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NEWTON'S OBSERVATIONS ABOUT THE FIELD OF A UNIFORM THIN SPHERICAL SHELL

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Dedicated to the memory of Professor Gottfried M. Köthe

1. INTRODUCTION

Newton observed [(1)] that with his law of gravitation the field outside a thin uniform spherical surface is the same as that of a particle at the center, having the same mass as the sphere. He also showed that the field is 0 inside a sphere [(2)] [(3)].

We find below all laws of central force having one or the other of these properties. We also formulate the analogous question in two and four (rather than just three) dimensions. We present the findings in a table where, for example, the entry $\frac{(1, r^3)}{r^2}$ in the place labelled «*outside, 3*» says that the fields that satisfy the condition outside every sphere in E^3 are governed by laws which have any linear combination of 1 and r^3 in the numerator, and r^2 in the denominator [(4)].

Dimension	*Outside*	*Inside*
2	$\frac{(1, r^2)}{r}$	$\frac{(1, \log r)}{r}$
3	$\frac{(1, r^3)}{r^2}$	$\frac{(1, r^2)}{r^2}$
4	$\frac{(1, r^2)}{r^3}$	$\frac{(1, r)}{r^3}$

2. THE FIELD CREATED BY A THIN UNIFORM SPHERE IN E^3

We assume a central force. A central force always has a potential. We assume a potential of the form $g(r)/r$, where now this function g takes the place of Newton's gravitational constant.

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(1) I. Newton, *Philosophiae Naturalis Principia Mathematica*, London, 1687, Theorem XXXI.

(2) *Op. cit.*, Theorem XXX.

(3) A.S. Ramsey, *Newtonian Attraction*, Cambridge. Univ. Press, 1940. See p. 46 *et seqq.* By "sphere" we mean the surface of a solid ball. We suppose it has a constant surface density and a total mass 4π.

(4) The results for E^3 can be, or are (for the interior problem) obtained from Ramsay, as noted below. I am grateful to N. Grossman for this reference.

Suppose a sphere S or radius a and center at the origin has a uniform surface density making the total mass equalt to 4π. Then the potential V at the point $(0,0,b)$ will be given by a constant plus

$$(2.1)\qquad V = \int\int_S \frac{g(s)}{s}\sin\theta d\theta d\phi,$$

$$= 2\pi\int_0^\pi \frac{g(s)}{s}\sin\theta d\theta,$$

where at $s^2 = a^2 + b^2 - 2ab\cos\theta$.

3. THE EXTERIOR FIELD IN E^3

We hypothesize that $V = 4\pi g(b)/b$ *plus a constant,* when $b > a$. Let this constant be called $4\pi K(a)$, as it may depend on the radius a.

The hypothesis takes the form [(5)]

$$(3.1)\qquad \frac{g(b)}{b} = \frac{1}{2ab}\int_{b-a}^{a+b} g(s)ds + K(a).$$

If we multiply this by $2ab$, and differentiate twice with respect to b, thrice with respect to a, and then set $a = 0$, we get $g''''(b) = 0$. [(6)] This means that g has to be of the form $g(r) = c_0 + c_1 r + c_2 r^2 + c_3 r^3$. If we insert this into 3.1, we find that it satisfies 3.1 only if $c_2 = 0$.

4. THE INTERIOR FIELD

In this case $b < a$ so we just change the lower limit in 3.1 to $a - b$, and erase the term $g(b)$. We multiply by $2ab$ and differentiate once, giving

$$(4.1)\qquad 0 = g(a+b) + g(a-b) + 2aK(a) = 0.$$

By setting $b = 0$ we see what $K(a)$ is, and put that back into 4.1. Differentiating twice more with respect to b and then setting $b = a$ gives $g''(a) = 0$. We conclude [(7)] that $g(r) = c_0 + c_1 r$.

A look at the proof shows to obtain this result, one need only assume that the field vanish on a neighborhood of the center of the sphere.

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(5) We assume that g is continuous. By the immediately following integral representation, it will be differentiable to all orders.

