is now a product kernel, namely, $(\sqrt{2\pi})^n M_A(\mathbf{t}) = \Pi_{j=1}^n M_{m_j}(t_j)$, $\mathbf{t} \in \mathbb{R}^n$.

5.2. Convergence theorems with rates for box splines

As seen in Example 1, the box spline is a multivariate generalization of the univariate B-spline. As observed in the univariate case, there hold convergence theorems with rates for certain linear combinations of translates of such B-splines. These results can be transferred by Lemma 3.3 to the multivariate setting in the case of the product of such linear combinations. In particular, the φ in Lemma 3.3 can be chosen as a spline of degree $n(r-1)$ for which the corresponding generalized sampling series approximates a function $g \in C^r(\mathbb{R}^n)$ with rate $\mathcal{O}(W^{-r})$. Our aim now is to generalize these results to more general box splines, even to such of degree $r-1$. For this purpose we need the following theorem established in [6].

Theorem 5.2. *Let $\varphi \in C(\mathbb{R}^n)$ be such that $m_r(\varphi) < \infty$ for some $r \in \mathbb{N}$. If, additionally, (5.1) holds and the moments*

$$\frac{1}{(\sqrt{2\pi})^n} \sum_{\mathbf{k}\in\mathbf{Z}^n} (\mathbf{t}-\mathbf{k})^{\mathbf{j}} \varphi(\mathbf{t}-\mathbf{k}) = 0 \tag{5.8}$$

for all $\mathbf{t} \in \mathbb{R}^n, \mathbf{j} \in \mathbb{N}_0^n$ with $0 < |\mathbf{j}| < r$, then

$$\|S_{\mathbf{W}}^{\varphi} g - g\|_C \le m_r(\varphi) \sum_{|\mathbf{s}|=r} \frac{\|D^{\mathbf{s}} g\|_{C(\mathbb{R}^n)}}{\mathbf{s}!\mathbf{W}^{\mathbf{s}}} \qquad (g \in C^r(\mathbb{R}^n); \mathbf{W} > 0), \tag{5.9}$$

$$\|S_{\mathbf{W}}^{\varphi} f - f\|_C \le M\omega_r(f; \mathbf{W}^{-1}) \qquad (f \in C(\mathbb{R}^n); \mathbf{W} > 0). \tag{5.10}$$

Now, if φ is a finite linear combination of translates of a box spline, then φ compact support so that $m_r(\varphi)$ is finite automatically, and conditions (5.1) and (5.8) are, by [6, Lemma 3.2], equivalent to

$$D^{\mathbf{j}} \varphi^\wedge(2\pi\mathbf{k}) = \begin{cases} 0, & \mathbf{k} \in \mathbf{z} \setminus \{0\} \\ \delta_{\mathbf{j}0}, & \mathbf{k} = 0 \end{cases} \qquad (0 \le |\mathbf{j}| < r). \tag{5.11}$$

Next to the multivariate counterpart of the results of (3.4) - (3.7).

Theorem 5.3. *Let $r, m \in \mathbb{N}, r \geq 2, m \geq n + r - 1$, and A be an $n \times m$-matrix with column vectors $A_\mu \in \mathbb{Z}^n \setminus \{\mathbf{0}\}$, and $\rho(A) \geq r - 1$. Furthermore, let $b_{\boldsymbol{\mu} r}$, with $\boldsymbol{\mu} \in G_r :=$ $\left\{\boldsymbol{\mu} \in \mathbb{N}_0^n; |\boldsymbol{\mu}| = 0, 2, \ldots, 2\left[\!\left[\frac{r-1}{2}\right]\!\right]\right\}$ be solutions of the linear system*

$$(5.12) \qquad (-1)^{|\boldsymbol{\mu}|} \sum_{\boldsymbol{\mu} \in G_r} b_{\boldsymbol{\mu} r} \boldsymbol{\mu}^{\boldsymbol{\nu}} = D^{\boldsymbol{\nu}}\left(\frac{1}{M_A^\wedge}\right)(\mathbf{0}) \qquad (\boldsymbol{\nu} \in G_r)$$

if there exist any. Then

$$(5.13) \qquad \varphi_{A,r}(\mathbf{t}) := b_{\mathbf{0}r} M_A(\mathbf{t}) + \frac{1}{2} \sum_{\boldsymbol{\mu} \in G_r \setminus \{\mathbf{0}\}} b_{\boldsymbol{\mu} r}\{M_A(\mathbf{t} + \boldsymbol{\mu}) + M_A(\mathbf{t} - \boldsymbol{\mu})\}$$

is a polynomial spline of degree $m - n$ with compact support, $(r - 2)$ times continuously differentiable, and satisfying the assumptions of Theorem 5.2. In particular, the estimates (5.9) and (5.10) hold for $\varphi(\mathbf{t}) := \varphi_{A,r}(\mathbf{t})$.

