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ON COMPLEMENTED SUBSPACE OF CONVERGENCE-FREE SPACES (*)

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Abstract. *The author proved in 1935 (Math. Annalen 111, 229-258) that two convergence-free spaces* λ_1 *and* λ_2 *of countable degree are isomorphic if and only if there exists a permutation of the coordinates which transforms* λ_1 *into* λ_2 *. The author gives a new exposition of this result and its consequences and formulates some unsolved problems on the structure of the complemented subspaces of spaces of countable degree.*

1. INTRODUCTION

When I came in 1929 to Otto Toeplitz as assistant he proposed that we should investigate infinite systems of linear equations whose matrix he called half finite. The underlying vector space was $\varphi \oplus \omega$, the first example of a convergence-free space different from φ and ω. Our results were published in 1931 (see [KT 1]).

The whole class of convergence-free spaces was defined in 1934 in our work on sequence spaces (see [KT 2]) and we proved some basic facts on these spaces. I was fascinated by this new class of sequence spaces and my student F. Menn and I developed this theory in our papers [M] and [K 1]. For the vast class of spaces of countable degree we were able to solve the isomorphism problem: two of these spaces are isomorphic if and only if there exists a permutation of the coordinates which transforms one space into the other.

In the last year I came back to these spaces and was able to develop their theory further (see [K 4] to [K 8]) using the results of the old paper [K 1]. But the terminology of [K 1] is no longer in use, so continuity means sequential continuity for example (which does not really matter since the spaces are bornological). But a reader interested in the results of [K 1] will have many difficulties to adapt the proofs to modern terminology. So I thought it worthwhile to give a new exposition of the results of [K 1] with some simplifications of the proof. I added some problems concerning the structure of the complemented subspaces which are the main tools of our investigations.

2. CONVERGENCE-FREE SPACES

We assume some knowledge of the theory of sequence spaces and we use the terminology of [K 2] § 30. A sequence space $\lambda \supset \varphi$ is called convergence-free if it contains with $x = (x_1, x_2, \ldots)$ every $y = (y_1, y_2, \ldots)$ with $y_i = 0$ if $x_i = 0$. The set $W \subset \mathbf{N}$ of all k with $x_k \neq 0$ is called the support of x. It is clear that a convergence-free space λ is determined by the class $\mathscr{W}$ of all supports W of its elements. Such a support is also called a W-set of λ. We write $\lambda = \lambda(\mathscr{W})$.

(*) This is G. Köthe's last article.

A class $\mathscr{W}$ of subsets W of $\mathbf{N}$ defines a convergence-free space $\lambda(\mathscr{W})$ if $\mathscr{W}$ has the following properties: a) $\mathscr{W}$ contains all the finite subsets of $\mathbf{N}$, b) with W is every subset of W in $\mathscr{W}$, c) $W_1 \cup W_2 \in \mathscr{W}$ if W_1 and W_2 are in $\mathscr{W}$.

If $\lambda(\mathscr{W})$ is convergence-free we call a set $F \subset \mathbf{N}$ an F-set of $\lambda(\mathscr{W})$ if $F \cap W$ is finite for all $W \in \mathscr{W}$. The class $\mathscr{F}$ of all F-set of $\lambda(\mathscr{W})$ has the properties a), b) and c) and defines therefore a convergence-free space $\lambda(\mathscr{F})$ which is the $\propto$ -dual $\lambda(\mathscr{W})^x$ of $\lambda(\mathscr{W})$. If we write $\mathscr{F} = (\mathscr{W})^x$ then we have $\lambda(\mathscr{W})^x = \lambda(\mathscr{W}^x)$. That a convergence-free space is perfect, $\lambda(\mathscr{W})^{xx} = \lambda(\mathscr{W})$, can also be expressed as $\mathscr{W}^{xx} = \mathscr{W}$.

It is natural to include the finite dimensional vector spaces in the class of perfect convergence-free spaces. The classical examples of convergence-free spaces are φ, ω, $\varphi \oplus \omega$. Examples of convergence-free spaces which are not perfect are given on p. 127 of [K 8].

In the following λ, μ will be convergence-free spaces, we will omit the defining class $\mathscr{W}$ in the notation and only speak of the defining W-sets of λ resp. μ. That λ is perfect means that λ is complete for the Mackey topology $T_k[\lambda^x]$ (cf. [K 2] § 30, 5.(9)) and obviously $\lambda[T_k(\lambda^x)]' = \lambda^x$, the $\propto$ -dual of a convergence-free space is its dual.

Before saying more on the topologies on a convergence-free space we introduce some notation.

If M is a subset of $\mathbf{N}$ then the sectional subspace λ_M of λ consists of all $x_M = (x_j)$, $j \in M$, $x \in \lambda$. λ_M is again convergence-free. Obviously $(\lambda_M)^x = (\lambda)^x_M$. If M is an infinite W-set of λ, λ_M can be identified with ω, if M is an infinite F-set of λ, λ_M can be identified with φ.

If x is an element of λ then the normal cover $\{x\}^n$ of x consists of all elements y with $|y_i| \leq |x_i|$, $i = 1, 2, \ldots$. Since λ is convergence-free $\{x\}^n$ is a subset of λ. The normal topology $T_n(\lambda^x)$ on λ is the topology of uniform convergence on the normal covers $\{u\}^n$ of the elements u of λ^x and is also defined by the seminorms $p_u(x) = \sum_{i=1}^{\infty} |u_i||x_i|$, $u \in \lambda^x$. These sums are always finite since the supports of u and x have a finite intersection.