(6) This same result can be obtained from Ramsey, *op. cit.* p. 65, ex. 57, by expanding his $\phi(r + a)$ in a Taylor series up to and including a^7. Our g is ϕ'''.

(7) Another way of showing this is presented by Ramsay, *op. cit.*, p. 66, ex. 49.

5. GRAVITY IN TWO DIMENSIONS

In the two dimensional case it seems to be easier to deal with the force rather that with potentials.

It is analogous to Newton's law that two particles in the plane are repelled by a force $G(r^2)/r$ along their line of centers. Suppose a circle S of radius a and center at the origin has a uniform surface density making the total mass equal to 2π. Then the gravitational field caused by S at the point $(b, 0)$ will have an x-component X given by

$$X = \int_0^{2\pi} \frac{G(s^2)(b^2 + s^2 - a^2)}{2bs^2} d\theta. \tag{5.1}$$

where $s^2 = a^2 + b^2 - 2ab\cos\theta$.

6. THE FIELD OUTSIDE THE CIRCLE

Define q and u by $a = bq$ and $s = bu$. Then $q < 1$. We seek those G for which X has the value $2\pi G(b^2)/b$, that is, for which

$$G(b^2) = \frac{1}{2\pi}\int_0^{2\pi} \frac{G(b^2u^2)(1 - q\cos\theta)}{1 + q^2 - 2q\cos\theta} d\theta, \tag{6.1}$$

where we have used, in some places, the relation $u^2 = q^2 + 1 - 2q\cos\theta$.

We now use u as the variable of integration, and obtain

$$G(b^2) = \int_{1-q}^{1+q} G(b^2u^2)K(u, q)\,du. \tag{6.2}$$

It is clear that K is continuous, nonnegative and actually positive in a neighborhood of 1. Further details will not be needed.

Let ξ denote the function whose value at x is x, so that for example e^{ξ} is the exponential function.

Let us define an integral transform

$$T(h; q)(x) = \int_{1-q}^{1+q} h(x + \log u)K(u, q)\,du. \tag{6.3}$$

Then our equation 6.2 amounts to requiring $T(h; q) = h$ for the function $h = G(e^{2\xi})$. Given h define G by $h\left(\frac{\log\xi}{2}\right)$. Then

$$T(h; q)(x) = \frac{1}{2\pi}\int_0^{2\pi} \frac{G(e^{2x}u^2)(1 - q\cos\theta)}{1 + q^2 - 2q\cos\theta} d\theta. \tag{6.4}$$

Let c be real and consider $h = e^{c\xi}$. Then $G = exp(c/2, \log\xi) = \xi^{c/2}$, and $G(e^{2x}u^2) = e^{cx}u^c$.

Proposition 1. *$T(e^{c\xi};q) = \Lambda(c)e^{c\xi}$ where*

$$\Lambda(c) = \frac{1}{2\pi}\int_0^{2\pi}\frac{u^c(1-q\cos\theta)}{u^2}d\theta.$$

Thus $T(e^{c\xi};q) = e^{c\xi}$ if [8] *$c = 0$ or 2.*

The verification of this may be left to the reader.

We need another very elementary observation.

Lemma. *Let h be a smooth function of a real variable and suppose $h''(x_0) - 2h'(x_0) > 0$. Then there is a linear combination k of 1 and $e^{2\xi}$ which has the same value as h at x_0 and has $k(x) < h(x)$ for $0 < |x - x_0| < \delta$ for some positive δ.*

We now state the result about the field outside the circle.