Proof. The assertions regarding polynomial degree, smoothness and support follow from the corresponding properties of M_A. Concerning (5.1) and (5.8), we just have to verify (5.11). Indeed, consider the Fourier transform of $\varphi_{A,r}$, namely,

$$(5.14) \qquad \varphi_{A,r}^\wedge(\mathbf{v}) = M_A^\wedge(\mathbf{v})\left\{b_{\mathbf{0}r} + \sum_{\boldsymbol{\mu} \in G_r \setminus \{\mathbf{0}\}} b_{\boldsymbol{\mu} r} \cos(\boldsymbol{\mu} \cdot \mathbf{v})\right\} := M_A^\wedge(\mathbf{v}) p(\mathbf{v}).$$

Then by (5.12) there results

$$p(\mathbf{0}) = \sum_{\boldsymbol{\mu} \in G_r} b_{\boldsymbol{\mu} r} = \frac{1}{M_A^\wedge(\mathbf{0})},$$

$$D^{\boldsymbol{\nu}} p(0) = (-1)^{|\boldsymbol{\nu}|/2} \sum_{\boldsymbol{\mu} \in G_r \setminus \{\mathbf{0}\}} b_{\boldsymbol{\mu} r} \boldsymbol{\mu}^{\boldsymbol{\nu}} = D^{\boldsymbol{\nu}}\left(\frac{1}{M_A^\wedge}\right)(\mathbf{0}) \quad (\boldsymbol{\nu} \in G_r \setminus \{\mathbf{0}\}).$$

If $|\boldsymbol{\nu}| < r$, $\boldsymbol{\nu} \in \mathbb{N}_0^n \setminus G_r$, then $|\boldsymbol{\nu}|$ is odd, and $D^{\boldsymbol{\nu}} p_r(\mathbf{0}) = 0 = D^{\boldsymbol{\nu}}(1/M_A^\wedge)(0)$, since $M_A^\wedge$ is symmetric in the origin by (5.5). So for $\mathbf{j} \in \mathbb{N}_0^n, |\mathbf{j}| < r$ we have by the Leibniz rule

$$D^{\mathbf{j}} \varphi_A^\wedge(\mathbf{0}) = \sum_{\mathbf{0} \leq \boldsymbol{\nu} \leq \mathbf{j}} \binom{\mathbf{j}}{\boldsymbol{\nu}} D^{\boldsymbol{\nu}} M_A^\wedge(\mathbf{0}) D^{\mathbf{j}-\boldsymbol{\nu}} p(\mathbf{0}) = \sum_{0 \leq \nu \leq j} \binom{j}{\nu} D^{\nu} M_A^\wedge(0) D^{j-\nu}\left(\frac{1}{M_A^\wedge}\right)(0)$$

$$= D^{\mathbf{j}}\left(M_A^\wedge \frac{1}{M_A^\wedge}\right)(\mathbf{0}) = \delta_{\mathbf{j}\mathbf{0}}.$$

Hence (5.11) follows for $\mathbf{k} = \mathbf{0}$.

