HOLOMORPHIC FUNCTIONS ON $C^I, I$ UNCOUNTABLE
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Dedicated to the memory of Professor Gottfried Köthe

Abstract. In this article we show that $\hat{H}(C^I)$, the (Fréchet) holomorphic functions on $C^I$, is complete with respect to the topologies $\tau_0, \tau_\omega$ and $\tau_\delta$. The same result for countable $I$ is well known (see [2]) since in this case $C^I$ is a Fréchet space. The extension to uncountable $I$ requires a different approach. For the compact open topology $\tau_0$ we use induction to reduce the problem to the countable case. Next we use the result for $\tau_0$ to reduce the problem for $\tau_\omega$ and $\tau_\delta$ to the case of homogeneous polynomials. Using a method developed for holomorphic functions on nuclear Fréchet spaces with a basis and, once more, the result for the compact open topology we complete the proof for $\tau_\omega$ and $\tau_\delta$. We refer to [2] for background information.

1. HOLOMORPHIC FUNCTIONS ON LOCALLY CONVEX SPACES

Let $E$ denote a locally convex space over $C$.

A $C$-valued function on a domain $\Omega$ is said to be holomorphic (or Fréchet holomorphic) if

(i) it is continuous;

(ii) its restriction to each finite dimensional section of $\Omega$ is holomorphic as a function of several complex variables.

A function which satisfies (ii) is said to be Gâteaux holomorphic. We let $H(\Omega)$ denote the vector space of all holomorphic functions on $\Omega$. The compact open topology on $H(\Omega), \tau_0$, is the topology of uniform convergence on the compact subsets of $\Omega$. A semi-norm $p$ on $H(\Omega)$ is said to be ported by the compact subset $K$ of $\Omega$ is for every open set $V, K \subset V \subset \Omega$, there exists $C(V) > 0$ such that

$$p(f) \leq C(V) ||f||_V$$

for all $f$ in $H(\Omega)$.

The $\tau_\omega$ topology on $H(\Omega)$ is the topology generated by the $\tau_\omega$-continuous semi-norms. A semi-norm $p$ on $H(\Omega)$ is said to be $\tau_\delta$-continuous if for every increasing open cover of $\Omega, (V_n)_{n=1}^\infty$, there exists a positive integer $n_0$ and $C > 0$ such that

$$p(f) \leq C||f||_{V_{n_0}}$$

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for all \( f \in H(\Omega) \).

The \( \tau_6 \) topology is the topology generated by all \( \tau_6 \)-continuous semi-norms on \( H(\Omega) \). We always have \( \tau_0 \leq \tau_w \leq \tau_6 \).

We let \( P(\mathbb{N}E) \) denote the (vector) subspace of \( H(E) \) consisting of all (continuous) \( n \)-homogeneous polynomials. By [2, proposition 2.41] \( \tau_w \) and \( \tau_6 \) induce the same topology on \( P(\mathbb{N}E) \) for all \( n \).

We shall need the following result which can be easily deduced from [2, definition 3.32, the remarks following this definition and proposition 3.36].

**Proposition 1.** Let \( E \) denote a locally convex space and suppose \((H(E), \tau_6)\) is complete. The following are equivalent:
(a) \((H(E), \tau_6)\) is complete,
(b) \((H(E), \tau_w)\) is complete,
(c) \((P(\mathbb{N}E), \tau_w)\) is complete for all \( n \).

2. **HOLOMORPHIC FUNCTIONS ON \( C^I \)**

A function \( f : C^I \to C \) is said to depend on finitely many variables if there exists a finite subset \( J \) of \( I \) such that

\[
 f((x_i)_{i \in I}) = f((y_i)_{i \in I})
\]

whenever \( x_i = y_i \) for all \( i \) in \( J \). By Liouville’s theorem every element of \( H(C^I) \) depends on finitely many variables and a Gâteaux holomorphic function on \( C^I \) is holomorphic if and only if it depends on finitely many variables. On \( H(C^I) \) (see [1]) we have \( \tau_0 < \tau_w < \tau_6 \).

Let \( I^{(N)} = \{ (m_i)_{i \in I} ; m_i \in \mathbb{Z}^+ \text{ and } m_i = 0 \text{ for all except a finite number of } i \} \). For \( a \in C \) we let \( a^0 = 1 \). For \( m = (m_i)_{i \in I} \in I^{(N)} \) we denote by \( z^m \) the \( |m| = \sum_i |m_i| \)-homogeneous polynomial which maps

\[
 (z_i)_{i \in I} \mapsto \prod_{i \in I} z_i^{m_i}.
\]

If \( P \) is an \( n \)-homogeneous polynomial on \( C^I \) then, since \( P \) depends on finitely many variables, there exists a set of scalars, \( (a_m)_{m \in I^{(N)}} \), with \( a_m = 0 \) for all but a finite number of elements of \( I^{(N)} \) such that

\[
 P((z_i)_{i \in I}) = \sum_{m \in I^{(N)}} a_m z^m.
\]

