Note di Matematica Vol. X, Suppl. n. 1, 73-102 (1990)

A STRONG HOMOLOGY THEORY SATISFIES A CLUSTERAXIOM

FRIEDRICH W. BAUER

Dedicated to the memory of Professor Gottfried Köthe

0. INTRODUCTION

For *ordinary* strong homology theories on compact metrizable spaces a clusteraxiom is part of the detinition (cf. [7]). In case of *generalized* homology theories on compacta the validity of a clusteraxiom is still of importance for the corresponding extension theorems (cf. [3]).

A strong (generalized) homology thcory h_* on a more general category of topologica1 spaces $\underline{K}$ (cf. definition 1.2) is also defined by means of some kind of continuity property *(the so-called chain continuity or c-continuity,* cf. [2] or definition 1.1. of this paper). Unlike the clusteraxiom (definition 2.1) this c-continuity is depending on a subcategory $\underline{P} \subset \underline{K}$ of «good spaces» (e.g. the category of polyhcdra of all $ANRs$). For the pair $\underline{K}$ = compacta, $\underline{P}$ = compact $ANRs$, a strong homology theory h_* is characterized by the validity of a strong excision axiom, a clusteraxiom and prescribcd $h_*|\underline{P}$ (cf. [2] theorems 4.1).

This is not any more true in gcneral. In the present paper we establish a proof of the assertion that under a mild assumption on the spaces of $\underline{P}$, every strong homology theory satisfies a clusteraxiom (theorem 2.2). The proof is surprisingly involved from the conceptual as well as from the technical point of view. This is the subject of § 4 - § 6.

I do not know of any example going bcyond the category of compacta where a converse holds (i.e. where c-continuity and a clustcraxiom turn out to be equivalent). Even for compact spaces this problem is open.

In § 3 we include an assertion about the extcnsion of a chain functor given on a category $\underline{P}$ over a larger category $\underline{K}$. This is done although this assertion (theorem 3.3) is not needed fora proof of the main theorcm 2.2, because it is a good example for dealing with problems of homology theories through chain functors.

In § 7 we give an example that even ordinary strong homology rel. to a category of compact $ANRs$ fails to be additive nor does it admit compact carriers.

Some facts and conventions about categories of topologica1 spaces, as well as about chain functors are relegated to an appendix § 8. In spite of this, the reader is assumed to be familiar with the more extensive treatment of this matcrial in [1]. We are exclusively dealing with the concept of strong homology defncd in dcfinition 1.2. There is (for ordinai-y homology theories) another one defined in [5], [8]. We will briefly retum-to this in § 1 remark 6) resp. § 2 remark 3).

1. STRONG HOMOLOGY THEORIES:

A *homology theory* $h_* = \{h_n, \partial, n \in \mathbb{Z}\}$ on a category of topologica1 spaces $\underline{K} \subset$ Top is a series of functors $h_n : \underline{K}^2 \to \underline{Ab}$, $n \in Z$, $\underline{K}^2$ the category of pairs associated with

$\underline{K}$, together with a natural transformation $\partial_n : h_n \to h_{n-1} \circ T$, $T(X,A) = (A, \emptyset) = A$, satisfying 1) a ***homotopy axiom, 2) an exactness axiom*** and ***3) an excision axiom.***

By a ***strong excision axiom*** we mean the statement: Suppose $(X, A), (X/A, *) \in \underline{K}^2$, $A \subset X$ closed, then the projection $p : (X, A) \to (X/A, *)$ induces an isomorphism

$$p_* : h_*(X,A) \overset{\approx}{\to} h_*(X/A, *).$$

Let $\underline{P} \subset \underline{K}$ be a prescribed subcategory (referred to as category of «good spaces») then a strong ***homology theory*** h_* ***on*** $\underline{K}$ *rel.* $\underline{P}$ is a homology theory on $\underline{K}$, satisfying a strong excision axiom and, in addition, a new kind of continuity axiom ***(the*** so-called *chain-continuity* or ***c-continuity*** which depends on the choice of the subcategory $\underline{P}$. This concept has been introduced for the first time in [2] and will be explained now: According to [1] theorem 8.1 (or theorem 8.3) there exists to h_* a chain functor $\underline{C}_*$ related to h_*. That means that $H_*(\underline{C}_*)(\ )$, the homology of $\underline{C}_*$, is naturally isomorphic to $h_*(\ )$ (by means of an isomorphism commuting with boundaries).

A chain functor $\underline{C}_*$ is c-continuous (rel. $\underline{P}$) whenever the following holds: Suppose $(X,A) \in \underline{K}^2$ and $g \in \underline{K}^2((X,A),(P,Q)) \mapsto c_g \in C_n(P,Q), (P,Q) \in \underline{P}^2$, is an assignment satisfying

$$r_\# c_{g_1} = c_{g_2}$$

whenever $g_i \in \underline{K}^2((X,A),(P_i,Q_i))$, $r : g_1 \to g_2$ in $\underline{P}^2$ (i.e. $r \in \underline{P}^2((P_1,Q_1), (P_2,Q_2))$, $rg_1 = g_2$).

Then we require:

C1) There exists a unique $c \in C_n(X, A)$ satisfying

$$g_\#(c) = c_g, \qquad g \in \underline{K}^2((X,A),(P,Q)).$$

C2) We have $c \in C_n'(X, A)$ whenever all $c_g \in C_n'(P, Q)$.

Definition 1.1. ***The homology theory*** h_* ***is called c-continuous (rel.*** $\underline{P}$***) whenever the following holds:***

Suppose $\underline{C}_*$ ***is any chainfunctor related to*** $h_* | \underline{P}$ ***(i.e.*** $\underline{C}_*$ ***is only defined on*** $\underline{P}$***) then there exists 1) a c-continuous chain functor*** ${}^s\underline{C}_*$ ***related to*** h_* ***(now on the larger category*** $\underline{K}$***) and 2) a weak equivalence*** $\nu : \underline{C}_* \subset {}^s\underline{C}_* | \underline{P}$ ***of chain functors (rel.*** $\underline{P}$***) (cf. defmition 8.2).***

We summarize:

Definition 1.2. ***A strong homology theory*** h_* ***rel.*** $\underline{P}$ ***is a homology theory on*** $\underline{K}$***, satisfying a strong excision axiom which is c-continuous rel.*** $\underline{P}$***.***

Remarks. 1) As we pointed out in the remark following definition 8.2 this concept of a strong homology theory and in particular of a weak equivalence tums out to be sufficient for the purpose of theorem 2.2. An *existence theorem* for a strong homology theory allows a stronger version of a weak equivalence. Moreover one can include the following assertion: let λ : ${}^1\underline{C}_* \rightarrow {}^1\underline{C}_*$ be a composition of weakly strict transformations

$$* \xrightarrow{\sigma} \cdot \quad \overset{\nu}{\underset{\zeta}{\rightleftarrows}} \quad *,$$

all $*$ being c- continuous chain functors, σ strict and

$$' \quad \overset{\nu}{\underset{\zeta}{\rightarrow}} \quad *$$

a full weak subfunctor. *Suppose that* h_* *is c-continuous and* $h_* \approx H_*({}^i\underline{C}_*)$, $i = 1, 2$, $\lambda_*|\underline{P}$ *an isomorphism, then* λ_* *itself is an isomorphism.*

This specifies the statement «$h_* \approx H_*({}^i\underline{C}_*)$» by telling more precisely what kind of chain transformations are expected to induce this isomorphism. It applies in particular to the case that $\underline{C}_*$ is a c-continuous chain functor on $\underline{K}$ such that $h_* \approx H_*(\underline{C}_*)$, $\nu : \underline{C}_* |\underline{P} \subset {}^s\underline{C}_*$ is a weak equivalence and $\overline{\nu} : H_*(\underline{C}_*) \rightarrow H_*{}^s(\underline{C}_*)$ an extension of $\overline{\nu}$ (over $\underline{K}$). However this property of a strong homology theory is not needed for theorem 2.2.

Conceming strong excision an *existence proof* of a strong homology theory ${}^sh_*$ rel. $\underline{P}$ (with prescribed $\underline{P}$ c $\underline{K}$, ${}^sh_*$ on $\underline{P}$) requires some additional assumptions on the relationship between $\underline{P}$ and $\underline{K}$ or some additional restrictions on the embedding *A c X in the* formulation of the strong excision axiom. In the present paper we are not dealing specifically neither with strong excision nor with the existence problem of a strong homology theory.

2) As pointed out in [2], ordinary singular homology (on $\underline{K}$ = Top with $\underline{P}$ = category of compact polyhedra) can be determined by a c-continuous chain functor (the flat chain functor with $C_*($ X, *A)* being the singular chains). However singular homology is nevertheless *not* c-continuous .

3) Every homology theory h_* defined on $\underline{K}$ is c-continuous rel. $\underline{K}$.

4) Let $\underline{P}$ be the category consisting of a single point $*$, $\underline{K}$ c *Top* a full subcategory, then every c-continuous homology theory (rel. $\underline{P}$) is trivial (i.e. one has always $h_*($ *X, A) = 0*).

5) Suppose that to each (X, *A)* $\in \underline{K}^2$ there exists a (P, Q) $\in \underline{P}^2$ such that (X, *A) c* (P, Q) (i.e. each pair can be embedded into a good pair) then it suffices in (1) to require the mere *existence* of a c $\in C_n($ X, *A)* satisfying (2). The *uniqueness* follows because by definition of a chain functor $\underline{C}_*$ the inclusion i : (X, *A)* c (P, Q) induces always a monomorphism $i_\# : C_n($ X, *A)* $\rightarrow C_n($ *P, Q)* .

6) Ju. Lisica and S. Mardesic (cf. [5]) resp. Z. R. Miminoshwili (cf. [8]) have developed another concept of an ordinary strong homology theory. For compact metric spaces (and coefficients in an abelian group G) their homology theory coincides with Steenrod-Sitnikov

homology theory, hence (in view of [2] theorem 4.1) with our concept of a strong homology theory.

1 do not know under what more general conditions on the categories $\underline{P}$ and $\underline{K}$ an isomorphism between the Lisica-Mardesic-Miminoshwili strong homology theory and the (ordinary) strong homology theory in the sense of definition 1.2 can be expected, cf. § 2 remark 2). It should be noticed that both concepts are defined by means of chain complexes, however a chain functor for a (non-ordinary) generalized homology theory is a rather involved instrument in comparison to the chain complexes which determine ordinary homology theories.

2. THE CLUSTERAXIOM

A *continuous* homology theory in general cannot exist because continuity and exactness are not compatible (cf. [4]). However it is well-known that there are weaker forms of continuity which do not collide with exactness. The most popular example is fumished by the clusteraxiom for metric compacta (cf. [7]): let (X_i, x_{i0}), $i = 1, 2, \ldots$ be a countable family of based spaces, then the *cluster* (or strong *wedge*) of these spaces is the wedge equipped with the strong topology:

$$\mathop{\mathrm{Cl}}_{i=1}^{\infty}(X_i, x_{i0}) = \lim_{\underset{n}{\leftarrow}} X_1 \vee \ldots \vee X_n .$$

Altematively one can define this space by requiring that a neighbourhood of the basepoint contains almost all spaces X_i.

We have a natura1 transformation for any homology theory

$$\lambda : h_* \left(\mathop{\mathrm{Cl}}_{i=1}^{\infty} (X_i, x_{i0})\ _* \right) \to \prod_{i=1}^{\infty} h_* (X_i, x_{i0})$$

induced by the projections $p_k : Cl_{i=1}^{\infty} X_i \to X_k$.