The following simple proposition is fundamental (for a proof see [KT 2], p. 221):

(1) *Let $\lambda[T_n]$ be a perfect convergence-free space equipped with the normal topology $T_n(\lambda^x)$. A subset M of λ is bounded if and only if the union W of the supports of the elements of M is a W-set of λ and M is bounded in every coordinate of W.*

This means that M is contained in λ_W isomorphic to ω and M is bounded in ω. Hence M is contained in the normal hull of an element, which is compact. Therefore a bounded subset M of a perfect λ is always relatively compact. Since λ^x is always perfect this implies:

(2) *On a convergence-free space λ the topologies $T_n(\lambda^x)$, the Mackey topology $T_k(\lambda^x)$ and the strong topology $T_b(\lambda^x)$ coincide.*

If in the following nothing is said on the topology of a convergence-free space λ it is tacitly assumed that it has the normal topology. (2) implies that every perfect convergence-free space is reflexive. We add that on p. 128 of [K 4] the nuclearity of every convergence-free space is proved. The name «convergence-free» is justified as the following remark shows:

(3) *A perfect sequence space* λ *is convergence-free if and only if for every* $x \in \lambda$ *and every* $u \in \lambda^x$ *the sum* $ux = \sum_{i=1}^{\infty} u_i x_i$ *contains only finitely many members* $u_i x_i \neq 0$.

Proof. We know that the condition is necessary. Consequently we suppose λ to be a perfect sequence space and that for every $x \in \lambda$ and every $u \in \lambda^x$ ux is a finite sum. Then uy is a finite sum for every y with the same support as x. Hence $y \in \lambda^{xx} = \lambda$ and λ is convergence-free.

I proved in [K 1], § 4, the following theorem:

(4) *Let* λ *be a perfect convergence-free space. If the perfect sequence space* μ *is isomorphic to* λ , $\lambda \cong \mu$, *then* μ *is convergence-free too.*

The following proof uses the ideas of the original proof in [K 1]. Let A be the isomorphism of λ onto μ, B its inverse, then $AB = I_\mu$, $BA = I_\lambda$, where I is the identity map on μ resp. λ and A and B can be written as infinite matrices (a_{ik}) and (b_{ik}) and I is the unit matrix (see [KT 2] § 6 or [K 2], p. 80/1.

If M, N are subsets of $\mathbf{N}$, then the matrix $A_{MN} = (a_{mn})$ with $m \in M$, $n \in N$, is called a section of A.

We assume now $\lambda \cong \mu$ and that μ is not convergence-free. (3) implies the existence of $y^o \in \mu$, $v^o \in \mu^x$, such that $v^o y^o = \sum_{i=1}^{\infty} v_i^o y_i^o$ converges but is not a finite sum. Since μ is normal we may assume that on an infinite subset K of $\mathbf{N}$ the v_i^o and the y_i^o are positive and zero outside K. We write $y^o = \sum_{k \in K} y_k^o e_k$, e_k the k-th unit vector. The $y_k^o e_k$ lie in the normal hull of y^o hence they constitute a bounded set C in μ. Hence $B(C)$ is bounded in λ.

It follows from (1) that the supports of the $b_k = Be_k$ lie in a W-set W of λ and are therefore contained in the sectional subspace $\lambda_W \cong \omega$ of λ.

The $b_k = Be_k$, $k \in K$, are columns of the matrix B and have coordinates $\neq 0$ only in the section B_{WK} of B. From $AB = I_\mu$ it follows:

(5) $$A_{KW} B_{WK} = I_K ,$$

where I_K is the identity map of the sectional subspace μ_K , $A_{KW} \in L(\lambda_W, \mu_K)$ and $B \in L(\mu_K, \lambda_W)$.

Since $\lambda_W \cong \omega$, the rows a_k of A_{KW} are in $\lambda_W^x \cong \varphi$, therefore finite. Hence A_{KW} is a rowfinite matrix. The a_k , $k \in K$ are linearly independent since from $a_{k_0} = \sum_{i=1}^{n} \alpha_i a_{k_i}$ and (5) would follow

$$1 = a_{k_0} b_{k_0} = \sum_{i=1}^{n} \alpha_i a_{k_i} b_{k_0} = 0 .$$

Let $a_{q_1}, a_{q_2}, \ldots$ be a subsequence of the a_k, $k \in K$, of strictly increasing length. Then the linear equations $a_{q_1} x = c_{q_1}$, $a_{q_2} x = c_{q_2}, \ldots$, have a common solution $x \in \lambda_W$ for arbitrary chosen positive c_{q_i}, $i = 1, 2, \ldots$ This means $A_{KW} x = c$, where c has the coordinates c_{q_i} on $Q = \{q_1, q_2, \ldots\} \subset K$. We may choose the c_{q_i} so that $\sum_{i=1}^{\infty} v^o_{q_i} c_{q_i}$ diverges. This means $v^o c_Q$ diverges, where c_Q is the restriction of c onto Q. But c_Q (with added zeros) is in μ and this contradicts $v^o \in \mu^x$.

3. SPACES OF COUNTABLE DEGREE

We define a class $\mathscr{C}$ of perfect convergence-free spaces in the following way:

a) The one-dimensional space $\lambda = K$ is in $\mathscr{C}$, where K is the real or complex field,

b) $\lambda_1 \oplus \lambda_2$ is in $\mathscr{C}$ if λ_1 and λ_2 are in $\mathscr{C}$,

c) the topological product $\prod_{n=1}^{\infty} \lambda_n$ and the topological direct sum $\bigoplus_{n=1}^{\infty} \lambda_n$ are in $\mathscr{C}$ if the λ_n are in $\mathscr{C}$.

We call $\mathscr{C}$ the class of spaces of countable degree.