Now let \( p \) denote \( \tau_w \)-continuous semi-norm on \( P(\mathbb{N}(C^I)) \). If \( b_m = p(z^m) \) for all \( m \) in \( I^{(N)} \) then

\[
 p \left( \sum_{m \in I^{(N)}} a_m z^m \right) \leq \sum_{m \in I^{(N)}} |a_m| b_m.
\]
for all \( \sum_{m \in I^{(N)}} a_m z^m \) in \( P^n(C^I) \). Let

\[
q \left( \sum_{m \in I^{(N)}} a_m z^m \right) = \sum_{m \in I^{(N)}} |a_m| b_m
\]

and, for each finite subset \( F \) of \( I^{(N)} \), let

\[
q_F \left( \sum_{m \in I^{(N)}} a_m z^m \right) = \sum_{m \in I^{(N)}} \sum_{m \in F} |a_m| b_m
\]

Clearly, by the Cauchy inequalities \( q_F \) is a \( \tau_0 \)-continuous semi-norm, \( q \) is always finite since each polynomial has only a finite number of non-zero terms and

\[
q = \sup_F q_F.
\]

Since \( \tau_0 \) is a barrelled topology on \( P^n(C^I) \) ([2, p. 24]) \( q \) is a \( \tau_0 \)-continuous semi-norm on \( P^n(C^I) \).

We summarize the above in the following proposition:

**Proposition 2.** If \( p \) is a \( \tau_0 \)-continuous semi-norm on \( P^n(C^I) \) then there exists a \( \tau_0 \)-continuous semi-norm \( q \) on \( P^n(C^I) \) and a collection of \( \tau_0 \)-continuous semi-norms \( (q_\alpha)_{\alpha \in A} \) such that:

(i) \( p \leq q \),

(ii) \( q = \sup_{\alpha \in A} q_\alpha \).

3. **COMPLETENESS OF** \((H(C^I), \tau_0)\)

**Proposition 3.** \((H(C^I); \tau_0)\) is complete.

**Proof.** Let \((f_\alpha)_{\alpha \in \Gamma}\) denote a Cauchy net in \((H(C^I), \tau_0)\). Since the Banach space \( C(K), K \) compact, with the supremum norm is complete there exists a function \( f \) on \( C^I \), continuous on compact subset of \( C^I \), such that \( f_\alpha \rightarrow f \) as \( \alpha \rightarrow \infty \), uniformly on compact sets. Since \( f_\alpha \rightarrow f \) uniformly on the finite dimensional compact subsets of \( C^I \) and each \( f_\alpha \) is holomorphic it follows that \( f \) is Gâteaux holomorphic. Hence, to complete the proof we must show that \( f \) is continuous. By our remarks in §2 this is equivalent to showing that \( f \) depends on a finite number of variables. Suppose otherwise. Let \( J_1 \) denote a non-empty finite subset of \( I \). Then
there exist \( x' = (x'_i)_{i \in I} \) and \( y' = (y'_i)_{i \in I} \) in \( C^I \) such that \( f(x' + y') \neq f(x') \) and \( y'_i = 0 \) for \( i \in J_1 \). Let \( \delta = |f(x' + y') - f(x')| \) and let \( K_1 = \{ (\omega_i)_{i \in I}; |\omega_i| \leq |x'_i| + |y'_i| \) for all \( i \) in \( I \). Then \( K_1 \) is a compact subset of \( C^I \), \( x' \) and \( x' + y' \) belong to \( K_1 \). Now choose \( \alpha \in \Gamma \) such that
\[
||f - f_\alpha||_{K_1} \leq \delta/8.
\]
Since \( f_\alpha \) is holomorphic it depends on a finite number of variables \( I_1 \). Let
\[
\overline{x}'_i = \begin{cases} 
  x'_i & \text{if } i \in I_1 \cup J_1, \\
  0 & \text{otherwise}
\end{cases}
\]
and let
\[
\overline{y}'_i = \begin{cases} 
  y'_i & \text{if } i \in I_1 \cup J_1, \\
  0 & \text{otherwise}
\end{cases}
\]
Then \( \overline{x}' \) and \( \overline{x}' + \overline{y}' \) belong to \( K_1 \) and since \( x' \) and \( \overline{x}' \) agree on \( I_1 \) and \( y' \) and \( \overline{y}' \) agree on \( I_1 \) we have
\[
f_\alpha(x' + y') = f_\alpha(\overline{x}' + \overline{y}') \quad \text{and} \quad f_\alpha(x') = f_\alpha(\overline{x}').
\]
Hence
\[
|f(\overline{x}' + \overline{y}') - f(\overline{x}')| \geq |f(x' + y') - f(x')| - |f(\overline{x}' + \overline{y}') - f_\alpha(\overline{x}' + \overline{y}')| - |f_\alpha(x' + y') - f_\alpha(\overline{x}' + \overline{y}')| \geq \delta/2
\]
both \( \overline{x}' \) and \( \overline{y}' \) have their support in \( I_1 \cup J_1 \) and \( \overline{y}'_i = 0 \) if \( i \in J_1 \). Let \( J_2 = I_1 \cup J_1 \). Using the same method we can find a finite subset \( I_2 \) of \( I \) and vectors \( \overline{x}^2 \) and \( \overline{y}^2 \) with support in \( I_2 \cup J_2 \) such that
\[
f(\overline{x}^2 + \overline{y}^2) \neq f(\overline{x}^2) \quad \text{and} \quad \overline{y}^2_i = 0 \quad \text{if } i \in J_2.
\]
By induction we can generate an increasing sequence of finite subset of \( I, (J_n)_{n=1}^{\infty} \), and sequences of vectors \( (\overline{x}^n) \) and \( (\overline{y}^n) \) in \( C^I \) such that
\[
\begin{align*}
(i) & \quad f(\overline{x}^n + \overline{y}^n) \neq f(\overline{x}^n) \quad \text{for all } n, \\
(ii) & \quad \overline{x}^n \text{ and } \overline{y}^n \text{ have their support in } J_{n+1}, \\
(iii) & \quad \overline{y}^n_i = 0 \quad \text{if } i \in J_n.
\end{align*}
\]
Let \( J = U_n J_n \). We now restrict all \( f_\alpha \) and \( f \) to the Fréchet space \( C^J \times 0^{I \setminus J} \). Since \( f_\alpha|_{C^I \times 0^{I \setminus J}} \rightarrow f|_{C^I \times 0^{I \setminus J}} \) uniformly on compact sets it follows that \( f|_{C^I \times 0^{I \setminus J}} \) is holomorphic and hence depends on a finite number of variables in \( J \). This is impossible, however, by (i), (ii) and (iii), since any finite subset of \( J \) is contained in some \( J_n \). This completes the proof.
4. COMPLETENESS FOR THE $\tau_w$ and $\tau_6$ TOPOLOGIES