Milnor's clusteraxiom requires that λ is an isomorphism (cf. [7]).

In general we can define the cluster $\mathop{\mathrm{Cl}}_{\alpha \in \underline{A}}(X_\alpha, x_{\alpha 0}) = (X, *)$ of any family of based spaces $(X_\alpha, x_{\alpha 0})$ in the same way as for countably many factors:

Let $(X', *) = \bigvee_{\alpha \in A} X_\alpha$ be the wedge, retopologized by requiring that a subset $U \subset X'$ is open whenever 1) $U \cap X_\alpha$ is open in X_α for all α and 2) that $* \in U$ implies that almost all X_α are containcd in U.

Altematively we can again define $\mathop{\mathrm{Cl}}_{\alpha \in \underline{A}}(X_\alpha, x_{\alpha 0})$ as $\lim_{\leftarrow (\alpha_1, \ldots, \alpha_n)} X_{\alpha_1} \vee \ldots \vee X_{\alpha_n}$ forany finite subset $\{\alpha_1, \ldots, \alpha_n\} \subset \underline{A}$, with obvious projections as bonding maps.

The fact that both definitions agrce is a simple exercise (cf. [9] proposition 44 conceming different dcfinitions of a cluster). Suppose that $\{(X_\alpha, x_{\alpha 0}) \mid \alpha \in \underline{A}\}$ is such an indexed family of based spaces in $\underline{K}_0$ (the category of based spaces in $\underline{K}$) such that $\mathop{\mathrm{Cl}}_{\alpha\in\underline{A}}(X_\alpha, x_{\alpha 0})$ is again a based space in $\underline{K}$ (hence an object of $\underline{K}_0$).

Let h_* be any homology theory on $\underline{K}$, then we have again a natural transformation

(1) $$\lambda : h_*\left(\mathop{\mathrm{Cl}}_{\alpha\in\underline{A}}(X_\alpha, *_{\alpha 0}), *\right) \to \prod_{\alpha\in\underline{A}} h_*(X_\alpha, x_{\alpha 0})$$

induced by the projections $p_\beta : \mathop{\mathrm{Cl}}_{\alpha\in\underline{A}} X_\alpha \to X_\beta$.

Definition 2.1. ***A homology theory h_* on $\underline{K}$ satisfies a clusteraxiom wheneverfor any indexedfamily $\{(X_\alpha, x_{\alpha 0}) \mid \alpha\in\underline{A}\}$ of based spaces such that $\mathop{\mathrm{Cl}}_{\alpha\in\underline{A}}(X_\alpha, x_{\alpha 0})\in\underline{K}_0$ the transformation λ is an isomorphism.***

Remarks. 1) Suppose that $\underline{K} = \underline{Com}$, the category of compacta, then the condition on $\{(X_\alpha, x_{\alpha 0}) \mid \alpha\in\underline{A}\}$ that $\mathop{\mathrm{Cl}}_{\alpha\in\underline{A}} X_\alpha \in \underline{K}_0$ is fulfilled if and only if $\underline{A}$ is countable.

2) Suppose $\underline{K} = \underline{Co}$, the category of compact spaces, then every family $\{(X_\alpha, x_{\alpha 0}) \mid \alpha \in \underline{A}\}$ satisfies the condition $\mathop{\mathrm{Cl}}_{\alpha\in\underline{A}} X_\alpha \in \underline{K}_0$, for any indexing set $\underline{A}$.

3) It is well-known that the clusteraxiom is a weaker form of continuity.

4) Let $\underline{K}$ be the category $\underline{Com}$, $\underline{P}$ the category of compact $ANRs$, h_* a given homology theory on $\underline{P}$. Then h_* satisfies a strong excision axiom (because every inclusion in $\underline{P}$ is a cofibration). According to a result in [3] there exists a, up to an isomorphism, unique extension $\bar{h}_*$ of h_* over $\underline{Com}$, which satisfies a clusteraxiom. This homology theory is a (generalized) *Steenrod-Sitnikov* ***homology theory.***

The following sections 4-6 are devoted to a proof of:

Theorem 2.2. ***Suppose $\underline{P} \subset \underline{K}$ is a full subcategory of locally contractible spaces, h_* a strong homology theory rel. $\underline{P}$ on $\underline{K}$, then h_* satisfies a clusteraxiom.***

In view of the preceding remarks 1), 2) and because $ANRs$ are locally contractible we have:

Corollary 2.3. ***Suppose $\underline{K} = \underline{Com}$, $\underline{P}$ = category of compact $ANRs$, then every strong homology theory h_* on $\underline{K}$ rel. $\underline{P}$ satisfies a clusteraxiom (for countably many summands).***

Corollary 2.4. ***Let $\underline{K} = \underline{Co}$, $\underline{P}$ as in corollary 2.3, then any strong homology theory h_* on $\underline{Co}$ rel. $\underline{P}$ satisfies a clusteraxiom (without any restriction on the number of summands in the cluster).***

Remarks. 1) For $\underline{K} = \underline{Com}$, $\underline{P}$, as before, we have proved in [2] Theorem 4.1 that a converse holds: every homology theory h_* on $\underline{K}$, satisfying a strong excision axiom and a clusteraxiom is a strong homology theory. Hence strong homology theories and Steenrod-Sitnikov homology theories coincide. The proof uses a non-trivial result from [3] and the existence of strong homology theories (for this particular case). The proof given in [2] for the fact that for compacta every strong homology theory satisfies a clusteraxiom (i.e. the proof of corollary 2.3) uses details of the construction of a strong homology theory. The present proof of theorem 2.2 is independent of any explicit constructional devices fora strong homology theory.

2) Unlike c-continuity the clusteraxiom does not depend upon the subcategory $\underline{P}$, nor does it refers to any chain functor related to that homology theory.

3) In [9] T. Watanabe verifies the clusteraxiom for a strong homology theory in the sense of Lisica, Mardesic [5] and Miminoshwili [8] on the category of compact spaces (even for strongly paracompact spaces, cf. [9] p. 194 conceming the definition). The author deals also with strong excision for this kind of homology. His results might be regarded as an indication that for ordinary homology theories on compact spaces both concepts of strong homology theories agree or are at least very closely related.

3. SINGULARISATION OF A CHAIN FUNCTOR

Suppose $\underline{P} \subset \underline{K}$ is a full subcategory of a category of topological spaces (for example the category of $ANRs$ in the category of all topological spaces) and $\underline{C}_*$ a chain functor on $\underline{P}$, then we would like to find a chain functor ${}^1\underline{C}_*$ on $\underline{K}$ such that ${}^1\underline{C}_*|\underline{P} = \underline{C}_*$. The solution of this problem is not needed for the proof of theorem 2.2 but, since it deserves some independent interest, included in this paper. The process used to construct ${}^1\underline{C}_*$ is similar to that of establishing singular homology. The additional difficulties appearing are due to the fact that the homology theory associated with $\underline{C}_*$ is not necessarily an ordinary one. In order to proceed we are obliged to impose some restriction on the relationship between $\underline{C}_*$ and $\underline{P}$:

(*) Let $(X_i, A_i) \in \underline{P}^2$ be pairs of good spaces, $(X, A) \in \underline{K}$, $c_i \in C_n(X_i, A_i)$, $f_i \in \underline{K}^2((X_i, A_i), (X, A))$ be given, $k : (X, A) \subset (Y, B)$ an inclusion. Assume that

$$(g\ k\ f_1)_\# c_1 = (g\ k\ f_2)_\# c_2$$

for any $g \in \underline{K}^2((Y,B),(P,Q))$, $(P,Q) \in \underline{P}^2$, then we have

$$(g' f_1)_\# c_1 = (g' f_2)_\# c_2$$

for any $K^2((X,A),(P,Q))$.

This condition is for example fulfilled for any chain functor $\underline{C}_*$, whenever we can find to any diagram

$$\begin{array}{ccc} (X,A) & \overset{k}{\subset} & (Y,B) \\ g' \downarrow & & \\ (P',Q') & & \end{array} \qquad (P',Q') \in \underline{P}^2$$

a commutative diagram

$$\begin{array}{ccccc} & & (X,A) & \overset{k}{\subset} & (Y,B) \\ & \swarrow g' & \downarrow g'' & & \downarrow g \\ (P',Q') & \leftarrow & (P'',Q'') & \overset{k'}{\subset} & (P,Q). \end{array} \qquad (P'',Q''),(P,Q) \in \underline{P}^2$$

If $(g'f_1)_\# c_1 \neq (g'f_2)_\# c_2$, then we have $(g''f_1)_\# c_1 \neq (g''f_2)_\# c_2$, hence, because k' is an inclusion, $(k'g''f_1)_\# c_1 \neq (k'g''f_2) c_2$, so that

$$(g\,k\,f_1)_\# c_1 \neq (g\,k\,f_2)_\# c_2$$

follows.

The determination of ${}^1\underline{C}_*$ starts with the definition of two categories:

1) $\underline{P}_{(X,A)}$: The objects are mappings g : (X, *A*) $\to$ **(P, Q)** $\in \underline{P}^2$, the morphisms commutativetriangles $r : g_1 \to g_2$, $g_i : (X,A) \to (P_i,Q_i)$, $r:(P_1,Q_1) \to (P_2,Q_2)$, $rg_1 = g_2$.

2) $\underline{C}_n$: The objects are chains c $\in C_n(P,Q)$, $(P,Q) \in \underline{P}^2$, the morphisms are mappings r as in 1) satisfying $r_\# c_1 = c_2$, $c_i \in C_n(P_i,Q_i)$. We define $F_n(X,A)$ to be the family of all functors

$$\phi : \underline{P}_{(X,A)} \to C_n$$

such that $\phi(g : (X,A) \to (P,Q)) \in C_n(P,Q)$. By setting

$$(\phi_1 + \phi_2)(g) = \phi_1(g) + \phi_2(g),$$

$F_n(X,A)$ is endowed with an abelian group structure. Let $f \in \underline{K}^2((X,A),(Y,B))$ be a functor, then the assignment

$$\begin{array}{ccc} F_n(X,A) & \to & F_n(Y,B) \\ \phi(\cdot) & \mapsto & f_\# \phi(\cdot) = \phi(\cdot f) \end{array}$$

tums

$$F_n : \underline{K}^2 \to \underline{Ab} \quad (= \text{category of abelian groups})$$

into a functor.

By setting

$$\mathrm{d} : F_n(X, A) \to F_{n-1}(X, A)$$
$$\phi \mapsto (g \mapsto \mathrm{d}\phi(g))$$

$F_*(X, A) = \{F_n(X, A), \mathrm{d},\}$ carries the structure of a chain complex and $F_* : \underline{K}^2 \to \underline{ch}$ becomes a functor into the category of chain complexes.

Suppose (X, $A) \in \underline{P}^2$, then

$$\nu : C_n(X, A) \to F_n(X, A)$$
$$c \mapsto \phi_c = (g \mapsto g_\# c)$$

is a natural transformation $\nu : C_* \to F_* | \underline{P}^2$. Suppose $c_1 \neq c_2 \in C_n(X, A)$, $(X, A) \in \underline{P}^2$, then $\nu(c_1)(1_{(X,A)}) = c_1 \neq c_2 = \nu(c_2)(1_{(X,A)})$ implying that ν is monic.