φ and ω are the first infinite dimensional spaces in $\mathscr{C}$ and obviously their duals. From b) and c) follows that with λ also λ^x is in $\mathscr{C}$ and that all spaces in $\mathscr{C}$ are perfect and convergence-free.

Instead of $\bigoplus_{n=1}^{\infty} \lambda_n$, $\lambda_n = \lambda$, we will write $\bigoplus_{\mathbb{N}} \lambda$ or $\varphi\lambda$; instead of $\prod_{n=1}^{\infty} \lambda_n$, $\lambda_n = \lambda$, we will write $\prod_{\mathbb{N}} \lambda$ or $\omega\lambda$.

(1) *$\mathscr{C}$ contains the following spaces of infinite dimension:*

$$\sigma_1 = \varphi, \ \sigma_1^x = \omega, \ \sigma_1 \oplus \sigma_1^x = \varphi \oplus \omega,$$

$$\sigma_\alpha = \bigoplus_{\beta<\alpha} \sigma_\beta, \ \sigma_\alpha^x = \prod_{\beta<\alpha} \sigma_\beta^x, \ \sigma_\alpha \oplus \sigma_\alpha^x, \ \alpha \ \textit{a limit ordinal},$$

$$\sigma_\alpha = \bigoplus_{\mathbb{N}} \sigma_{\alpha-1}^x = \varphi\sigma_{\alpha-1}^x, \ \sigma_\alpha^x = \prod_{\mathbb{N}} \sigma_{\alpha-1} = \omega\sigma_{\alpha-1}, \ \sigma_\alpha \oplus \sigma_\alpha^x, \ \alpha \ \textit{not a limit ordinal},$$

where α is any countable ordinal number.

σ_α and σ_α^x are called the simple, $\sigma_\alpha \oplus \sigma_\alpha^x$ the composite normal forms of degree α. A finite dimensional space has degree zero.

Our intention now is to show that every infinite dimensional $\lambda \in \mathscr{C}$ can be permuted in one of these normal forms.

We use the word «permutation» in a more general sense than the usual one which means a permutation of the numbers of $\mathbb{N}$. If we look at $\sigma_2 = \bigoplus_{\mathbb{N}} \omega = \varphi\omega$, then the natural way to describe its elements x is to write them as sequences of sequences and so to denote their coordinates with indices of $\mathbb{N} \times \mathbb{N}$. Obviously we may write them also as sequences by writing $\mathbf{N}$ as the union $\mathbf{N} = \bigcup_{i=1}^{\infty} \mathbf{N}_i$ of infinitely many disjoint infinite subsets. The passage from the first representation of $\varphi\omega$ to the second one is here also called a permutation.

It is important to realize that the elements of the spaces listed in (1) may all be written as simple sequences but their natural representation is given by iterated sequences and a permutation of one representation into the other will mean a rather complicated rearrangement of the coordinates.

We will write $\lambda \approx \mu$ if we obtain μ from λ by a permutation of the coordinates in the above sense. Such a permutation is a special kind of isomorphism between the spaces.

(2) *If* λ *is any space of (1) then* $\lambda \oplus \lambda \approx \lambda$.

This is obvious for φ, ω, $\varphi \oplus \omega$, so we can use transfinite induction. Let first α be a limit ordinal. Then

$$\sigma_\alpha = \bigoplus_{\beta<\alpha} \sigma_\beta \approx \bigoplus_{\beta<\alpha} \left(\sigma_\beta \oplus \sigma_\beta\right) \approx \bigoplus_{\beta<\alpha} \sigma_\beta \oplus \bigoplus_{\beta<\alpha} \sigma_\beta = \sigma_\alpha \oplus \sigma_\alpha.$$

If α is not a limit ordinal, the proof is similar, so (2) is true for all σ_α. By duality we get $\sigma_\alpha^x \oplus \sigma_\alpha^x \approx \sigma_\alpha^x$, finally $(\sigma_\alpha \oplus \sigma_\alpha^x) \oplus (\sigma_\alpha \oplus \sigma_\alpha^x) \approx \sigma_\alpha \oplus \sigma_\alpha^x$.

(3) *If* λ, μ *are normal forms of degree* α *resp.* β *and if* $\alpha < \beta$ *then* $\lambda \oplus \mu \approx \mu$. *If* $\alpha = \beta$ *and* $\lambda \neq \mu$ *then* $\lambda \oplus \mu$ *is permutable in a normal form of degree* α.

The second statement is a trivial consequence of (2). To prove the first statement we use transfinite induction on β. For $\beta = 1$ nothing is to prove. Again it will be sufficient to prove (3) for $\mu = \sigma_\beta$. From $\lambda \oplus \sigma_\beta \approx \sigma_\beta$ follows: $\lambda^x \oplus \sigma_\beta^x \approx \sigma_\beta^x$ and λ^x represents with λ all normal forms of degree $\alpha < \beta$. Finally $\lambda \oplus (\sigma_\alpha \oplus \sigma_\alpha^x) \approx \sigma_\alpha \oplus \sigma_\alpha^x$.

i) Let first β be a limit ordinal. From $\alpha < \beta$ follows $\alpha + 1 < \beta$ and $\lambda \oplus \sigma_{\alpha+1} \approx \sigma_{\alpha+1}$ by assumption. Hence in $\sigma_\beta = \bigoplus_{\gamma<\beta} \sigma_\gamma$ we may replace $\sigma_{\alpha+1}$ by $\lambda \oplus \sigma_{\alpha+1}$ which implies $\lambda \oplus \sigma_\beta \approx \sigma_\beta$.