Theorem. *If G is such that for h defined as $G(e^{2\xi})$, $T(h;q) = h$, then h is a linear combination of 1 and $e^{2\xi}$, and $G(r^2)$ is a linear combination of 1 and r^2.*

Proof. Suppose $h'' - 2h'$ is not the 0 function. Then it is not 0 at some x_0. It suffices to deal with the case mentioned in the lemma. Let k be the osculating approximation mentioned there, so $h(x_0) = k(x_0)$. Pick q so small that if $1 - q < u < 1 + q$ then $|\log u| < \delta$. Then as far as the integral transform 6.3 is concerned, $h(x)$ might as well be positively greater than $k(x)$ *everywhere,* except for x_0. Hence $T(h;q)(x_0) \geq T(k;q)(x_0)$ because K is nonnegative. But actually this inequality will be strong: $T(h;q)(x_0) > T(k;q)(x_0)$, because K is positive on a *neighborhood* of 1. Presumably $T(h;q)(x_0) = h(x_0)$, and by the earlier lemma, $T(k;q)(x_0) = k(x_0)$. Thus we have $h(x_0) > k(x_0)$ which is a contradiction.

Thus $h'' - 2h' = 0$, and the theorem is substantially proved.

7. THE FIELD INSIDE THE CIRCLE

We return to 5.1 and let $b = pa, p < 1$. Since the field inside is supposed to be 0 we get

$$0 = \int_0^{\pi}\frac{G(a^2u^2)(p-\cos\theta)}{p^2+1-2p\cos\theta}d\theta,$$

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[8] And, indeed, only if c has those values, because, as T. Liggett pointed out to me, Λ is convex. This explains why the fields admitted in the table of sec. 1 are linear combination of only *two.* One could also argue on the basis of the theorem of Choquet and Deny, if one were willing to assume *a priori* that h were bounded on one side. See G. Choquet and J. Deny, *Sur l'équation de convolution $\mu = \mu * \sigma$.* Comptes Rendues Paris 1960, **250,** 799-801. Part 1, Math. Sci.

where $u^2 = p^2 + 1 - 2p\cos\theta$. We also reduced the 2π to π.

We now integrate this equation with respect to p from $p = 0$ to $p = P$. In the resulting double integral we reverse the order of integration, and then eliminate the variable p in favor of w where $w^2 = a^2(p^2 + 1 - 2p\cos\theta)$, θ being fixed during this step. The result is

$$0 = \int_0^{\pi} \int_a^{a\sqrt{1+P^2-2P\cos\theta}} \frac{G(w^2)}{w} dw d\theta.$$

We differentiate this equation with respect to a, cancel two expressions $\sqrt{1 + P^2 - 2P\cos\theta}$ which arise and obtain

$$0 = \frac{1}{\pi} \int_0^{\pi} \{G[a^2(1 + P^2 - 2P\cos\theta)] - G[a^2]\} d\theta.$$

Therefore

(7.1) $$G(a^2) = \frac{1}{\pi} \int_0^{\pi} G[a^2(1 + P^2 - 2P\cos\theta)] d\theta$$

should be true for all $P < 1$, for the desired functions G. Defining $g(x) = G(e^{2x})$ as before, this equation takes the form

(7.2) $$h(x) = U(h, P)(x)$$

where

(7.3) $$U(h, P)(x) = \frac{1}{\pi} \int_0^{\pi} h(x + \log\sqrt{1 + P^2 - 2P\cos\theta}) d\theta.$$

It is easy to see that $U(1, P) = 1$, and that the kernel [(9)] involved in the definition of the transform U is nonnegative.

Now

$$U(\xi, P)(x) = x + \frac{1}{\pi} \int_0^{\pi} \log\sqrt{1 + P^2 - 2P\cos\theta}) d\theta.$$

This latter integral is [(10)] 0. So $U(\xi, P) = \xi$.

Reasoning as in sec. 6 about linear combinations, this time of 1 and ξ, we deduce that for and h satisfying 7.2 one must have $h'' = 0$. Thus, for the field to behave in the required way inside the circle, it is necessary and sufficient that $G(r^2) = c + k\log r$ or the law of central force be $(c + k\log r)/r$.

(9) Like the K in the last section.

(10) See Burington's tables, Fourth Edition, page 106.