Weset $F_n'(X, A) = \{\phi \in F_n(X, A) | \phi(g) \in C_n', g \in \underline{P}_{(X,A)}\}$. Let ${}^1C_n(X, A)$ be the subgroup of $F_n(X, A)$ generated by all those $\phi \in F_n(X, A)$ which are of the form $f_\# \phi_c$, for some $c \in C_n(X', A')$, $(X', A') \in \underline{P}^2$, $f \in \underline{K}^2((X', A'), (X, A))$.

Correspondingly we defme ${}^1C_n'(X, A)$ c ${}^1C_n(X, A)$ by means of $F_n'(X, A)$ and get a natural inclusion $\ell : {}^1C_*' \subset {}^1C_*$.

Let $f \in \underline{K}^2((X', A'), (X, A))$ be a mapping then we have the associated $f' \in \underline{K}^2(X', X)$ and set

(1)
$$\varphi_\# : {}^1C_n'(X, A) \to {}^1C_n(X)$$
$$f_\# \phi_c \mapsto f'_\# \phi_{\varphi_\#(c)} = (g \mapsto (gf')_\# \varphi_\#(c))$$

Let $f \in \underline{K}^2(X'$, X) be a mapping , X' $\in \underline{P}$, (X, $A) \in \underline{K}^2$ a pair, then we set $\tilde{f} \in \underline{K}^2((X', f^{-1}(A)), (X, A))$ whenever $(X', f^{-1}(A)) \in \underline{P}^2$ resp. $\tilde{f} \in \underline{K}^2((X', \emptyset), (X, A))$ otherwise as associated maps. We defme

(2)
$$\kappa_\# : {}^1C_n(X) \to {}^1C_n'(X, A)$$
$$f_\# \phi_c \mapsto (g \mapsto (g\tilde{f})_\# \kappa_\#(c)),$$
$$\|$$
$$\tilde{f}_\# \phi_{\kappa_\#(c)}$$

Let $f \in \underline{K}(A', A)$, $A' \in \underline{P}$ be a mapping $(X, A) \in \underline{K}^2$, then we have the associated $\hat{f} \in \underline{K}^2((A', A'), (X, A))$ and define

$$(3) \quad \begin{aligned} i' : {}^1C_n(A) &\to {}^1C'_n(X, A) \\ f_\# \phi_c &\mapsto (g \mapsto (g\hat{f})_\# i'(c)). \\ &\quad \| \\ &\quad \hat{f}_\# \phi_{i'(c)} \end{aligned}$$

We observe that:

Lemma 3.1. *1) (1) - (3) define chain mappings; moreover i', ℓ are natural transformations.*
2) We have

$$\nu(\ell(c)) = \ell\nu(c)$$

$$\nu(\varphi_\#(c)) = \varphi_\# \nu(c)$$

$$\nu(\kappa_\#(c)) = \kappa_\# \nu(c)$$

$$\nu(i'(c)) = i'\nu(c)$$

whenever both sides are dejined.

We have

$$\varphi_\# \kappa_\# f_\# \phi_c = \varphi_\# f_\# \phi_{\kappa_\#(c)} = (\tilde{f})'_\# \phi_{\varphi_\# \kappa_\#(c)}$$
$$= f_\# \phi_{\varphi_\# \kappa_\#(c)},$$

because $(\tilde{f})' = f$. Since there exists a chain homotopy

$$\mathrm{d} D(c) + D(\mathrm{d} c) = \varphi_\# \kappa_\#(c) - c$$

we are enabled to define

$$(f_\# \Delta)(g) = (gf)_\#(D(c)),$$

$f_\# \Delta \in {}^1C_{n+1}(X)$, satisfying

$$\mathrm{d}\, f_\# \Delta + f_\# \Delta \mathrm{d} = \varphi_\# \kappa_\# f_\# \phi_c - f_\# \phi_c.$$

Similarly we have:

$$j_\# \varphi_\# f_\# \phi_c = (jf')_\# \phi_{\varphi_\#(c)}$$
$$= (fj')_\# \phi_{\varphi_\#(c)}$$
$$= f_\# \phi_{j'_\# \varphi_\#(c)},$$

where $j : X \to (X,A)$, $j' : X' \to (X', A')$ are the inclusions. Since $j'_\# \varphi_\# \simeq \ell$, we have again

$$j_\# \varphi_\# f_\# \phi_c \simeq \ell f_\# \phi_c.$$

Let $f \in \underline{K}(A', A)$, $A' \in \underline{P}$, $i : A \to X$ be given, then we have $\widetilde{if} = \hat{f}$, hence

$$\kappa_\# i_\# f_\# \phi_c = \hat{f}_\# \phi_{\kappa_\#(c)} = \hat{f}_\# \phi_{i'(c)} = i' f_\# \phi_c$$

because $(\kappa_\# : C_n(A') \to C_n(A', A')) = \kappa_\# \bar{i} = i'$, $\bar{i} = 1_{A'} : A' \to A'$.

Lemma 3.2. *$^1\underline{C}_*$ as defined above is a chain functor.*

Proof. We have defined $^1C_*, {}^1C'_*, \ell, \varphi_\#, \kappa_\# i'$, and verified that $\varphi_\# \kappa_\# \simeq 1$, $j_\# \varphi_\# \simeq \ell$, $\kappa_\# \imath_\# = i'$.

All other necessary details (like for example the existence of chain homotopies D_H for a homotopy $H : (X, A) \times I \to (Y, B)$ are established by the same methods. The verification of the relations between these mappings, as required in the definition of a chain functor (cf. § 8) can be immediately accomplished. The fact that an inclusion $k : (X, A) \subset (X, B)$ induces a monomorphism follows from (*).

We summarize:

Theorem 3.3. *Let $\underline{P} \subset \underline{K}$ be a pair of categories of topological spaces $\underline{C}_*$ a chain functor on $\underline{P}$ satisfying (*). Then there exists a chain functor $^1\underline{C}_*$ on $\underline{K}$ such that $^1\underline{C}_*|\underline{P} \approx \underline{C}_*$.*

Proof. We have an inclusion $\nu : \underline{C}_* \hookrightarrow {}^1\underline{C}_*|\underline{P}^2$. Let $f_\# \phi_c \in {}^1\underline{C}_*(X, A)$ be a chain, $(X,A) \in \underline{P}^2$, $f \in \underline{P}^2((X', A'), (X, A))$, then weconclude

$$f_\# \phi_c = \phi_{f_\#(c)} = \nu(f_\#(c)).$$

Hence ν is surjective. Since ν^{-1} is also a transformation of chain functors, ν is an isomorphism, as asserted.

Remark. The classical process of singularization of an ordinary homology theory is also performed on the chain level. It fails for generalized homology theories defined on an arbitrary category $\underline{P}$ to provide us with a homology theory on a larger category $\underline{K}$ because of the absence of a singular complex functor $|S| : \underline{K} \to \underline{P}$. Moreover there has to be much more structure of the chain functor $\underline{C}_*$ to be the transported to the larger category $\underline{K}$.

We have a derived prehomology theory $H_*({}^1\underline{C}_1)(\ )$ on $\underline{K}$ (we do not investigate the validity of any excision axiom, therefore in accordance with the terminology in [1] the name *prehomology*).

Corollary 3.4. *Let $\underline{E}_*$ be another chain functor on $\underline{P}$ and*

$$\mu : H_*(\underline{C}_*)(\ ) \approx H_*(\underline{E}_*)(\ )$$

an isomorphism **of** *prehomology theories (i.e. a natura1 isomorphism offunctors commuting with the boundary operator). Then there exists an extension*

$$\overline{\mu} : H_*({}^1\underline{C}_*)(\ ) \approx H_*({}^1\underline{E}_*)(\ )$$

being defined on $\underline{K}$.

Proof. Let $f_\# \phi_z \in Z_n({}^1C_*(X,A))$, $z \in Z_n(C_*(X',A'))$, $f \in \underline{K}^2((X',A'),(X,A))$, $(X', A') \in \underline{P}^2$ be given, then we take a $\widetilde{z} \in Z_n(E_*(X', A'))$, $\widetilde{z} \in \mu(\{z\})$ and define

$$\overline{\mu}\{f_\# \phi_z\} = \{f_\# \phi_{\widetilde{z}}\}.$$

This is obviously independent of the choice of the representative z, $\widetilde{z}$ in their homology classes and gives a natural isomorphism of homology groups. Suppose $z = \ell(z') + q_\#(a)$, $z' \in C'_n(X', A')$, $dz' \in im(i' : C_{n-1}(A') \to C'_{n-1}(X', A'))$, (cf. § 8 conceming the notation), then we have by detinition

$$\partial\{z\} = \{i'^{-1} d z'\} \in H_{n-1}(\underline{C}_*)(A').$$

So we take

$$\widetilde{z} = \ell(\widetilde{z}') + q_\#(\widetilde{a}),\ \widetilde{z}' \in E'_n(X',A'),\ d\widetilde{z}' \in im(i' : E_{n-1}(A') \to E'_{n-1}(X',A')),$$

and deduce (because μ is commuting with boundaries)

$$\{i'^{-1} d \widetilde{z}'\} \in \mu(\{i'^{-1} d z'\}) = \partial\mu(\{z\}).$$

Hence we have

$$\partial\{f_\# \phi_z\} = \{\check{f}_\# \phi_{i'^{-1} d z'}\},$$

$\check{f} i A' \to A$ associated with f, and therefore $\overline{\mu}\partial\{f_\# \phi_z\} = \overline{\mu}\{\check{f}_\# \phi_{i'^{-1} d z'}\} = \{\check{f}_\# \phi_{i'^{-1} d \widetilde{z}'}\}$, $\partial\overline{\mu}\{f_\# \phi_z\} = \partial\{f_\# \phi_{\widetilde{z}}\} = \{\check{f}_\# \phi_{i'^{-1} d \widetilde{z}'}\}$ ensuring that ji und ∂ commute.

4. OUTLINE OF THE PROOF OF THEOREM 2.2

Suppose h_* is a strong homology theory rel. $\underline{P}$ and $\underline{C}_*$ any chain functor related to h_* on $\underline{P}$. We find by defmition a c-continuous chain functor ${}^s\underline{C}_*$ related to h_* (on $\underline{K}$) as well as a weak equivalence $\nu : \underline{C}_* \subset {}^s\underline{C}_*$. As a result we are allowed, by surpressing ν in our notation, to talk about chains $c \in C_*$ on pairs $(X,A) \in \underline{K}^2$ (for example by taking $f_\# \nu(c') \in {}^sC_n(X,A)$, $c' \in C_n(X',A')$, $(X',A') \in \underline{P}^2$, $f \in \underline{K}^2((X',A'),(X,A))$. Now assume that we enlarge for some reason the chain functor $\underline{C}_*$ to a chain functor ${}^1\underline{C}_*$ (both being defined von $\underline{P}$) such that the inclusion $\underline{C}_* \subset {}^1\underline{C}_*$ induces an isomorphism of homology theories on $\underline{P}$. Again we can consider ${}^1\underline{C}_*$ as being defined on $\underline{K}$. It may happen that cycles ${}^1z$ in $Z_n({}^1C_*(X,A))$ determine homology classes $\{{}^1z\}$ in $H_n({}^1\underline{C}_*)(X,A)$ which are not in the image of $H_n(\underline{C}_*)(X,A))$. Similarly a cycle $z \in Z_n(C_*(X,A))$ may bound in ${}^1C_*(X,A)$ but not in $C_*(X,A)$. This observation is basic for the detection of an inverse to

$$\lambda : h_*\left(\underset{\alpha\in\underline{A}}{\mathrm{Cl}}(X_\alpha, x_{\alpha 0}), *\right) \to \prod_{\alpha\in\underline{A}} h_*(X_\alpha, x_{\alpha 0})$$

Let $\{\zeta_\alpha\} \in \prod_{\alpha\in\underline{A}} h_n(X_\alpha, x_{\alpha 0})$ be an element, $z_\alpha \in \zeta_\alpha$, $z_\alpha \in Z_n(C_*(X_\alpha, x_{\alpha 0}))$ a family of cycles. If there does not exist a «sum» $z = \sum_{\alpha\in A} z_\alpha \in Z_n(\underset{\alpha\in\underline{A}}{\mathrm{Cl}}(X_\alpha, x_{\alpha 0}), *)$ satisfying $p_{\alpha\#} z = z_\alpha$, $p_\alpha : \underset{\alpha\in\underline{A}}{\mathrm{Cl}} X_\alpha \to X_\alpha$ the projection), then we invent such a cycle, enlarging the originally given chain functor by these new «sums». We must take care that on $\underline{P}^2$ the homology is not changed by the introduction of these new cycles.