ii) If β is not a limit ordinal then $\beta = \delta + 1$. If $\alpha < \delta$ then by assumption $\lambda \oplus \sigma_\delta^x \approx \sigma_\delta^x$ and $\sigma_\beta = \sigma_{\delta+1} = \bigoplus_{\mathbf{N}} \sigma_\delta^x \approx \lambda \oplus \bigoplus_{\mathbf{N}} \sigma_\delta^x \approx \lambda \oplus \sigma_\beta$.

iii) We treat now the case that λ has the degree δ. We assume first that δ is a limit ordinal. Now for every $\gamma < \delta$ we have $\sigma_\gamma^x \oplus \sigma_\delta \approx \sigma_\delta$, hence $\sigma_\delta^x \oplus \sigma_\gamma \approx \sigma_\delta^x$. It follows $\sigma_{\delta+1} = \bigoplus_{\mathbf{N}} \sigma_\delta^x \approx \bigoplus_{\gamma<\delta}(\sigma_\delta^x \oplus \sigma_\delta) \approx \bigoplus_{\mathbf{N}} \sigma_\delta^x \oplus \bigoplus_{\gamma<\delta} \sigma_\gamma = \sigma_{\delta+1} \oplus \sigma_\delta$.

This settle the case $\lambda = \sigma_\delta$. For $\lambda = \sigma_\delta^x$ we have $\sigma_{\delta+1} = \bigoplus_{\mathbf{N}} \sigma_\delta^x \approx \sigma_\delta^x \oplus \sigma_{\delta+1}$ and $(\sigma_\delta \oplus \sigma_\delta^x) \oplus \sigma_{\delta+1} \approx \sigma_{\delta+1}$ follows immediately.

If δ is not a limit ordinal we have by assumption $\sigma_\delta^x \approx \sigma_\delta^x \oplus \sigma_{\delta-1}$ and therefore $\sigma_{\delta+1} = \bigoplus_{\mathbf{N}} \sigma_\delta^x \approx \bigoplus_{\mathbf{N}}(\sigma_\delta^x \oplus \sigma_{\delta-1}^x) \approx \sigma_{\delta+1} \oplus \sigma_\delta$, which settles the case $\lambda = \sigma_\delta$.

Again $\sigma_{\delta+1} \approx \sigma_\delta^x \oplus \sigma_{\delta+1}$ and $(\sigma_\delta \oplus \sigma_\delta^x) \oplus \sigma_{\delta+1} \approx \sigma_{\delta+1}$ are trivial.

Remark. If λ is of finite dimension then $\lambda \oplus \mu \approx \mu$ for every μ of (1).

Using (2) and (3) we are able to prove (Hauptsatz 1 in [K 1]):

(4) *Every infinite dimensional space of $\mathscr{C}$ is permutable in one of the normal forms of (1).*

Proof. $\mathscr{C}$ is generated by the two operations a) and b). We have to show that by performing these operations we always obtain spaces permutable in normal forms.

If λ and μ are permutable in normal forms then by (2) and (3) this is true also for $\lambda \oplus \mu$. Secondly we have to show that if all λ_i, $i = 1, 2, \ldots$, are permutable in normal forms this is true also for $\bigoplus_{i=1}^{\infty} \lambda_i$ and $\prod_{i=1}^{\infty} \lambda_i$. Using duality one sees that it is sufficient to prove this for $\bigoplus_{i=1}^{\infty} \lambda_i$. So (4) will follow from

(5) *Let $\lambda = \bigoplus_{i=1}^{\infty} \nu_{\beta_i}$ with ν_{β_i} a normal form of degree β_i or finite dimensional and let α be the smallest ordinal with $\beta_i < \alpha$ for all i. Then λ is permutable in a normal form of degree $\leqq \alpha$.*

We prove this by transfinite induction on α. A finite dimensional ν_{β_i} has the degree $\beta_i = 0$, then obviously (5) is true for $\alpha = 1$ since every infinite direct sum of finite dimensional spaces is finite dimensional or equal to φ.

We have to look at two cases.

(i) In $\lambda = \bigoplus_{i=1}^{\infty} \nu_{\beta_i}$ there is a greatest $\beta_i = \beta$, so that $\alpha = \beta + 1$. Then

$$\lambda \approx \bigoplus_{\beta_i < \beta} \nu_{\beta_i} \oplus \bigoplus_{\beta_i = \beta} \nu_{\beta_i}. \tag{6}$$

By assumption the first sum in (6) is permutable in a normal form of degree $\leqq \beta$. If the second sum in (6) is finite, then it follows from (2) and (3) that it is permutable in a normal form of degree β and therefore λ is permutable in a normal form of degree $\leqq \beta$.

Next we assume that the second sum in (5) is infinite. It consists of finite or infinite sums of the spaces σ_β, σ_β^x, $\sigma_\beta \oplus \sigma_\beta^x$. Now $\bigoplus_{\mathbf{N}} \sigma_\beta \approx \sigma_\beta$, $\bigoplus_{\mathbf{N}} \sigma_\beta^x = \sigma_{\beta+1}$, and using (2) and (3) we see that $\bigoplus_{\beta_i = \beta} \nu_{\beta_i}$ is permutable in a normal form of degree β or $\beta + 1$ and (5) is proved in the case (i).

(ii) The second case will be $\lambda = \bigoplus_{i=1}^{\infty} \nu_{\beta_i}$ where α is a limit ordinal. (4) will be proved if we can show that in this case $\lambda \approx \sigma_\alpha$.

A space ν_{β_i} may be finite dimensional or σ_{β_i} or $\sigma_{\beta_i}^x$ or $\sigma_{\beta_i} \oplus \sigma_{\beta_i}^x$. As in (i) we can get rid of the finite dimensional spaces and permute λ in such a way that for any ordinal β_i there is only one σ_{β_i} or one $\sigma_{\beta_i}^x$ or one $\sigma_{\beta_i} \oplus \sigma_{\beta_i}^x$ in the permuted space.