To show that $D^{\mathbf{j}}\varphi_{A,r}(2\pi\mathbf{k}) = 0$ for all $\mathbf{k} \in \mathbb{Z}^n \setminus \{\mathbf{0}\}, \mathbf{j} \in \mathbb{N}_0^n, 0 \leq |\mathbf{j}| < r$, we will show that $2\pi\mathbf{k}, \mathbf{k} \neq \mathbf{0}$, is an r-fold zero of $M_A^\wedge$. This would yield (5.11) in view of (5.14). Now, by (5.5),

$$M_A^\wedge(2\pi\mathbf{k}) = \frac{1}{(\sqrt{2\pi})^n} \prod_{j=1}^{m} sinc(\mathbf{k} \cdot A_j).$$

Here $sinc(\mathbf{k} \cdot A_j) = 0$ if and only if $\mathbf{k} \cdot A_j \neq 0$ because the entries of A are integers. Since $\rho(A) \geq r - 1$, for fixed $\mathbf{k} \in \mathbb{Z}^n \setminus \{\mathbf{0}\}$ there is at least one column A_{j_1} among the first $m - (r-1)$ columns of A such that $\mathbf{k} \cdot A_{j_1} \neq 0$. Deleting this one there is again a column A_{j_2} among the first $m - (r-1)$ columns of the remaining matrix for which $\mathbf{k} \cdot A_{j_2} \neq 0$. This method yields at least r such columns of A, so that $2\pi\mathbf{k}$ is an r-fold zero of $M_A^\wedge$. Thus (5.11) is verified for $\mathbf{k} \neq \mathbf{0}$, and the proof is complete.

As will have been observed, the existence of solutions of the linear system (5.12) is postulated in Theorem 5.3. In fact, the authors assume that it would be quite difficult to establish the existence and uniqueness of solutions of (5.12) [(1)]. However, this is easier for the following, less elegant linear combination of translates of a box spline. It will be stated for completeness.

Theorem 5.4. *Let r, m, A be given as in Theorem 5.3, and let $b^*_{\boldsymbol{\mu}r}, \boldsymbol{\mu} \in \mathbb{N}_0^n$ with $|\mu| \leq r_0 := 2[\![(r-1)/2]\!]$ be the unique solutions of the linear system (Vandermond type)*

$$(5.15) \qquad (-1)^{[\![|\nu|]\!]} \sum_{0 \leq |\boldsymbol{\mu}| \leq r_0} b^*_{\boldsymbol{\mu}r} \boldsymbol{\mu}^{\boldsymbol{\nu}} = D^{\boldsymbol{\nu}} \left(\frac{1}{M_A^\wedge}\right)(\mathbf{0}) \quad (\boldsymbol{\nu} \in \mathbb{N}_0^n; |\boldsymbol{\nu}| \leq r_0).$$

Then

$$\varphi^*_{A,r}(\mathbf{t}) := b^*_{\mathbf{0}r} M_A(\mathbf{t}) + \frac{1}{2} \sum_{0 < |\boldsymbol{\mu}| \leq r_0} b^*_{\boldsymbol{\mu}r} \{M_A(\mathbf{t} + \boldsymbol{\mu}) - M_A(\mathbf{t} - \boldsymbol{\mu})\}$$

is a polynomial spline of degree $m - n$ with compact support, $(r-2)$ times continuously differentiable, and satisfying the assumptions of Theorem 5.2. In particular, the estimates (5.9) and (5.10) hold for $\varphi(\mathbf{t}) := \varphi^*_{A,r}(\mathbf{t})$.

The proof is similar to that of Theorem 5.3; concerning the existence and uniqueness of the $b^*_{\mu r}$ see, e.g., [11].

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(1) The authors would like to thank Dr. R.A. Lorentz (Birlinghoven, St. Augustin) and their colleague Professor W. Plesken (Aachen) for useful correspondence and discussion concerning this question.

Above we have forced the uniqueness of the solutions $b^*_{\boldsymbol{\mu}r}$ by requiring more conditions than necessary. In fact, we only need, instead of (5.15),

$$(-1)^{|\boldsymbol{\nu}|/2} \sum_{0\le|\boldsymbol{\mu}|\le r_0} b^*_{\boldsymbol{\mu}r}\boldsymbol{\mu}^{\boldsymbol{\nu}} = D^{\boldsymbol{\mu}}\left(\frac{1}{M_A^\wedge}\right)(\mathbf{0}) \qquad (\boldsymbol{\nu} \in \mathbb{N}_0^n, |\nu| \le r_0, |\nu| \; even).$$

Note also that the choice of the translation vectors $\boldsymbol{\mu}$ was arbitrarty; the $\boldsymbol{\mu}$ in (5.12) could just as well be replaced by $\boldsymbol{\mu}/2$ or more general vectors for which there exist solutions of the associated linear system.