This is accomplished by observing that for a good space $(Y, *)$ one has a contractible neighborhood of the base point U. So let $f : \underset{\alpha\in\underline{A}}{\mathrm{Cl}} X_\alpha \to Y$ be a based mapping, then all but a finite number of the cycles $\{f_\# z_\alpha\}$ are lying inside U, hence they bound in $(Y, *)$ not contributing to the homology class of $f_\# z$. Therefore $f_\# z$ is homologous to $\sum_{i=1}^k f_\# z_{\alpha_i}$ for a finite sum, which is already present in $Z_n(C_*(Y,*))$.

This procedure is to remedy the fact that λ for $H_*(\underline{C}_*)$ (instead of h_*) is eventually not epic.

Suppose that $z \in Z_n(C_*(\underset{\alpha\in\underline{A}}{\mathrm{Cl}}(X,), *)$ lies in the kemel of λ, hence we have $X(z) = \{z_\alpha\}$, $z_\alpha = d x_\alpha$, $x_\alpha \in C_{n-1}(X,, x_{\alpha 0})$ (now with λ on the chain level). Then we proceed with $\{x_\alpha\}$ in the same way as before with $\{z_\alpha\}$, inventing (if necessary) a «sum» $x = \sum_{\alpha\in\underline{A}} x_\alpha \in {}^1C_{n+1}(\underset{\alpha\in\underline{A}}{\mathrm{Cl}} X_\alpha, *)$, satisfying $p_{\alpha\#} x = x_\alpha$, $d x^\nu = z$. While the new cycles $z = \sum_{\alpha\in\underline{A}} z_\alpha$ are counterimages of $\{z_\alpha\} \in \prod_{\alpha\in\underline{A}} Z_n(C_*(X_\alpha, x_{\alpha 0}))$, ensuring that

$\lambda\{z\} = \{\{z_\alpha\}\} \in \prod_{\alpha\in\underline{A}} h_n(X_\alpha, x_{\alpha 0})$, the new chains $x = \sum_{\alpha\in\underline{A}} x_\alpha \in {}^1C_{n+1}(\underset{\alpha\in A}{\mathrm{Cl}} X_\alpha, *)$ have the property d $x = z$, so that λ (for H_* c ${}^1\underline{C}_*$) is now monic.

As a result the basic issues in the proof of theorem 2.2 are 1) inventing sums $\sum_{\alpha\in A} x_\alpha$ in such a way that the resulting ${}^1\underline{C}_*$ carries the structure of a chain functor (one must for example determine induced chains $f_\#(\sum_{\alpha\in\underline{A}} x_\alpha)$ for any $f : (\underset{\alpha\in\underline{A}}{\mathrm{Cl}} X_\alpha, *) \to (Y, B) \in \underline{P}^2$ in a canonical way) and 2) to make sure that by these new chains the homology of $\underline{C}_*$ (on $\underline{P}$) is not altered.

In this process we have to deal with the following technical problem: it may happen that for given $\{z_\alpha\}$ there exist already different cycles $z', z'' \in Z_n(C_*(\underset{\alpha\in\underline{A}}{\mathrm{Cl}}$ X,, $*)$ satisfying $p_{\alpha\#} z' = p_\alpha z'' = z_\alpha$. If $z_\alpha = 0$ unless $\alpha_1, \ldots, \alpha_m$, then we have always the finite sum $z = \sum_{i=1}^m z_{\alpha_i}$ (omitting inclusions from our notation) and every other z' with $p_{\alpha_i} z' = z_{\alpha_i}$ must be (not necessarily equal but) homologous to z .

In particular there might exist a z $\neq$ 0 in $Z_n(C_*(\underset{\alpha\in\underline{A}}{\mathrm{Cl}} X_\alpha, *)$ such that $p_{\alpha\#} z = 0$ for all $\alpha \in \underline{A}$.

This problem is treated in § 6.

5. PROOF OF THEOREM 2.2 (FIRST PART)

We resume the notation of § 4 and consider the functor C_* originally defined on $\underline{P}$ as a functor on $\underline{K}$.

Suppose $\underset{\alpha\in\underline{A}}{\mathrm{Cl}} X_\alpha \in \underline{K}_0$ is a cluster of based spaces {X,) in $\underline{K}_0$ and let (Y, B) $\in \underline{P}^2$ be any pair.

A X-set $\{x_\alpha, f_\alpha\} = \{x_\alpha\}$ is 1) a family of chains $x_\alpha \in C_n(Y$, B) , 2) a family f_α : (X,, $x_{\alpha 0}$) -t (Y, B) such that $f = \underset{\alpha\in\underline{A}}{\mathrm{Cl}} f_\alpha : (\underset{\alpha\in\underline{A}}{\mathrm{Cl}} X_\alpha, *) \to$ (Y, B) is defined and 3) a family $\{\hat{x}_\alpha\}, \hat{x}_\alpha \in C_n(X_\alpha, x_{\alpha 0})$ satisfying $f_{\alpha\#} \hat{x}_\alpha = x_\alpha$.

A X-set is called ***inessential,*** whenever there exists a $\hat{x} \in C_n(\underset{\alpha\in A}{\mathrm{Cl}} X_\alpha, *)$ satisfying $p_{\alpha\#} \hat{x}$ = 5,. In this case the X-set $\{x_\alpha\}$ and x = $f_\# \hat{x}$ *are associated.* Notice that although $\hat{x}$ determins {x,} , the converse must not be true: there might exist different $x \in C_n($ Y, B) which are associated with the same X-set. In particular a X-set $\{x_\alpha\}$ with $\hat{x}_\alpha$ = 0 for almost all $\alpha \in \underline{A}$ isa inessential and associated with $\sum_{i=1}^k x_\alpha = x$, where $x_\alpha = 0$ for $\alpha \neq \alpha_1, \ldots, \alpha_k$.

A X-set which is not inessential is called *essential.* If {x,} is a X-set, then so is {dx,} . If $\{x_\alpha\}$ *and* $\{\mathrm{d}\, x_\alpha\}$ are essential, we call $\{x_\alpha\}$ *fully essential.*

In what follows we have to distinguish two cases:

In the course of this section we

1) ***Assume that*** $x^1, x^2 \in C_n(\underset{\alpha\in A}{\mathrm{Cl}} X_\alpha, *)$, $p_{\alpha\#} x^1 = p_{\alpha\#} x^2$ *for all* $\alpha \in \underline{A}$ *implies* $x^1 = x^2$.

In particular two chains associatcd with the same X-set are equal.

Let $\tilde{\Gamma}_n$(Y, B) be the set of all essential X-sets, thcn we dcfine

$$d\{x_\alpha\} = \begin{cases} z \ldots & \text{z associatcd with } \{d\, x_\alpha\} \\ \{d\, x_\alpha\} \ldots \{x_\alpha\} & \text{fully essential} \end{cases}$$

Let $g \in \underline{P}^2$ ((Y, B) , (Y', B')) bc a mapping, thcn we observe:
$\{x_\alpha\}$ = (essential, fully essential) X-set $\Rightarrow$
$\{g_\#(x_\alpha)\}$ = (essential, fully essential) X-set
x associated with $\{x_\alpha\} \Rightarrow g_\#$ x associated with $\{g_\#$ x,$\}$.
Hence we are able to dcfine

$$g_\# : \tilde{\Gamma}_n(Y, B) \to \tilde{\Gamma}_n(Y', B')$$
$$\{x_\alpha\} \mapsto \{g_\#(x_\alpha)\},$$

commuting with boundaries.

From now on we writc Γ_n((Y, B); {X,}) instcad of $\tilde{\Gamma}_n(Y, B)$ in order to specify the given family of spaces. Suppose we have another family of indexed spaces $\{X^1_\alpha\}$ (same indexing set $\underline{A}$) in $\underline{K}_0$, $\underset{\alpha\in\underline{A}}{\mathrm{Cl}} X^1_\alpha \in \underline{K}_0$ and a family of mappings

$$r_\alpha : (X_\alpha, x_{\alpha 0}) \to (X^1_\alpha, x^1_{\alpha 0}) \qquad \text{giving rise to a}$$
$$r : \left(\underset{\alpha\in\underline{A}}{\mathrm{Cl}} X_\alpha, *\right) \to \left(\underset{\alpha\in\underline{A}}{\mathrm{Cl}} X^1_\alpha, *\right)$$

We are defining an cquivalence relation in the union $\cup\Gamma_n$((Y, B) , $\{X'_\alpha\}$) (taken over all such families $\{X'_\alpha\}$):

Suppose we have

$$\{x^i_\alpha\} \in \Gamma_n((Y, B); \{X^i_\alpha\}), f^i_\alpha : (X^i_\alpha, x^i_{\alpha 0}) \to (Y, B),$$
$$f^i = \underset{\alpha\in\underline{A}}{\mathrm{Cl}} f^i_\alpha, \hat{x}^i_\alpha \in C_n(X^i_\alpha, x^i_{\alpha 0}), f^i_\alpha \hat{x}^i_\alpha = x^i_\alpha,$$
$$i = 1, 2.$$
$$t_\alpha : (X^1_\alpha, x^1_{\alpha 0}) \to (X^2_\alpha, x^2_{\alpha 0}), t = \underset{\alpha\in\underline{A}}{\mathrm{Cl}} t_\alpha,\ t_\alpha f^1_\alpha = f^2_\alpha.$$

Then we set

$$\{x_\alpha^1\} \sim \{x_\alpha^2\}$$

whenever $t_{\alpha\#}\,\widehat{x}_\alpha^1 = \widehat{x}_\alpha^2$ for all $\alpha \in \underline{A}$.

This relationship generates an equivalence relation.

If $\{x_\alpha^1\}$ is associated with z, then $\{x_\alpha^2\}$ is also associated with z. We call $\Gamma_n(Y, B) = \cup\Gamma_n((Y,B);\{X'_\alpha\})/\sim$, observing that:

$$\{x_\alpha^1\} \sim \{x_\alpha^2\} \Rightarrow \begin{cases} \{\mathrm{d}\,x_\alpha^1\} \sim \{\mathrm{d}\,x_\alpha^2\} \\ \{g_\# x_\alpha^1\} \sim \{g_\# x_\alpha^2\} \end{cases}$$

for any $g \in \underline{P}^2((Y,B),(Y',B'))$.

By an abuse of notation we still write $\{x_\alpha\} \in \Gamma_n(Y, B)$ for the equivalence class.