It follows that we have to prove $\lambda \approx \sigma_\alpha$ only for spaces of the form $\lambda = \bigoplus_{i=1}^\infty \sigma_{\beta_i}$ and $\lambda = \bigoplus_{i=1}^\infty \sigma_{\beta_i}^x$ with different β_i with supremum α.

a) We treat first the case $\lambda = \bigoplus_{i=1}^\infty \sigma_{\beta_i}$. We note

$$\bigoplus_{\beta \leqq \gamma} \sigma_\beta \approx \sigma_\gamma \quad \textit{for every ordinal } \gamma. \tag{7}$$

If γ is a limit ordinal (7) means by the definition of σ_γ in (1) that $\sigma_\gamma \oplus \sigma_\gamma = \sigma_\gamma$. If $\gamma = \gamma_0 + n$, γ_0 a limit ordinal, (7) follows immediately from (7) for γ_0 by using (2) and (3). In the general case $\lambda = \bigoplus_{i=1}^\infty \sigma_{\beta_i}$, the β_i constitute a subset of the well ordered set of all $\beta < \alpha$ and as such can be well ordered in $\gamma_1 < \gamma_2 < \ldots \gamma_\delta < \ldots$. To prove $\lambda \approx \sigma_\alpha$ it will be sufficient to fill up the gaps between two consecutive γ_δ which is possible using

$$\bigoplus_{\delta < \beta \leqq \gamma} \sigma_\beta \approx \sigma_\gamma. \tag{8}$$

We prove (8) by transfinite induction on γ. For $\gamma = \delta + 1$ (8) follows from (3). If γ is not a limit ordinal we have by assumption $\sigma_\gamma \approx \sigma_{\gamma-1} \oplus \sigma_\gamma \approx \bigoplus_{\delta<\beta\leqq\gamma-1} \sigma_\beta \oplus \sigma_\gamma$ which is (8). If γ is a limit ordinal then by using (3) and (7) we obtain

$$\sigma_\gamma \approx \sigma_\gamma \oplus \sigma_\gamma \approx \bigoplus_{\beta \leqq \gamma} \sigma_\beta \approx \bigoplus_{\beta \leqq \delta} \sigma_\beta \oplus \bigoplus_{\delta < \beta \leqq \gamma} \sigma_\beta \approx \sigma_\delta \oplus \bigoplus_{\delta < \beta \leqq \gamma} \sigma_\beta \approx \bigoplus_{\delta < \beta \leqq \gamma} \sigma_\beta$$

and (8) is proved.

b) It remains to prove $\lambda = \bigoplus_{i=1}^\infty \sigma_{\beta_i}^x \approx \sigma_\alpha$ where the limit ordinal α is the supremum of the β_i. Corresponding to (8) we have for a limit ordinal α

$$\bigoplus_{\beta<\alpha} \sigma_\beta^x \approx \sigma_\alpha \quad \text{and} \quad \bigoplus_{\beta \leqq \alpha} \sigma_\beta^x \approx \sigma_\alpha \oplus \sigma_\alpha^x. \tag{9}$$

We remark first that $\bigoplus_{\beta<\alpha} \sigma_\alpha \approx \bigoplus_{\beta<\alpha} \sigma_{\beta+1}$ and $\bigoplus_{\beta<\alpha} \sigma_\beta^x \approx \bigoplus_{\beta<\alpha} \sigma_{\beta+1}^x$. In the right sides the σ_β resp. σ_β^x, β a limit ordinal, are missing. But from $\sigma_{\beta+1} \approx \sigma_\beta \oplus \sigma_{\beta+1}$ and $\sigma_{\beta+1}^x \approx \sigma_{\beta+1}^x \oplus \sigma_\beta^x$ the two equivalences follow. This implies the first statement of (9):

$$\bigoplus_{\beta<\alpha} \sigma_\beta^x \approx \bigoplus_{\beta<\alpha} \sigma_{\beta+1}^x \approx \bigoplus_{\beta<\alpha} \left(\sigma_{\beta+1}^x \oplus \sigma_\beta\right) \approx \bigoplus_{\beta<\alpha} \sigma_{\beta+1}^x \oplus \bigoplus_{\beta<\alpha} \sigma_\beta \approx \bigoplus_{\beta<\alpha} \sigma_\beta^x \oplus$$

$$\oplus \bigoplus_{\beta<\alpha} \sigma_{\beta+1} \approx \bigoplus \left(\sigma_\beta^x \oplus \sigma_{\beta+1}\right) \approx \bigoplus_{\beta<\alpha} \sigma_{\beta+1} \approx \sigma_\alpha.$$

The second statement of (9) is a trivial consequence of the first.

The general case $\lambda = \bigoplus_{i=1}^{\infty} \sigma_{\beta_i}^{x}$ can be treated in the same way. We write again $\lambda = \bigoplus_{\delta} \sigma_{\gamma_\delta}^{x}$, $\gamma_1 < \gamma_2 < \dots \gamma_\delta < \dots$, where the γ_δ are a well ordered set of indices with supremum α. Using the result of a) and the relation $\sigma_{\gamma_{\delta+1}}^{x} \approx \sigma_{\gamma_{\delta+1}}^{x} \oplus \sigma_{\gamma_\delta}^{x}$ we have

$$\lambda \approx \bigoplus_{\delta} \sigma_{\gamma_\delta}^{x} \approx \bigoplus_{\delta} \sigma_{\gamma_{\delta+1}}^{x} \approx \bigoplus_{\delta} \left(\sigma_{\gamma_{\delta+1}}^{x} \oplus \sigma_{\gamma_\delta}\right) \approx \bigoplus_{\delta} \sigma_{\gamma_{\delta+1}}^{x} \oplus \bigoplus_{\delta} \sigma_{\gamma_\delta} \approx$$

$$\approx \bigoplus_{\delta} \left(\sigma_{\gamma_\delta}^{x} \oplus \sigma_{\gamma_{\delta+1}}\right) \approx \bigoplus_{\delta} \sigma_{\gamma_\delta} \approx \sigma_\alpha,$$

which proves (5) and also (4).