Proposition 4. $(H(C^I); \tau_w)$ and $(H(C^I); \tau_6)$ are complete locally convex spaces.

Proof. By propositions 1 and 3 it suffices to show that $(P(n(C^I)), \tau_w)$ is complete for all $n$. Let $(P_{\alpha})_{\alpha \in \Gamma}$ denote a $\tau_w$ Cauchy net in $(P(n(C^I)), \tau_w)$. Since $\tau_w \geq \tau_0$, proposition 3 implies that there exists a polynomial $P$ in $P(n(C^I))$ such that $P_{\alpha} \to P$ in $(P(n(C^I)), \tau_0)$ as $\alpha \to \infty$. Let $p$ denote a $\tau_w$-continuous semi-norm on $P(n(C^I))$. By proposition 2 we may suppose in the following argument that

$$p = \sup_{\beta \in B} p_\beta$$

where each $p_\beta$ is a $\tau_0$-continuous semi-norm and $B$ is some indexing set. Given $\epsilon > 0$ there exists $\alpha_0 \in \Gamma$ such that $p(P_{\alpha_1} - P_{\alpha_2}) \leq \epsilon$ for all $\alpha_1, \alpha_2 \geq \alpha_0$. Hence $p_\beta(P_{\alpha_1} - P_{\alpha_2}) \leq \epsilon$ for all $\beta \in B$ and all $\alpha_1, \alpha_2 \geq \alpha_0$. Since $p_\beta$ is $\tau_0$-continuous and $P_{\alpha} \to P$ as $\alpha \to \infty$ in the compact open topology we have

$$p_\beta(P_{\alpha} - P) \leq \epsilon \text{ for all } \beta \in B \text{ and all } \alpha \geq \alpha_0.$$ 

Hence

$$p(P_{\alpha} - P) = \sup_{\beta \in B} p_\beta(P_{\alpha} - P) \leq \epsilon$$

and $P_{\alpha} \to P$ in $(P(n(C^I)), \tau_w)$ as $\alpha \to \infty$. This completes the proof.

5. BALANCED DOMAINS IN $C^I$

If $U$ is a balanced open subset of a locally convex space $E$ and $\tau$ is a locally convex topology on $H(U)$ then $(H(U), \tau)$ is said to be T.S. (Taylor series) complete if for any sequence $(P_n)_{n=0}^{\infty}, P_n \in P(n)E$ all $n, \sum_{n=0}^{\infty} p(P_n) < \infty$ for every $\tau$-continuous seminorm $p$ implies $\sum_{n=0}^{\infty} P_n \in H(U)$ [2, p. 128]. The hypothesis in proposition 1 are used to show that $(H(E), \tau_0)$ is T.S. complete and from this it follows that $(H(E), \tau_w)$ and $(H(E), \tau_6)$ are also T.S. complete. Now, if $U$ is a balanced open subset of $C^I$ and $(P_n)_{n=0}^{\infty}$ is a sequence of continuous polynomials, $P_n \in P(n)E$ all $n$, then since each polynomials only depends on finitely many variables the sequence $(P_n)_{n=0}^{\infty}$ only depends on countably many variables and hence, using the fact that $C^N, N$ countable, is a Fréchet space we see that $(H(U), \tau_0)$ is T.S. complete for any balanced open subset $U$ of $E$. Propositions 3 and 4 thus imply the following.
Proposition 5. If $U$ is a balanced domain in $C^I$ then $(H(U), \tau)$ is complete for $\tau = \tau_0, \tau_\infty$ and $\tau_S$. 
REFERENCES