Let $\{x_\alpha^i\} \in \Gamma_n((Y, B); \{X_\alpha^i\})$, $i = 1, 2$, be given.

Then we form $X'_\alpha = X_\alpha^1 \vee X_\alpha^2$, $t_\alpha^i : (X_\alpha^i, x_{\alpha 0}^i) \subset (X'_\alpha, x'_{\alpha 0})$ (the inclusion) and f'_α : $(X'_\alpha, x'_{\alpha 0}) \to (Y, B)$ (defined by $f_\alpha^1 \vee f_\alpha^2$). We conclude that with $\widetilde{x}_\alpha^i = t_{\alpha\#}^i\, x_\alpha^i$, we have $\{x_\alpha^i\} \sim \{\widetilde{x}_\alpha^i\}$.

As a result we can assume without loss of generality that up to an equivalence, every finite set of elements in $\cup\Gamma_n((Y, B); \{X'_\alpha\})$ is lying in the *same* $\Gamma_n((Y, B); \{X'_\alpha\})$. Let $F_n(Y, B)$ be the free abelian group generated by the elements of $\Gamma_n(Y, B)$. We establish a quotient group of $F_n(Y, B) \oplus C_n(Y, B)$ by introducing the following relations:

R1) Suppose $\{x_\alpha^i, f_\alpha\} \in \Gamma_n((Y, B); \{X'_\alpha\})$ (same mappings f_α!) $i = 1, 2$, are such that $x_\alpha^1 = x_\alpha^2$ unless $\alpha = \alpha_1, \ldots, \alpha_m$, then we set

$$\{x_\alpha^1\} - \{x_\alpha^2\} = \sum_{i=1}^{m} x_{\alpha_i}^1 - x_{\alpha_i}^2 \in C_n(Y, B).$$

R2) Suppose again $\{x_\alpha^i, f_\alpha\} \in \Gamma_n((Y, B); \{X'_\alpha\})$, $i = 1, 2$, then we set

$$\{x_\alpha^1\} - \{x_\alpha^2\} = \{x_\alpha^1 - x_\alpha^2\},$$

provided the right-hand term is in $\Gamma_n((Y, B); \{X'_\alpha\})$ defined.

The quotient group ${}^1C_n(Y, B) = F_n(Y, B) \oplus C_n(Y, B)/R$ has the following properties:

'1) There exists a boundaty $\mathrm{d} : {}^1C_n(Y, B) \to {}^1C_{n-1}(Y, B)$ which coincides with the boundaty on $C_n(Y, B)$ whenever both are defined. One has $\mathrm{d}^2 = 0$, hence ${}^1C_*(Y, B)$ is a chain complex.

*2) There are induced mappings $g_\# : {}^1C_n(Y,B) \to {}^1C_n(Y',B')$, $g \in \underline{P}^2((Y,B),(Y',B'))$, turning ${}^1C_* : \underline{P}^2 \to \underline{ch}$ (= category of chain complexes) into a functor.

*3) The natural mapping $C_n(Y, B) \to F_n(Y, B) \oplus C_n(Y, B) \to {}^1C_n(Y, B)$ is a monomorphism, i.e. the equivalence relation R does not identify different elements of $C_n(Y, B)$.

*4) An inclusion $i : (Y, B) \subset (Y', B')$ induces a monomorphism $i_\# : {}^1C_n(Y, B) \to {}^1C_n(Y', B')$.

*5) ${}^1C_n$ satisfies a homotopy axiom (cf. § 8): To each $(Y, B) \in \underline{P}^2$ there exists a natural chain homotopy $D_{(Y,B)} : {}^1C_n(Y,B) \to {}^1C_{n+1}(Y \times I, B \times I)$ between the inclusions $i_{0\#}, i_{1\#} : {}^1C_n(Y,B) \to {}^1C_{n+1}(Y \times I, B \times I)$.

*6) We have

$$ {}^1C_n(Y) = {}^1C_n(Y,\emptyset) = C_n(Y). $$

Proof. Ad * 1) : The boundary d on Γ_n as well as on C_n induces one in $\Gamma_n \oplus C_n$, which is immediately seen to respect the relations in R.

Ad *2) : Follows by the same kind of argument.

Ad *3) : An identification of two chains $x^1 \neq x^2 \in C_n(Y, B)$ by means of R can only happen if there are different chains associated with a given X-set, but this is in case 1 (with which we are dealing now) excluded.

Ad *4) : Suppose $\{x^1_\alpha, f^1_\alpha\}, \{x^2_\alpha, f^2_\alpha\} \in \Gamma_n((Y, B), \{X'_\alpha\})$ are identified under i, then at first $if^1_\alpha = if^2_\alpha$ implies $f^1_\alpha = f^2_\alpha$. Hence $i_\# (\{x^1_\alpha\}) = i_\# \{x^2_\alpha\}$ implies $\{x^1_\alpha\} = \{x^2_\alpha\}$ and $\Gamma_n(i) : \Gamma_n(Y, B) \to \Gamma_n(Y', B')$ is a monomorphism. Since $i_\# : C_n(Y, B) \to C_n(Y', B')$ is a monomorphism by definition,

$$ i_\# : F_n(Y, B) \bigoplus C_n(Y, B) \to F_n(Y',B') \bigoplus C_n(Y',B') $$

is a monomorphism. Now we have

$$ i_\# R(Y,B) = R(Y',B') \cap i_\# \left(F_n(Y,B) \bigoplus C_n(Y,B)\right) $$

implying the assertion *4) .

Ad *5) : We have $D_{(Y,B)} = D$ already defined on C_* and set

$$ D\{x_\alpha\} = \{Dx_\alpha\} $$

whenever this is essential resp.

$$ D\{x_\alpha\} = D(x) $$

if x is associated with $\{Dx_\alpha\}$. So we obtain a $D = D_{(Y,B)}$ for $F_n \oplus C_n$ in a natural way. Since this D respects obviously all the relations involved, the assertion follows.

Ad '6) : Is an immediate consequence of $\Gamma_n(Y, 0) = \emptyset$.

In order to complete the definition of a chain ftmctor we define ${}^1C'_n(Y, B) = C'_n(Y, B)$ and resume all remaining items (like φ, κ, i', ρ) from $\underline{C}_*$, endowing ${}^1\underline{C}_*$ with the structure of a chain functor, containing $\underline{C}_*$ as a subfunctor.

The most important assertion about the inclusion $\nu : \underline{C}_* \subset {}^1\underline{C}_*$ is:

*7): ν induces an isomorphism of homology theories in $\underline{P}$.

Proof. We are obliged to prove that every cycle in ${}^1C_*(Y, B)$, $(Y, B) \in \underline{P}^2$, is holomogous to a cycle in $C_*(Y, B)$ and that a cycle $z \in C_n(Y, B)$ which bounds in ${}^1C_{n+1}(Y, B)$ is already bounding in $C_{n+1}(Y, B)$.

Let to this end $(Y, B) \in \underline{P}^2$ be a pair and consider a cycle

$$\tilde{z} = \sum_{i=1}^{m} a_i\{x_\alpha\} + c \in Z_n({}^1C_*(Y,B))$$

$a_i \in \mathbf{Z}$, $c \in C_n(Y, B)$, then we can assume without loss of generality (because of the remark following the definition of Γ_n) that, up to an equivalence, all $\{x_\alpha\}$ are lying in a fixed $\Gamma_n((Y, B); \{X'_\alpha\})$. Using the relationships R1), R2), $\tilde{z}$ can be written in the form

$$\tilde{z} = \{x_\alpha\} + \{y_\alpha\} + c,$$

where $\{x_\alpha\}$ is fully essential, $\{y_\alpha\}$ is essential but *not* fully essential (i.e. one has $d\{y_\alpha\} = z_1 \in Z_{n-1}(C_*(Y,B))$) and $c \in C_n(Y,B)$.

We have $d\tilde{z} = 0$, so that $d\{x_\alpha\}$ does not drop out against dc or $d\{y_\alpha\}$. Hence we conclude that $\{x_\alpha\}$ does not appear; displaying a $\tilde{z}$ of the form

$$\tilde{z} = \{y_\alpha\} + c,$$

with not fully essential $\{y_\alpha\}$.

Since Y is locally contractible, there exists a family $\{U_\alpha\}$ of neighborhoods of the base-point $f(*) \in B \subset Y$ such that 1) $\cap U_\alpha = f(*)$, 2) $f(X'_\alpha) \subset U_\alpha$, 3) all but a finite number of these U_α are contractible.

Let $g : (\operatorname{Cl}_{\alpha\in A} U_\alpha,\ f(*)) \to (Y, B)$ be induced by the inclusions $U_\alpha \subset Y$ and let $U_{\alpha_1}, \dots, U_{\alpha_m}$ be those neighborhood which are eventually *not* contractible. We split $\{y_\alpha\}$ into

$$\{y_\alpha\} = \{y'_\alpha\} + \{y''_\alpha\}$$

$$y'_\alpha = \begin{cases} y_\alpha \ldots \alpha \neq \alpha_1, \ldots, \alpha_m \\ 0 \ldots \alpha = \alpha_1, \ldots, \alpha_m \end{cases}$$

$$y''_\alpha = \begin{cases} y_\alpha \ldots \alpha = \alpha_1, \ldots, \alpha_m \\ 0 \ldots \alpha \neq \alpha_1, \ldots, \alpha_m, \end{cases}$$

observing that $\{y''_\alpha\} = c_1 \in C_n(Y,B)$, calling $d\{y'_\alpha\} = z_1 \in Z_{n-1}(C_*(Y,B))$.

According to *5) we have

$$d\,D(\{y'_\alpha\}) + D(z_1) = \{y'_\alpha\} - *,$$

* denoting a bounding cycle in $C_n(*,*)$. So we conclude that the cycle $\{y'_\alpha\} - D(z_1) = d\,\tilde{x}$ is bounding (in ${}^1C_*(Y,B)$) and that

$$\tilde{z} = z + d\,\tilde{x}$$

$$z = D(z_1) + c + c_1 \in Z_n(C_*(Y,B))$$

is a cycle homologous to $\tilde{z}$.

As a result $\nu : \underline{C}_* \subset {}^1\underline{C}_*$ induces an epimorphism. The argument for verifying that ν induces a monomorphism is similar: Let $z = d\tilde{x}$, $z \in Z_n(C_*(Y,B))$, $\tilde{x} = \{x_\alpha\} + c \in {}^1C_{n+1}(A,B)$, then we find again $\{x'_\alpha\}$ (as before $\{y'_\alpha\}$), $\tilde{x} = \{x'_\alpha\} + c'$, $d\{x'_\alpha\} = z'$ and deduce the existence of a $x \in C_{n+1}(Y,B)$ such that $\tilde{x} = x + d\tilde{x}_1$, hence $z = d\,x$ is already bounding in $C_*(Y,B)$.