Remark. (9) shows that for a limit ordinal α one could define σ_α also as $\sigma_\alpha = \bigoplus_{\beta<\alpha} \sigma_\beta^{x}$.

An easy consequence of (4) is

(10) *Every sectional subspace λ_M of a normal form λ is permutable in a normal form of degree less or equal to the degree of λ.*

By using transfinite induction and using (5) one sees easily that (10) is true for $\lambda = \sigma_\alpha$. It follows from (1) that λ and λ^{x} have the same degree. Since $(\lambda_M)^{x} = (\lambda^{x})_M$ duality implies (10) for the $\lambda = \sigma_\alpha^{x}$ and then (10) follows for the $\lambda = \sigma_\alpha \oplus \sigma_\alpha^{x}$.

4. ALL NORMAL FORMS ARE DIFFERENT

So far we have proved that every infinite dimensional $\lambda \in \mathscr{E}$ is permutable in one of the spaces listed in 3.(1). We will now show that two different spaces of 3.(1) are not permutable into each other so that a space $\lambda \in \mathscr{E}$ is permutable in exactly one of the spaces 3.(1) and has therefore also a uniquely defined degree α.

The following exposition is very close to § 3 of [K 1].

(1) *σ_α is not permutable in a sectional subspace of σ_α^{x} and conversely σ_α^{x} is not permutable in a sectional subspace of σ_α.*

We prove this again by transfinite induction. For $\alpha = 1$ this is true since obviously φ is not permutable in a sectional subspace of ω and conversely. We assume (1) for $\beta < \alpha$.

(i) *σ_α^{x} is not permutable in a sectional subspace of σ_α.*

We prove this first for a limit ordinal α. We assume there exists a permutation P of the coordinate indices which maps $\sigma_\alpha^{x} = \prod_{\beta<\alpha} \sigma_\beta^{x}$ in a sectional subspace of $\sigma_\alpha = \bigoplus_{\beta<\alpha} \sigma_\beta$.

Now by 3.(7) $\bigoplus_{\gamma \leqq \beta} \sigma_\gamma \approx \sigma_\beta$ and by assumption σ_β^x is not permutable in a sectional subspace of σ_β. Hence there exists a coordinate index δ_β of σ_β^x whose image $\delta'_\beta = P(\delta_\beta)$ is not a coordinate index of $\bigoplus_{\gamma \leqq \beta} \sigma_\gamma$. Obviously the set M of the δ_β, $\beta < \alpha$, is a W-set of σ_α^x but its image $P(M)$ is not a W-set of σ_α because every W-set of σ_α is a W-set of a finite sum of σ_β, $\beta < \alpha$. This contradiction proves (i) for a limit ordinal α.

The proof for $\sigma_{\delta+1}^x = \prod_{\mathbf{N}} \sigma_\alpha$ and $\sigma_{\delta+1} = \bigoplus_{\mathbf{N}} \sigma_\alpha^x$ is similar.

(ii) *σ_α is not permutable in a sectional subspace of σ_α^x.*

If σ_α were permutable in $(\sigma_\alpha^x)_M$, then σ_α^x would be permutable in the sectional subspace $(\sigma_\alpha^x)_{M^x} = (\sigma_\alpha)_M$ which contradicts (i).

From (1) follows now:

(2) *No simple normal form is permutable in another simple normal form.*

This follows for σ_α and σ_α^x immediately from (1). If λ, μ are normal forms of degree $\alpha < \beta$ then by 3.(3) λ is permutable in a sectional subspace of μ^x and by (1) μ is not permutable in λ.

(3) *A simple normal form is not permutable in a composite normal form.*

Every composite normal form $\sigma_\alpha \oplus \sigma_\alpha^x$ is permutable in its dual $\sigma_\alpha^x \oplus \sigma_\alpha$ which is not true for simple normal forms.

(4) *$\sigma_\beta \oplus \sigma_\beta^x$ is not permutable in $\sigma_\alpha \oplus \sigma_\alpha^x$ for $\alpha < \beta$.*

We assume that $\sigma_\beta \oplus \sigma_\beta^x$ is permutable in $\sigma_\alpha \oplus \sigma_\alpha^x$. Then σ_β is permutable in a sectional subspace of $\sigma_\alpha \oplus \sigma_\alpha^x$. Then we have $\sigma_\beta = \nu_1 \oplus \nu_2$, where ν_1, ν_2 are permutable in a sectional subspace μ_1 resp. μ_2 of σ_α resp. σ_α^x. By 2.(10) ν_1 and ν_2 are permutable in normal forms of degree $\leqq \beta$. Using 3.(2) and 3.(3) one sees that at least one of these normal forms has to be σ_β and we have the situation that a space permutable in σ_β is a sectional subspace of σ_α or σ_α^x, that means of σ_β^x and this contradicts (1).

Collecting our results we obtain ([K 1]§ 3, Hauptsatz 2)

(5) *Every space of countable degree and infinite dimension is permutable in one of the normal forms of 3.(1) and these spaces are all different in the sense that none of them is permutable in another one.*

5. THE ISOMORPHIC CLASSIFICATION

Our aim is now to prove that our classification relative to permutations is also the classification of the spaces of $\mathscr{C}$ relative to isomorphism. The main tool for the proof is a class of complemented subspaces. We note two simple facts on complemented subspaces.