Example 2. With the choice $m := n + r - 1$, A given by

$$(5.16) \qquad A = \begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & 2 & 3 & \dots & m \\ 1 & 4 & 9 & \dots & m^2 \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & 2^{n-1} & 3^{n-1} & \dots & m^{n-1} \end{pmatrix},$$

the methods of Theorems 5.3 and 5.4 yield polynomial splines of degree $r-1$ satisfying the assumptions of Theorem 5.2.

Example 3. Another possibility for the choice of A is the following: For a non-singular $n \times n$-matrix B with column vectors in $\mathbb{Z}^n \setminus \{\mathbf{0}\}$, let $B_r := (BB \dots B)$ be an r-fold repetition of B. Since $\rho(B_r) = r-1$, this matrix satisfies the assumptions of Theorems 5.3 and 5.4 as well. If, in particular,

$$(5.17) \qquad B = E := \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix},$$

then $M_{E_r}(\mathbf{t}) = (\sqrt{2\pi})^n \Pi_{\lambda=1}^n M_r(t_\lambda)$, and Theorems 5.3 and 5.4 yield two types of linear combinations of translates of this product kernel satisfying the estimates (5.9) and (5.10).

In cases like the last one, if $M_A^\wedge$ is symmetric in each variable, i.e., if $M_A^\wedge(v_1,\dots,v_{j-1},v_j,v_{j+1},\dots,v_n) = M_A^\wedge(v_1,\dots,v_{j-1},-v_j,v_{j+1},\dots,v_n)$, $1 \le j \le n$, there is a third possibility for constructing such linear combinations, using a linear system even simpler than (5.12).

Theorem 5.5. *Let r, m, A be given as in Theorem 5.3 such that, additionally, $M_A^\wedge$ is symmetric in each variable. Further, let $c_{\boldsymbol{\mu}r}$ for $\boldsymbol{\mu} \in H_r := \{\boldsymbol{\mu} \in \mathbb{N}_0^n; 0 \leq |\boldsymbol{\mu}| \leq r_0\}, r_0 := [\![(r-1)/2]\!]$, be the unique solutions of the linear system*

$$(5.18) \qquad (-1)^{|\boldsymbol{\nu}|} \sum_{\boldsymbol{\mu} \in H_r} c_{\boldsymbol{\mu}r} \boldsymbol{\mu}^{2\boldsymbol{\nu}} = D^{2\boldsymbol{\nu}} \left(\frac{1}{M_A^\wedge} \right)(\mathbf{0}) \qquad (\boldsymbol{\nu} \in H_r).$$

Then

$$(5.19) \qquad \chi_{A,r}(\mathbf{t}) := c_{\mathbf{0}r} M_A(\mathbf{t}) + 2^{-n} \sum_{\boldsymbol{\mu} \in H_r \setminus \{\mathbf{0}\}} c_{\boldsymbol{\mu}r} \sum_{\boldsymbol{\zeta} \in \{-1,1\}^n} M_A(\mathbf{t} + \boldsymbol{\zeta}\boldsymbol{\mu})$$

is a polynomial spline of degree $m - n$ with compact support, $(r-2)$ times continuously differentiable, again satisfying the estimates (5.9) and (5.10) for $\varphi(\mathbf{t}) := \chi_{A,r}(\mathbf{t})$.

Proof. Concerning the existence and uniqueness of the $c_{\boldsymbol{\mu}r}$, note that (5.18) is also of Vandermond type and see e.g. [11]. Again we need just show that $\chi_{A,r}$ satisfies (5.11). As above,

$$\chi_{A,r}^\wedge(\mathbf{v}) = M_A^\wedge(\mathbf{v}) \left\{ c_{\mathbf{0}r} + 2^{-n} \sum_{\boldsymbol{\mu} \in H_r \setminus \{\mathbf{0}\}} c_{\boldsymbol{\mu}r} \sum_{\boldsymbol{\zeta} \in \{-1,1\}^n} \exp\left((-i\boldsymbol{\zeta}\boldsymbol{\mu}) \cdot \mathbf{v}\right) \right\}$$

$$= M_A^\wedge(\mathbf{v}) \left\{ c_{\mathbf{0}r} + \sum_{\boldsymbol{\mu} \in H_r \setminus \{\mathbf{0}\}} c_{\boldsymbol{\mu}r} \prod_{\lambda=1}^{n} \cos(\mu_\lambda v_\lambda) \right\} := M_A^\wedge(\mathbf{v}) q(\mathbf{v}).$$