In order to complete the proof of theorem 2.2 (stili for the case 1)) we argue as in § 4: Let $\underset{\alpha\in \underline{A}}{\mathrm{Cl}}\, X_\alpha \in \underline{K}_0$ be given, $\{z_\alpha\} \in \prod_{\alpha\in\underline{A}} Z_n(C_*(X_\alpha, x_{\alpha 0}))$, being inessential, then we find the associated $z \in Z_n(C_*(Y,B))$ satisfying $\lambda\{z\} = \{\{p_{\alpha\#}z\}\} = \{\{z_\alpha\}\} \in \prod_{\alpha\in\underline{A}} h_n(X_\alpha, x_{\alpha 0})$. If however $\{z_\alpha\}$ is essential, then we have a $\tilde{z} \in Z_n({}^1C_*(\underset{\alpha\in\underline{A}}{\mathrm{Cl}}\, X_\alpha, *))$ such that $\lambda\{\tilde{z}\} = \{\{z_\alpha\}\}$. So λ is epic. If $z \in Z_n(C_*(\underset{\alpha\in A}{\mathrm{Cl}}\, X_\alpha, *))$ has the property that $\lambda(z) = \{p_{\alpha\#}z\} = \{d\,x_\alpha\}$, $x_\alpha \in C_{n+1}(X_\alpha, x_{\alpha 0})$, we argue with $\{x_\alpha\}$ as before with $\{z_\alpha\}$. This settles the first case.

6. PROOF OF THEOREM 2.2 (SECOND PART)

We treat case 11) where the assumption of I) is not necessarily true. This will be accomplished by 1) replacing the given chain functor $\underline{C}_*$ by a new one $\underline{E}_*$, giving the same homology as $\underline{C}_*$ and 2) restricting the class of X-sets (with $\underline{E}_*$) in such a way that on one hand 1) is fulfilled and on the other there are still sufficiently many X-sets available to perform the same constructions as in case 1).

The construction of $\underline{E}_*$ is accomplished in several steps:

Resuming thenotations of § 5 conceming {X,} , $\{X'_\alpha\}$, $r : (\operatorname{Cl}_{\alpha\in\underline{A}} X_\alpha, *) \to (\operatorname{Cl}_{\alpha\in\underline{A}} X'_\alpha, *)$, $f_\alpha : (X'_\alpha, x'_{\alpha 0}) \to$ (Y,B) etc. wedefine

$$G_n^{(1)}\left(\operatorname{Cl}_{\alpha\in\underline{A}} X_\alpha, *\right) = C_n\left(\operatorname{Cl}_{\alpha\in\underline{A}} X_\alpha, *\right)$$

$$G_n^{(1)}(X_\alpha, x_{\alpha 0}) = C_n(X_\alpha, x_{\alpha 0}) \bigoplus F\left(\left\{(c, p_\alpha) \mid c \in C_n\left(\operatorname{Cl}_{\alpha\in\underline{A}} X_\alpha, *\right)\right\}\right)$$

where we generally denote by F(M) the free abelian group generated by the set M . Dealing with pairs like (c, p,) we agree to identify (0, p,) with 0.

$$G_n^{(1)}\left(\operatorname{Cl}_{\alpha\in\underline{A}} X'_\alpha, *\right) = C_n\left(\operatorname{Cl}_{\alpha\in\underline{A}} X'_\alpha, *\right) \bigoplus F\left(\left\{(c, r) \mid c \in C_n\left(\operatorname{Cl}_{\alpha\in\underline{A}} X_\alpha, *\right)\right\}\right)$$

$$G_n^{(1)}(X'_\alpha, x'_{\alpha,0}) = C_n(X'_\alpha, x'_{\alpha 0}) \bigoplus$$

$$\bigoplus F\left(\left\{(c', p'_\alpha) \mid c' \in C_n\left(\operatorname{Cl}_{\alpha\in\underline{A}} X'_\alpha, *\right)\right\} \cup \left\{(c, p'_\alpha r) \mid c \in C_n\left(\operatorname{Cl}_{\alpha\in\underline{A}} X_\alpha, *\right)\right\} \cup\right.$$

$$\left.\cup \{(x_\alpha, r_\alpha) \mid x_\alpha \in C_n(X_\alpha, x_{\alpha 0})\}\right).$$

For (Y, $B) \in \underline{K}^2$ we Set:

$$E_n^{(1)}(Y, B) = C_n(A, B) \bigoplus F\left(\bigcup_{\{X'_\alpha\}} \{(c', f_\alpha p'_\alpha)\} \cup \{x'_\alpha, f_\alpha)\} \mid x'_\alpha \in C_n(X'_\alpha, x'_{\alpha 0})\right)$$

where the union is taken over all families $\{X'_\alpha\}$ as in § 5 (where {X,} plays a distinguished role).

We have a boundary:

$$d_n : E_n^{(1)}(Y, B) \to E_{n-1}^{(1)}(Y, B)$$

(resp. for $G_n^{(1)}$) by applying d to the first components of the pairs (c, p,) etc. and preserving theoriginal d on C_*.

To each g $\in \underline{K}^2$ ((Y, $B)$, (Y', B')) we have the induced

$$g_\# : E_n^{(1)}(Y, B) \to E_n^{(1)}(Y', B')$$

defined by

$$c \mapsto g_{\#}(c) \quad \text{on} \quad C_*$$
$$(c', f_\alpha p'_\alpha) \mapsto (c', g f_\alpha p'_\alpha)$$
$$(x'_\alpha, f_\alpha) \mapsto (x'_\alpha, g f_\alpha).$$

Similarly we define:

$$G_n^{(1)}(p_\alpha) = p_{\alpha\#} : G_n^{(1)}\left(\underset{\alpha\in \underline{A}}{\text{Cl}} X_\alpha, *\right) \to G_n^{(1)}(X_\alpha, x_{\alpha 0})$$
$$c \mapsto (c, p_\alpha)$$
$$G_n^{(1)}(p'_\alpha) = p'_{\alpha\#} : G_n^{(1)}\left(\underset{\alpha\in \underline{A}}{\text{Cl}} X'_\alpha, *\right) \to G_n^{(1)}(X'_\alpha, x'_{\alpha 0})$$
$$c' \in C_n\left(\underset{\alpha\in \underline{A}}{\text{Cl}} X'_\alpha, *\right) \mapsto (c', p'_\alpha)$$
$$(c, r) \mapsto (c', p'_\alpha r)$$

resp. $f_{\#}$, $r_{\#}$.

We have a natural inclusion

$$G_*^{(1)} \subset E_*^{(1)}$$

whenever both sides are defined, e.g.

$$G_n^{(1)}(X'_\alpha, x'_{\alpha 0}) \subset E_n^{(1)}(X'_\alpha, x'_{\alpha 0})$$
$$x'_\alpha \mapsto (x'_\alpha, 1)$$
$$(c', p'_\alpha) \mapsto (c', 1 p'_\alpha).$$

In a second step we introduce certain connecting chains A , fumishing us with homologies between pairs (x, f) and the element $f_{\#}$ x :

$$G_{n+1}^{(2)}\left(\underset{\alpha\in \underline{A}}{\text{Cl}} X_\alpha, *\right) = G_{n+1}^{(1)}\left(\underset{\alpha\in \underline{A}}{\text{Cl}} X_\alpha, *\right)$$
$$G_{n+1}^{(2)}(X_\alpha, x_{\alpha 0}) = G_{n+1}^{(1)}(X_\alpha, x_{\alpha 0}) \bigoplus F(\{\Delta(c, p_\alpha)\})$$

where we require

$$d\Delta(x, p_\alpha) = (c, p_\alpha) - p_{\alpha\#} c - \Delta(dc, p_\alpha)$$

and throughout

$$\Delta(c,\cdot) = 0.$$

$$G^{(2)}_{n+1}\left(\underset{\alpha\in\underline{A}}{\mathrm{Cl}}\, X'_\alpha, *\right) = G^{(1)}\left(\underset{\alpha\in\underline{A}}{\mathrm{Cl}}\, X'_\alpha, *\right) \bigoplus F(\{\Delta(c,r)\}),$$

$$d\Delta(c,r) = (c,r) - r_\# c - \Delta(dc,r),$$

$$G^{(2)}_{n+1}(X'_\alpha, x'_{\alpha 0}) = G^{(1)}_{n+1}(X'_\alpha, x'_{\alpha 0}) \bigoplus F\{\Delta(c', p'_\alpha)\} \cup$$

$$\cup \{\Delta(c, p'_\alpha r, p'_\alpha)\} \cup \{\Delta(x_\alpha, r_\alpha)\}$$

where the first and the third A -chain satisfies a relation as before while we require

$$d\Delta(c, p'_\alpha r, p'_\alpha) = (c, p'_\alpha r) - (r_\# c, p'_\alpha) - \Delta(dc, p'_\alpha r, p'_\alpha)$$

and in all cases again $\Delta(0,\cdot,\cdot) = \Delta(0,\cdot) = 0$.

$$E^{(2)}_{n+1}(Y,B) = E^{(1)}_{n+1}(Y,B) \bigoplus F(\bigcup_{\{X'_\alpha\}} \{\Delta(c, f_\alpha p'_\alpha)\} \cup$$

$$\cup \{\Delta(c', f_\alpha p'_\alpha, p'_\alpha)\} \cup \{\Delta(x'_\alpha, f_\alpha)\}).$$

$$d\Delta(c', f_\alpha p'_\alpha, p'_\alpha) = (c', f_\alpha p'_\alpha) - (p'_{\alpha\#} c', f_\alpha) - \Delta(dc', f_\alpha p'_\alpha, p'_\alpha),\ (Y,B) \in \underline{P}^2.$$

The boundary d in these new groups is obvious; let $g \in \underline{P}^2((Y,B),(Y',B'))$ be a mapping, then we set

$$g_\# \Delta(c', f_\alpha g'_\alpha) = \Delta(c', g f_\alpha g'_\alpha)$$

$$g_\# \Delta(c', f_\alpha p'_\alpha, p'_\alpha) = \Delta(c', g f_\alpha p'_\alpha, p'_\alpha)$$

and in a similar form for the remaining cases, providing us also with induced mappings

$$p_{\alpha\#},\ p'_{\alpha\#},\ f_\#,\ r_\#, \quad \text{for } G^{(2)}_n \text{ and } E^{(2)}_n.$$

Again we have an inclusion $G^{(2)}_* \subset E^{(2)}_*$ whenever both sides are defined.

We recall the definition of an algebra; cone, *cone*(K_*) over a chain complex K_*. In particular we have the possibility to erect the cone over the subcomplex $K_* \subset L_*$: $L_* \cup$ *cone*(K_*) (cf. [1] § 4).

So we define for $(Y,B) \in \underline{P}^2$

$$E_n(Y,B) = E^{(2)}_*(Y,B) \cup cone(D_*(Y,B))$$

where $D_*(Y, B)$ is the subcomplex of $E_*^{(2)}(Y, B)$ which is generated by all A chains (containing for example chains of the form $(z, fp_\alpha) - (fp_\alpha)_\# z$, $z \in Z_n(C_*(\text{CI } X,, *)))$. Since formation of the cone is a natural process, we have induced mappings

$$g_\# : E_*(Y,B) \to E_*(Y',B'), g \in \underline{K}^2((Y,B),(Y',B')).$$

We claim:

* 1) There exists a natura1 isomorphism

$$\mu_* : H_*(C_*(Y,B)) \approx H_*(E_*(Y,B))$$

induced by the inclusion (on the category $\underline{K}$)

$$\mu : C_* \subset E_*.$$

Proof. Let $\tilde{z} \in Z_n(E_*(Y, B))$ *be* a cycle

$$\tilde{z} = c + \sum_{i=1}^{\ell} a_i c_i', f_{\alpha_i} p_{\alpha_i} + \sum_{j=1}^{m} a_j'(x_{\alpha_j}', f_{\alpha_j}) + \sum_{k=1}^{m} a_k'' \Delta_k + \rho, a_i, a_j', a_k'' \in \mathbb{Z},$$

Δ_k a Δ-chain, $\rho \in cone D_* \setminus E_*^{(2)}$, then we can pull down all brackets by A-chains to elements in C_*. By adding them up we find a $\tilde{z}' \sim \tilde{z}, \tilde{z}' = c' + \sum b_i \Delta_i' + \rho$. We have

$$-\mathrm{d} c' = \sum b_i \mathrm{d} \Delta_i' + \mathrm{d} \rho.$$

Since $p \notin E_*^{(2)}$, dp does not contribute summands in $E_*^{(1)}$ hence all brackets $b(c, f)$, $b \in \mathbb{Z}$, appearing in $\sum b_i \mathrm{d} \Delta_i' + \mathrm{d}\, p$ must sum up to zero. On the other hand every summand $c_1 \in C_{n-1}$ in this sum comes together with such a bracket. Therefore also these c_1 sum up to zero, implying that -dc' = 0 and that $\sum b_i \Delta_i' + p \in Z_n(cone D_*)$ where it is bounding.