(1) *Let* $E = E_1 \oplus E_2$ *be a complementary decomposition of the locally convex space* E. *For a subspace* $F \subset E$ *with* $F \supset E_1$ *this induces a complementary decomposition* $F = E_1 \oplus E_2 \cap F$.

This is obvious.

(2) *Let* $E[T_k(E')]$ *be locally convex and reflexive and let* $E = E_1 \oplus E_2$ *be a complementary decomposition of* E. *Then* E_1 *and* E_2 *are reflexive and we have*

$$a) \quad E[T_k(E')] = E_1[T_k(E_1')] \oplus E_2[T_k(E_2')],$$

$$b) \quad E'[T_k(E)] = E_1'[T_k(E_1)] \oplus E_2'[T_k(E_2)] \cong$$

$$\cong E_2^{\perp}[T_k(E_2)] \oplus E_1^{\perp}[T_k(E_1)].$$

Proof. The given decomposition has the form $E[T_k(E')] = E_1[T_k(E')] \oplus E_2[T_k(E')]$. But $T_k(E')$ coincides with $T_k(E_1')$ on E_1 by [K 2], § 22, 5.(4), which implies a). Hahn-Banach implies $E' = E_1' \oplus E_2'$, hence b) holds.

The reflexivity of $E_1[T_k(E_1')]$ follows easily: a closed subspace of a reflexive space is semireflexive, hence $E_1[T_k(E_1')]$ is semireflexive by a) and its dual is semireflexive by b). This implies the reflexivity of E_1 by a well known proposition ([K 2], § 23, 5.(3)).

We come now to the first part of the main result of [K 1]:

(3) *Let* λ *be an infinite dimensional complemented subspace of* σ_α *(resp.* σ_α^x*) and let* λ *be isomorphic to a perfect convergence-free space* σ. *Then* σ *is permutable in* σ_α *(resp.* σ_α^x*) or into a normal form of degree* $\beta < \alpha$.

Proof. α) This is true for $\sigma_1 = \varphi$. In this case every infinite dimensional closed subspace is complemented and isomorphic to φ. We prove (3) by transfinite induction for α. We write $\sigma_\alpha = \bigoplus_{i=1}^{\infty} \nu_{\beta_i}$, where ν_{β_i} is a simple normal form of degree $\beta_i < \alpha$. In this way we have not to distinguish between α a limit ordinal or not.

By assumption $\sigma_\alpha = \lambda \oplus \mu$ and there exists an isomorphism A such that $A(\sigma) = \lambda$. Let e_i be a unit element in σ and $Ae_i = a_i$ for all i. Each a_i has a length n in σ_α which means $a_i \in \bigoplus_{i=1}^{n} \nu_{\beta_i}$ but not in $\bigoplus_{i=1}^{n-1} \nu_{\beta_i}$. We define $M_k = \{i \in \mathbb{N}, a_i \text{ of length } k\}$. We note that M_k can be infinite, finite or even empty. We have $\mathbb{N} = \bigcup_{k=1}^{\infty} M_k$ with pairwise disjoint M_k. Hence σ contains $\bigoplus_{k=1}^{\infty} M_k$, the σ_{M_k} sectional subspaces of σ. Obviously $\lambda_k = A(\sigma_{M_k})$ contains only elements of length $\leqq k$.

We show that $\sigma = \bigoplus_{k=1}^{\infty} \sigma_{M_k}$. Otherwise σ would contain an element x with a support N with $N \cap M_k$ non empty for infinitely many k. Since σ_{M_k} is convergence-free there exist unit elements e_{p_l}, p_l in M_l such that $\sum_l e_{p_l} \in \sigma$. Then $A(\sum_l e_{p_l}) = \sum_l a_{p_l}$ should be in

λ which is impossible since the partial sums of $\sum_l a_{Pl}$ have lengths going to infinity and do not converge in σ_α. Hence

$$\sigma = \bigoplus_{k=1}^{\infty} \sigma_{M_k} \quad \text{and} \quad \lambda = A(\sigma) = \bigoplus_{k=1}^{\infty} A(\sigma_{M_k}) = \bigoplus_{k=1}^{\infty} \lambda_k .$$

Now λ is complemented in σ_α, λ_k is complemented in λ, hence λ_k is complemented in σ_α. We have $\lambda_k \subset \bigoplus_{i=1}^k \nu_{\beta_i} \subset \sigma_\alpha$. It follows from (1) that λ_k is complemented in $\bigoplus_{i=1}^k \nu_{\beta_i}$, which is by 3.(3) permutable in a normal form of degree $\beta < \alpha$. By assumption (3) is true for $\beta < \alpha$, hence $A^{-1}(\lambda_k) = \sigma_{M_k}$ is a space of degree $\gamma \leqq \beta < \alpha$. It follows that $A^{-1}(\lambda) = \sigma = \bigoplus_{k=1}^\infty \sigma_{M_k}$ is permutable in σ_α or in a space of degree $< \alpha$.

β) We assume now $\sigma_\alpha^x = \lambda \oplus \mu$, λ isomorphic to σ. Then by (2) we have $\sigma_\alpha = \lambda^x \oplus \mu^x$ and λ^x is isomorphic to σ^x. Then by α) σ^x is permutable in σ_α or in a space of lesser degree. Hence σ is permutable in σ_α^x or in a normal form of degree $< \alpha$. This finishes the proof of (3).