As in the proof of Theorem 5.3, $2\pi\mathbf{k}$ is again an r-fold zero of $M_A^\wedge$ for all $\mathbf{k} \neq \mathbf{0}$, so that $D^{\mathbf{j}} \chi_{A,r}^\wedge(2\pi\mathbf{k}) = 0$ for all $0 \leq |\mathbf{j}| < r_0, \mathbf{k} \neq \mathbf{0}$. Further, there holds

$$q(\mathbf{0}) = \sum_{\boldsymbol{\mu} \in H_r} c_{\boldsymbol{\mu}r} = \left(\frac{1}{M_A^\wedge} \right)(\mathbf{0}),$$

and for $\mathbf{j} = 2\boldsymbol{\nu}, \boldsymbol{\nu} \in H_r \setminus \{\mathbf{0}\}$,

$$D^{\mathbf{j}} q(\mathbf{0}) = D^{2\boldsymbol{\nu}} q(\mathbf{0}) = (-1)^{|\boldsymbol{\nu}|} \sum_{\boldsymbol{\mu} \in H_r \setminus \{\mathbf{0}\}} c_{\boldsymbol{\mu}r} \boldsymbol{\mu}^{2\boldsymbol{\nu}} = D^{2\boldsymbol{\nu}} \left(\frac{1}{M_A^\wedge} \right)(\mathbf{0}).$$

If $\mathbf{j} \in \mathbb{N}_0^n$ is not of the form $\mathbf{j} = 2\boldsymbol{\nu}$, i.e., $\mathbf{j}$ has at least one odd entry, then $D^{\mathbf{j}} q(\mathbf{0}) = 0 = D^{\mathbf{j}}(1/M_A^\wedge)(\mathbf{0})$ because $M_A^\wedge$ is symmetric in each variable. Thus $D^{\mathbf{j}} q(\mathbf{0}) = D^{\mathbf{j}}(1/(M_A^\wedge)(\mathbf{0})$

for $\mathbf{j} \in \mathbb{N}_0^n, 0 < |\mathbf{j}| \le r-1$; (5.11) now follows by the same argument as in the proof of Theorem 5.3.

In case $A = E_r$, defined via (5.17), we have at least three methods for constructing linear combinations of translates of $M_A = M_{E_r}$ satisfying (5.9) and (5.10). First we can take the product kernel $\varphi_r(\mathbf{t}) := \Pi_{j=1}^n \psi_r(t_j), \mathbf{t} \in \mathbb{R}^n$, and ψ_r as in Section 3. The second method gives the kernel $\varphi_{A,r}$ of Theorem 5.3, and the third the kernel $\chi_{A,r}$ of Theorem 5.5. In case $r = 2$ and arbitrary $n \in \mathbb{N}$ these different methods yield the same kernel, namely,

$$\varphi_2(\mathbf{t}) = \varphi_{A,2}(\mathbf{t}) = \chi_{A,2}(\mathbf{t}) = (\sqrt{2\pi})^n M_A(\mathbf{t}) = \prod_{j=1}^n M_2(t_j) \quad (\mathbf{t} \in \mathbb{R}^n).$$

For $n = 3, r = 3, 4$ these kernels read

$$\varphi_3(\mathbf{t}) = \prod_{j=1}^3 \left\{ \frac{5}{4} M_3(t_j) - \frac{1}{8}\{M_3(t_j+1) + M_3(t_j-1)\} \right\}$$

$$\begin{aligned}\varphi_{A,3}(\mathbf{t}) = (\sqrt{2\pi})^3 \Big\{ \frac{19}{16} M_A(\mathbf{t}) - \frac{1}{32}\{ &M_A(\mathbf{t}+(2,0,0)) + M_A(\mathbf{t}-(2,0,0)) \\ &+ M_A(\mathbf{t}+(0,2,0)) + M_A(\mathbf{t}-(0,2,0)) + \\ &+ M_A(\mathbf{t}+(0,0,2)) + M_A(\mathbf{t}-(0,0,2))\}\Big\}\end{aligned}$$