Therefore we obtain a $\tilde{z}'' = c' \in Z_n(C_*(Y, B))$, $\tilde{z}'' \sim \tilde{z}$, ensuring that μ_* is an epimorphism.

The proof that μ_* is a monomorphism is similar:

Suppose $z \in Z_n(C_*(Y,B))$, $z = \mathrm{d}\tilde{x}$,

$$\tilde{x} = \sum a_i(c_i', f_{\alpha_i} p_{\alpha_i}') + \sum a_j'(x_{\alpha_j}', f_{\alpha_j}) + \sum a_k'' \Delta_k + \rho,$$

then we have again

$$\tilde{x} = c' + \sum b_i \Delta_i' + \rho + \mathrm{d} \sum b_e' \mathrm{d} \Delta''_e,$$

where all sums are finite and Δ_k, Δ_i', Δ_e'' are A -chains.

Since $\mathrm{d}\tilde{x} = z$ in $Z_n(C_*(Y, B))$ we deduce $z - \mathrm{d}c' = \sum b_i \mathrm{d}\Delta_i' + \mathrm{d}\rho$ and by the same argument as before that $\sum b_i \Delta_i' + \rho$ is a bounding cycle, implying

$$z = \mathrm{d}c'.$$

Hence z is already bounding in $C_*(Y, B)$.

By setting $E_n'(Y, B) = C_n'(Y, B)$, taking $\varphi, \kappa, \rho, i'$ from $\underline{C}_*$ and by observing that (by definition) $E_n(Y, \emptyset) = C_n(Y, \emptyset)$, we turn $\underline{E}_*$ into a chain functor, giving the same homology as $\underline{C}_*$. The validity of a homotopy axiom is immediate. Now we proceed *almost* as in § 5: A *restricted* X-set (X,-set) is a X-set {e,} such that the associated $\hat{e}_\alpha \in E_n(X_\alpha', x_{\alpha 0}')$ are already contained in the subgroup $G_n^{(2)}(X_\alpha', x_{\alpha 0}')$ (and not in the first summand C_*). We have

'2) Let {e,} be an inessential X,-set, associated with e^1, e^2 and suppose that $\hat{e}^1$, $\hat{e}^2 \in E_n(\underset{\alpha \in A}{\mathrm{Cl}}\, X_\alpha', x_{\alpha 0}')$ are the corresponding elements satisfying

$$f_\# \hat{e}^i = e^i, \quad i = 1, 2,$$

then we have

$$\hat{e}^1 = \hat{e}^2.$$

In other words: The X,-sets satisfy the assumption of case 1).

Proof. According to the definition of induced maps we have:

$$p_{\alpha\#}'(c^1, r) = p_{\alpha\#}'(c^2, r)$$

implies

$$(c^1, p_\alpha' r) = (c^2, p_\alpha' r)$$

hence

$$(c^1, r) = (c^2, r).$$

In the same way we obtain

$$p_{\alpha\#}'(c'^1) = p_{\alpha\#}'(c'^2) \Rightarrow c'^1 = c'^2, \quad p_{\alpha\#}' = E_*(p_\alpha').$$

resp. for the A -chains.

Now we repeat the argumentation of case 1) in $5:

Let $\{\tilde{z}_\alpha\} \in \prod_{\alpha\in \underline{A}} Z_n(E_*(X_\alpha, x_{\alpha 0}))$ be given, then we can according to * 1) replace $\tilde{z}_\alpha$ in its homology class by a $z_\alpha \in Z_n(C_*(X_\alpha, x_{\alpha 0}))$. By setting (Y, B) = $(X'_\alpha, x'_{\alpha 0})$ = $(X_\alpha, x_{\alpha 0})$, we have $(z_\alpha, 1) \in G_n^{(1)}(X_\alpha, x_{\alpha 0}) \subset E_n(X_\alpha, x_{\alpha 0})$, providing us with a X,-set $\{(z_\alpha, 1)\}$ to which there exists a $\check{z} \in {}^1\underline{E}_n(\underset{\alpha\in\underline{A}}{\mathrm{Cl}}\, X_\alpha, *)$ suchthat X(Z) = {(z,, 1)). Let $\tilde{z} \in Z_n(E_*(\underset{\alpha\in A}{\mathrm{Cl}}\, X_\alpha, *)$ bea cycle such that $E_*(p_\alpha)\tilde{z}$ = d $\tilde{x}_\alpha$. According to * 1) we find a cycle z $\in C_*(\underset{\alpha\in A}{\mathrm{Cl}}$ X,, $*$) such that (z, 1) $\sim$ z $\sim \tilde{z}$ andd chains $\hat{y}_\alpha \in E_{n+1}$ (X,, $x_{\alpha 0}$) satisfying d$\hat{y}_\alpha = C_*($p,) z . As in the proof of * 1) we detect chains $x_\alpha \in C_{n+1}$ (X,, $x_{\alpha 0}$) such that d$x_\alpha = C_*(p_\alpha)z$. As aresult wededuce $E_*(p_\alpha)(z, 1) = d((\Delta(z, p_\alpha, 1) + (x_\alpha, 1)) = d\tilde{y}_\alpha$. Since $\{\tilde{y}_\alpha\}$ is a X,-set, we find a $\tilde{y} \in {}^1E_n(\underset{\alpha\in\underline{A}}{\mathrm{Cl}}\, X_\alpha, *)$ such that

$$\tilde{z} - (z, 1) = d\tilde{y}.$$

SO

$$\lambda : h_*\left(\underset{\alpha\in\underline{A}}{\mathrm{Cl}}\, X_\alpha, *\right) \to \prod_{\alpha\in\underline{A}} h_*(X_\alpha, x_{\alpha 0})$$

is an isomorphism.

7. ADDITIVITY AND COMPACT CARRIER

We can use the considcrations of § 4 to settle the following two questions below:

Suppose $\underline{K}$ is any category of topological spaces containing 1) the category $\underline{P}$ of compact *AN Rs* 2) the category of finite dimensiona1 locally compact spaces as subcategories. Let ${}^sH_*$ ( ) be ordinary strong homology theory rel. $\underline{P}$ with integcr cocfficients.

Question 1: Is ${}^sH_*$ additive?

Question 2: Does ${}^sH_*$ have compact carricrs?

The answer to both questions is negative. More prccisely:

Proposition 7.1. *There exist compact metric spaces X, $\in \underline{K}$, an index n and an element* $\zeta \in {}^sH_n(\sum_{i=1}^\infty X_i)$ *(*$\sum_{i=1}^\infty X_i$ *denoting the free union of the spaces* X_i*) such that*

$$\zeta \notin im\left({}^sH_n\left(\sum_{i=1}^N X_i\right) \to {}^sH_n\left(\sum_{i=1}^\infty X_i\right)\right) \tag{1}$$

for any finite N .

Proof. For convenience we rcplace X_i by the based space $X_i^+ = (X_i^+, *)$ and $\sum_{i=1}^\infty X_i$ by the wedge

$$x = (X, *) = \bigvee_{i=1}^\infty (X_i^+, *).$$

We have

$$^sH_*(X,*) \approx {}^sH_*\left(\sum_{i=1}^{\infty} X_i\right)$$

$$^sH_*(X_i^+,*) \approx {}^sH_*(X_i)$$

Let $^sC_*$ be any c-continuous chain functor related to $^sH_*$. On compacta $^sC_*$ coincides with a c-continuous chain functor giving ordinary Steenrod-Sitnikov homology theory. It is well-known that there exist compacta X_i and cycles $z_i \in {}^sC_n(X_i^+,*)$ $z_i \not\sim 0$, such that $z_i \sim 0$ on each $(X_i^+,*) \subset (P,*) \in \underline{P}_0$. On the homology leve1 that amounts to the assertion that $\zeta_i = \{z_i\} \in {}^sH_n(X_i^+,*)$ is not trivial but $k_*\zeta_i = \{k_\# z_i\} \in {}^sH_n(\boldsymbol{P},*)$, $\boldsymbol{k} : (X_i^+,*) \subset (P,*)$, vanishes. Take for example the Sitnikov chain functor (cf. [1] § 9) and a cycle z in the solenoid Y = X_i which bounds on each enveloping ***ANR*** uncoherently, i.e. without bounding a chain $x \in {}^sC_{n+1}$ (Y) . The fact that this exists is standard.

In the same way in which we invented «sums» in § 4, we get a cycle $z = \sum_{i=1}^{\infty} z_i \in {}^sC_n(X,*)$ which has the property that $p_{i\#}\, z \in Z_n({}^sC_*(X_i^+,*))$ (p_i : X $\rightarrow (X_i^+,*)$ the projection) is notbounding. Hence $\zeta = \{z\} \in {}^sH_n(X,*)$ is a homology class satisfying (1).

It is immediately clear, that 7.1 provides us with a negative answer to both questions.

Remark. 1) $^sH_*$ was probably the candidate where one would most likely expect a positive answer to the two questions.

2) In [6] the authors come to the conclusion that for strong homology in the sense of J. Lisica and S. Mardesic [5] the questions 1), 2) are undecidable.

8. APPENDIX

We collect some definitions and conventions which are constantly used in the course of the present paper:

a) category of topological spaces: A full subcategory $\underline{K}$ c Top such that:

1) $\emptyset \in \underline{K}$, 2) X $\in \underline{K} \Rightarrow$ X x $I \in \underline{K}$, by $\underline{K}_0$ we denote the category of based spaces (X, x_0) , $X \in \underline{K}$, and require 3) $(X_i, x_{i0}) \in \underline{K}_0$, $i = 1,\ldots,m \Rightarrow \bigvee_{i=1}^{m} (X_i, x_{i0}) \in \underline{K}_0$.

b) chain functors: The explicit definition is contained in [1] to which we refer. A chain functor is a functor

(1) $\underline{C}_* : \underline{K} \longrightarrow \underline{ch}$ = (category of chain complexes)

with much additional structure:
there are functors

(2) $C_*, C'_* : \underline{K}^2 \longrightarrow \underline{ch},$

natural inclusions $\ell : C'_* \subset C_*$,

$$i' : C_*(A) \subset C'_*(X,A), (X,A) \in \underline{K}^2$$

and non-natural chain mappings

$$\varphi_\# : C'_*(X,A) \longrightarrow C_*(X)$$
$$\kappa_\# : C_*(X) \longrightarrow C'_*(X,A)$$

together with chain homotopies resp. relations:

$$D1) \quad : \varphi_\# \kappa_\# \simeq 1 \;:\; C_*(X) \longrightarrow C_*(X)$$
$$j_\# \varphi_\# \simeq \ell \;,\; j : X \subset (X,A)$$
$$\kappa_\# i_\# = i' \;,\; i : A \subset (X,A).$$

We have a diagram with exact upper row

$$\begin{array}{lccccccc} (\overline{S}) & 0 \to C_*(A) & \xrightarrow{i'} & C'_n(X,A) & \xrightarrow{p} & C''_n(X,A) & \to 0 \\ & \| & & \varphi_\# \downarrow\uparrow \kappa_\# & & \ell & \\ (S) & C_*(A) & \xrightarrow{i_\#} & C_n(X) & \xrightarrow{j_\#} & C_n(X,A) & \end{array}$$

and a (natural) mapping

$$\psi : H_*(C''_*(X,A)) \to H_*(C_*(X,A)),$$

defined by

$$\psi\{z''\} = \{\ell(z') + q_\# a\},$$
$$d z' \in im(i' : C_*(A) \to C'_*(X,A))$$
$$p(z') = z'', \quad q : (A,A) \subset (X,A), \quad a \in C_n(A,A)$$

such that $da = -s_\# i'^{-1} d z'$, $s : A \subset (A,A)$.