8. GRAVITY IN E^4

It is appropriate to let the law of central force be $G(r^2)/r^3$. We introduce spherical coordinates $x = r\sin\theta\sin\phi\sin\psi$, $y = r\sin\theta\sin\phi\cos\psi$, $z = r\sin\theta\cos\phi$, $w = r\cos\theta$. Then the w-component of the force of a sphere of mass $2\pi^2$, with center at the origin and radius a, on a unit mass at $(0,0,0,b)$ will be

$$(8.1)\qquad W = 4\pi\int_0^\pi \frac{G(a^2+b^2-2ab\cos\theta)}{[a^2+b^2-2ab\cos\theta]^2}(b-a\cos\theta)\sin^2\theta d\theta,$$

where $s^2 = a^2+b^2-2ab\cos\theta$.

Let $G=1$. When $a>b$, the integral 8.1 is 0. When $b>a$, it is $2\pi^2/b^3$.

The proof is tedious but can be left to the reader. The formula [11] is relevant.

9. THE FIELD OUTSIDE THE SPHERE

Define a and u by $a = bq$ and $s = bu$. Then $q < 1$. We seek those G for which W has the value $2\pi^2 G(b^2)/b^3$, that is, for which

$$(9.1)\qquad G(b^2) = \frac{2}{\pi}\int_0^\pi \frac{G(b^2u^2)(1-q\cos\theta)\sin^2\theta d\theta}{(q^2+1-2q\cos\theta)^2}.$$

Here $u^2 = q^2+1-2q\cos\theta$. We now use u as a variable of integration, and obtain an equation literally like 6.2, except of course for a new kernel K. We can use 6.3 to define new T, for which 6.4 holds by virtue of 8.2. To get an analogue of Proposition 1 we must study 9.1 when $G(u^2)$ is set equal to u^c. Let

$$\frac{2}{\pi}\int_0^\pi \frac{u^c(1-q\cos\theta)\sin^2\theta d\theta}{(q^2+1-2q\cos\theta)^2} = \Lambda(c).$$

Then $T(e^{c\xi};q) = \Lambda(c)e^{c\xi}$. Moreover, $\Lambda(0)$ and $\Lambda(4) = 1$. For $c = 0$ this is essentially 8.2 for the case $a < b$. For $c = 4$, it is obvious.

Dealing with $h'' - 4h'$ as in sec. 6, we conclude that a field satisfying our conditions outside of the sphere in R^4, must have $G(r^2)$ be a linear combination of 1 and r^2.

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[11] Burington, *op. cit*, No. 213.

10. THE FIELD INSIDE THE SPHERE IN R^4

Now $b < a$. Set $b = ap$ in 8.1, and set the integral equal to 0. Let us take $\int_0^P \dots dp$, and change the order of integration. We change to a new variable in the dp integral, $w = a\sqrt{1 + p^2 - 2p\cos\theta}$. In this change, θ is held constant. There results

$$(10.1) \qquad \int_0^\pi \sin^2\theta d\theta \int_a^{a\sqrt{1+P^2-2P\cos\theta}} \frac{G(w^2)\,dw}{w^3} = 0,$$

where a factor a^2 has been discarded. We now take the partial derivative with respect to a. Introducing the variable $u = \sqrt{1 + P^2 - 2P\omega\theta}$, we can write the result as

$$G(a^2) = \frac{2}{\pi}\int_0^\pi \frac{G(a^2u^2)\sin^2\theta d\theta}{u^2}.$$

This result is analogous to 7.1. Just as we did there, we can introduce an integral transform

$$U(h, P)(x) = \frac{2}{\pi}\int_0^\pi h(x + \log u)\frac{\sin^2\theta}{u^2}d\theta.$$

For $h = e^{c\xi}$, the transform is $e^{c\xi}\Lambda(c)$ where this factor is

$$\frac{2}{\pi}\int_0^\pi \frac{u^c\sin^2\theta}{u^2}d\theta.$$

Obviously $c = 2$ makes this $= 1$. Now 8.2 says that $G = 1$ satisfies our conditions for the field inside the sphere, so $c = 0$ must also make the factor $= 1$. Now $h = e^{c\xi}$ corresponds to $G(r^2) = r$. Thus our $G(r^2)$ must be a linear combination of 1 and r.

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