$$\begin{aligned}\chi_{A,3}(\mathbf{t}) = (\sqrt{2\pi})^3 \Big\{ \frac{7}{4} M_A(\mathbf{t}) - \frac{1}{8}\{ &M_A(\mathbf{t}+(1,0,0)) + M_A(\mathbf{t}-(1,0,0)) \\ &+ M_A(\mathbf{t}+(0,1,0)) + M_A(\mathbf{t}-(0,1,0)) + \\ &+ M_A(\mathbf{t}+(0,0,1)) + M_A(\mathbf{t}-(0,0,1))\}\Big\}\end{aligned}$$

$$\varphi_4(\mathbf{t}) = \prod_{j=1}^3 \left\{ \frac{4}{3} M_4(t_j) - \frac{1}{6}\{M_4(t_j+1) + M_4(t_j-1)\} \right\}$$

$$\begin{aligned}\varphi_{A,4}(\mathbf{t}) = (\sqrt{2\pi})^3 \Big\{ \frac{5}{4} M_A(\mathbf{t}) - \frac{1}{24}\{ &M_A(\mathbf{t}+(2,0,0)) + M_A(\mathbf{t}-(2,0,0)) \\ &+ M_A(\mathbf{t}+(0,2,0)) + M_A(\mathbf{t}-(0,2,0)) + \\ &+ M_A(\mathbf{t}+(0,0,2)) + M_A(\mathbf{t}-(0,0,2))\}\Big\}\end{aligned}$$

$$\chi_{A,4}(\mathbf{t}) = (\sqrt{2\pi})^3 \left\{ 2M_A(\mathbf{t}) - \frac{1}{6}\{M_A(\mathbf{t}+(1,0,0)) + M_A(\mathbf{t}-(1,0,0)) \right.$$

$$+M_A(\mathbf{t}+(0,1,0)) + M_A(\mathbf{t}-(0,1,0)) +$$

$$\left. +M_A(\mathbf{t}+(0,0,1)) + M_A(\mathbf{t}-(0,0,1))\} \right\}.$$

All of these examples are polynomial splines of degree $n(r-1)$. Observe that whereas the kernels φ_3, φ_4 are of product type, none of the $\varphi_{A,r}, \chi_{A,r}$ are so.

Let us conclude with some examples of linear splines, so splines of minimal degree, in two dimensions. These can be obtained by considering any of the two matrices,

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & -1 \end{pmatrix}.$$

Both satisfy the assumptions of Theorem 5.3. Further, B satisfies the assumptions of Theorem 5.5 as well. One has

$$\varphi_{A,r}(\mathbf{t}) = 2\pi \left\{ \frac{9}{8} M_A(\mathbf{t}) - \frac{1}{24}\{M_A(\mathbf{t}+(1,1)) + M_A(\mathbf{t}-(1,1))\} \right.$$

$$-\frac{1}{96}\{M_A(\mathbf{t}+(2,0)) + M_A(\mathbf{t}-(2,0)) +$$

$$\left. +M_A(\mathbf{t}+(0,2)) + M_A(\mathbf{t}-(0,2))\} \right\}$$

$$\varphi_{B,r}(\mathbf{t}) = 2\pi \left\{ \frac{53}{48} M_B(\mathbf{t}) - \frac{1}{32}\{M_B(\mathbf{t}+(2,0)) + M_B(\mathbf{t}-(2,0))\} \right.$$

$$\left. -\frac{1}{48}\{M_B(\mathbf{t}+(0,2)) + M_B(\mathbf{t}-(0,2))\} \right\}$$

$$\chi_{B,r}(\mathbf{t}) = 2\pi \left\{ \frac{17}{12} M_B(\mathbf{t}) - \frac{1}{8}\{M_B(\mathbf{t}+(1,0)) + M_B(\mathbf{t}-(1,0))\} \right.$$

$$\left. -\frac{1}{12}\{M_B(\mathbf{t}+(0,1)) + M_B(\mathbf{t}-(0,1))\} \right\}.$$

It would of course be possible to give concrete examples of linear splines in arbitrary dimensions by using matrices of this type. One may also employ A of (5.16) with $m = n + 1$.

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