We require:

D2) ψ is an epimorphism; there exists a $\rho : im\, j_* \to H_*(C''_*(X,A))$ satisfying $\psi\rho = 1 : im\, j_* \to im\, j_*$ and

$$p_*\kappa = \rho j_*, \quad \kappa = \kappa_{\#*} : H_n(C_*(X)) \to H_n(C'_*(X,A)).$$

Let $\overline{\partial} : H_n(C'_*(X, A)) \to H_{n-1}$ (C,(A)) be the boundary operator associated with $(\overline{S})$, then we have

$$ker\ \psi \subset ker\, \overline{\partial}.$$

Moreover

*) All inclusions f : (X, A) ⊂ (Y, B) induce monomorphisms

**) The complex C_*(X, X) = $\underline{C}_*(X)$ is acyclic.

If $C_* : \underline{K}^2 \dashrightarrow \underline{ch}$ or altematively $\underline{C}_* : \underline{K} \to \underline{ch}$ carries all this structure, then it is called a *D-functor.*

A D-functor is called a *chain functor* whenever it satisfies

Cl) To each homotopy $H : f_0 \simeq f_1$: (X, A) → (Y, B) in $\underline{K}^2$ there exists a natura1 chain homotopy

$$D(H) : C_*(f_0) \simeq C_*(f_1).$$

C2) Denoting by $0 \in \underline{K}$ resp. $0 \in \underline{ch}$ the zero objects one has C_*(0) = 0 .

The following ***axiom of** carrier* is not explicitely used in this paper although it would be quite easy to endow all chain functors ${}^1\underline{C}_*$, $\underline{E}_*$ constructed in § 3-6 with carriers:

C3) To each c ∈ $\underline{C}_n(X)$ there exists a space $\overline{X} \subset$ X (nof necessarily $\overline{X} \in \underline{K}$) satisfying

a) to each subspace X' ⊂ X , $\underline{K} \ni$ X' ⊃ $\overline{X}$ there exists a c' ∈ $\underline{C}_n(X')$ such that

$$j_\# c' = c, \qquad j : X' \subset X.$$

b) Suppose that $X' \in \underline{K}$, $X' \subset$ X , c' ∈ $\underline{C}_n$(X') such that $j_\#$ c' = c holds, then we have $\overline{X} \subset X'$.

One can replace

1) Cl) by the appearently weaker:

Cl') To each (X, A) ∈ $\underline{K}^2$ there exists a chain homotopy $D_{(X,A)} : C_*(X, A) \to C_{n+1}(X \times I, A \times I)$ between $i_{0\#}$, $i_{1\#}$, i'_t : $(X, A) \subset (X \times I, A \times I)$, $t = 0, 1$, which is natura1 in the sense that g ∈ $\underline{K}^2$ ((X, A) , (Y, B)) renders the diagram

$$\begin{array}{ccc} C_n(X,A) & \xrightarrow{D_{(X,A)}} & C_{n+1}(X \times I, A \times I) \\ g_\# \downarrow & & \downarrow (g \times 1)_\# \\ C_n(Y,B) & \xrightarrow{D_{(Y,B)}} & C_{n+1}(Y \times I, B \times I) \end{array}$$

commutative;

and

2) the existence of g in D2) by the requirement

$$ker\ j_* \subset ker\, p_*\, k.$$

Definition 8.1. *1) The derived homology $H_*(\underline{C}_*)(\ )$ of a chain funcor $\underline{C}_*$ is defined by $H_*(\underline{C}_*)(X,A) = H_*(\underline{C}_*(X,A))$ while the boundary operator $\partial : H_*(\underline{C}_*)(X,A) \to H_{*-1}(\underline{C}_*)(A)$ stems from $\overline{\partial}$ in D2).*

2) A given homology theory $h_* = \{h_*, \partial\}$ is related to a chain functor $\underline{C}_*$ whenever there exists a natural isomorphism $H_*(\underline{C}_*)(\ ) \approx h_*(\ )$ of homology theories (i.e. a natura1 isomorphism of functors commuting with boundaries).

We need the concept of a ***transformation between chain functors:*** Let ${}^i\underline{C}_*$, $i = 1,2$ be two chain functors on $\underline{K}$ and let

$$\lambda_{(X,A)} = \lambda : {}^1C'_*(X,A) \to {}^2C'_*(X,A),$$
$$(X,A) \in \underline{K}^2$$
$$\lambda'_{(X,A)} = \lambda' : {}^1C'_*(X,A) \to {}^2C'_*(X,A)$$

be families of chain mappings which are additive, natural, compatible with ℓ and i' ***but only up togiven chain homotopies*** in ${}^2C_*$ such that every cy cle formed by these chain homotopies is bounding in ${}^2C_*$.

Then we talk about a ***transformation of chain functors*** $\lambda : {}^1\underline{C}_* \to {}^2\underline{C}_*$.

We call λ ***strict*** whenever the chain homotopies associated with naturality, additivity and l, i' are trivial.

This concept of a transformation is 1) sufficient to ensure that there exists an induced natural transformation $\lambda_* : H_*({}^1\underline{C}_*)(\ ) \to H_*({}^2\underline{C}_*)(\ )$ of homology theories, 2) general enough to comprise all transformations appearing in practice.

Strict transformations are in particular valuable whenever ${}^2\underline{C}_*$ is c-continuous (rel. $\underline{P}$). In this case a strict transformation $\lambda|\underline{P}$ allows a unique extension over $\underline{K}$. The fact that most interesting transformations are not strict can be remedied in the following way: A family $\widehat{\underline{C}}_* = \{C_*, C'_* : \underline{K}^2 \to$ eh; $\varphi_\#, \kappa_\#, \rho_*, i', l\}$ satisfying all requirements of a chain functor with the exception of the condition *) and the condition that i', l are monic is called a ***weak chain functor.***

Such a weak chain functor appears only as ***full weak subfunctor*** $\widehat{\underline{C}}_*$ of a chain functor $\underline{C}_*$, which means that there exist transformations

$$\widehat{\underline{C}}_* \underset{\zeta}{\overset{\eta}{\rightleftarrows}} \underline{C}_*$$

satisfying 1) $\zeta\eta =$ identity (i.e. η is monic, 2) η induces an isomorphism of homology groups, 3) ζ is suict. 4) $\widehat{C}_*$ is c-continuous whenever $\underline{C}_*$ is.

It turns out that many interesting tmnsformations $\lambda : {}^1\underline{C}_* \to {}^2\underline{C}_*$, with c-continuous ${}^2\underline{C}_*$, although not being strict themselves, factorize over full weak subfunctors by means of *strict* transformations (so-called weakly strict transformations).

Definition 8.2. ***A weak** equivalence* ν ***:*** ${}^1\underline{C}_* \subset {}^2\underline{C}_*$ ***is a** monic transformation **of** chain* ***jiinctors together with a** left* ***inverse*** $\mu : {}^2\underline{C}_* \to {}^1\underline{C}_*$, $\mu\nu = 1$, ***inducing an isomorphism of homology theories.***

Remarks. 1) For ***the*** purpose of an ***existence theorem*** of a strong homology theory one is able to enhance the concept of weak equivalence by requiring that ν is weakly strict. However in our context the weaker concept of 8.2 is sufficient.

2) The relationship between different chain functors ${}^1\underline{C}_*$, ${}^2\underline{C}_*$ which are related to the same homology theory (cf. definition 8.1, 2)) is rather complicatcd and cannot be described simply by natura1 transformations between these chain functors.

3) For establishing an existence theorem for strong homology theories we have to restrict ourselves to closed pairs (X, ***A)*** $\in \underline{K}^2$. In our context, i.e. for deducing theorem 2.2, this is unnecessary.

The main objective of [1] is to presenta proof of

Theorem 8.3. *Each* ***homology theory*** h_* ***on*** $\underline{K}$ ***is related to a** chain* ***jììctor*** $\underline{C}_*$.

It tums out that we can say much more about special properties of $\underline{C}_*$, e.g. $\underline{C}_*$ can be assumed to be a free chain complex.

The basic issue of theorem 8.1 is *not* simply to find a canonically defined chain complex $C_*(X, A)$ such that $h_*(X, A) \approx H_*(C_*(X, A))$; this can be achieved in a trivial way (define $C_n(X, A) = h_n(X, A)$ and let all boundaries $d : C_n \to C_{n-1}$ be trivial). The additional structure of a D- resp. at last of a chain functor is introduced to ensure that this isomorphism becomes an isomorphism of ***homology*** theories, i.e. it must commute with boundary operators $\partial : h_*(X, A) \to h_{*-1}(A)$ resp. for $H_*(\underline{C}_*)$. For ordinary homology theories (i.e. those satisfying a dimension axiom) we can confine ourselves to *flat chain functors* which are characterized by the property that $\psi : H_*(C''_*(X, A)) \to H_*(C_*(X, A))$ is always an isomorphism.

REFERENCES

[1] F. W. BAUER, *Generalized homology theories and chain complexes*, Annali di Matematica pura ed applicata (IV) Vol. CLV (1989). 143-191.

[2] F. W. BAUER, *Strong homology theories; Proceedings of «Geometric Topology and Shape Theory»*, Lecture Notes Vol. 1283 Springer Verlag (1984), 3-29.

[3] F. W. BAUER, *Extensions of generalized homology theories*, Pacific Journal of Math. 128(1), (1987). 25-61.

[4] S. EILENBERG, N. STFENROD. *Foundations of Algebraic Topology*, Princeton Univ. Press (1952).

[5] J. LISICA, S. MARDESIC, *Steenrod-Sitnikov homology for arbitrary spaces*, Bull. Amer. Math. Soc. 9(2) (1983). 207-210.

[6] S. MARDESIC, A. V. PRASOLOV, *Strong homology is not additive*, Trans AMS 307(2) (1988). 725-744.

[7] J. MILNOR, *On the Steenrod homology theory*, Berkeley (unpublished manuscript) (1960).

[8] Z. R. MIMINOSHVILI, *On the sequence of exact and half-exact homologies of arbitrary spoces.* (Russian) Bull. Akad. Nauk Gruz. SSR 113(1) (1984), 41-44.

[9] T. WATANABE, *Cech homology, Steenrod homology and strong homology I*, Glasnik Mat. 22(42) (1987), 187-238.

Received April 14, 1990

F. W. Bauer
Johann Wolfgang Goethe Universität
Fachbereich Mathematik
6 Frankfurt a.M.
Germany