(4) *Let λ be an infinite dimensional complemented subspace of $\sigma_\alpha^x \oplus \sigma_\alpha$ and let λ be isomorphic to a perfect convergence-free space σ; then σ is permutable in a normal form of degree $\beta < \alpha$ or in σ_α or σ_α^x or $\sigma_\alpha^x \oplus \sigma_\alpha$.*

Proof. We write $\sigma_\alpha^x \oplus \sigma_\alpha$ as $\sigma_\alpha^x \oplus \bigoplus_{i=1}^\infty \nu_i$, ν_i of degree $\beta_i < \alpha$. We have $\lambda = A(\sigma)$. An $x \in \sigma_\alpha^x \oplus \sigma_\alpha$ has length M if $x \in \sigma_\alpha^x \oplus \bigoplus_{i=1}^n \nu_i$ but not in $\sigma_\alpha^x \oplus \bigoplus_{i=1}^{n-1} \nu_i$. The $x \in \sigma_\alpha^x$ have the length 0. Let again $a_i = Ae_i$, $i = 1, 2, \ldots$, e_i the unit elements of σ. We define $M_k = \{i \in \mathbb{N}, a_i \text{ of length } k\}$. Then $\mathbb{N} = \bigcup_{k=0}^\infty M_k$, the M_k pairwise disjoint. The same argument as in the proof of (3) shows that $\sigma = \bigoplus_{k=0}^\infty \sigma_{M_k}$ and $\lambda = \bigoplus_{k=0}^\infty \lambda_k$ with $\lambda_k = A(\sigma_{M_k})$ and the λ_k are complemented in $\sigma_\alpha^x \oplus \sigma_\alpha$ and in $\sigma_\alpha^x \oplus \bigoplus_{i=1}^k \nu_i$. By 3.(3) $\sigma_\alpha^x \oplus \bigoplus_{i=1}^k \nu_i$ is permutable in σ_α^x. Hence by (3) $A^{-1}(\lambda_k) = \sigma_{M_k}$ is permutable in σ_α^x or in a space of lesser degree. Therefore σ is permutable in $\varphi\sigma_\alpha^x = \sigma_{\alpha+1}$ or a normal form of lesser degree. But if $\sigma_\alpha^x \oplus \sigma_\alpha$ were to contain a complemented subspace $\lambda \cong \sigma_{\alpha+1}$ then the dual space, which is again $\sigma_\alpha^x \oplus \sigma_\alpha$, would contain a complemented subspace $\lambda^x \cong \sigma_{\alpha+1}^x$, which is impossible since $\sigma_{\alpha+1}^x$ is not isomorphic to a complemented subspace of $\sigma_{\alpha+1}$ by (3).

(3) and (4) imply the main result of [K 1], Hauptsatz 3:

(5) *Every space of the class $\mathscr{C}$ is permutable in one of the normal forms of 3.(1) and two of these normal forms are never isomorphic.*

The first statement is contained in 4.(5). Now we assume that for $\beta < \alpha$ σ_β is isomorphic to σ_α. Since σ_β is complemented in σ_β it follows from (3) that σ_α is a sectional subspace of σ_β which contradicts 3.(10). So all σ_α are pairwise non isomorphic. The same argument shows that two σ_α^x are not isomorphic and that no σ_α is isomorphic to a σ_β^x.

Since no σ_α is isomorphic to its dual σ_α^x but $\sigma_\alpha \oplus \sigma_\alpha^x$ is isomorphic to its dual, no simple normal form is isomorphic to a composite normal form. Finally by (4) two different composite normal forms are not isomorphic.

6. COMMENTS AND PROBLEMS

We formulate our results in a different way which will relate them to ideas of Banach and Mazur. Let σ be a space of $\mathscr{E}$. We introduce the c-dimension of σ, $dim_c(\sigma)$, as the class of all normal forms of $\mathscr{E}$ to which a complemented subspace of σ is isomorphic. From 5.(3), 5.(4) and 3.(1) it follows immediately

(1) *$dim_c(\sigma_\alpha)$ resp. $dim_c(\sigma_\alpha^x)$ is the class of all normal forms of degree $\beta < \alpha$ and σ_α resp. σ_α^x.*

Obviously the c-dimension is invariant for isomorphisms and two spaces of $\mathscr{E}$ are isomorphic if and only if they have the same c-dimension. $\lambda \in \mathscr{E}$ is isomorphic to a complemented subspace of $\mu \in \mathscr{E}$ if and only if $dim_c(\lambda) \subset dim_c(\mu)$.

The analogue to the linear dimension of Banach and Mazur in the case of Banach spaces is $dim_\ell(\sigma)$, the class of all normal forms to which a closed subspace of σ is isomorphic. It may be that $dim_c(\sigma) = dim_\ell(\sigma)$ for all $\sigma \in \mathscr{E}$. This depends on the positive solution of the following

Problem 1. Suppose $\sigma \in \mathscr{E}$. Is every closed subspace H of σ which is isomorphic to a space of $\mathscr{E}$, complemented in σ?

This is true for $\varphi, \omega, \varphi \oplus \omega, \varphi\omega$ and $\omega\varphi$, since in these spaces every closed subspace is complemented (see [H]), but in $\varphi\omega \oplus \omega\varphi$ and the spaces of higher degree the question is open.

We give another application of our results. Pietsch introduces in [P], 10.1, the notation of an absolute equicontinuous basis of a locally convex space. In perfect sequence spaces equipped with the normal topology the unit vectors e, $i = 1, 2, \ldots$, constitute an absolute equicontinuous basis. Pietsch proves that every complete locally convex space E with an absolute equicontinuous basis is isomorphic to a perfect sequence space $S(E)$ and the isomorphism S of E onto $S(E)$ is given by $Sx = S(\sum_{i=1}^{\infty} x_i a_i) = (x_1, x_2, \ldots)$ for every $x \in E$.

Let now E be a $\lambda \in \mathscr{E}$. Since λ is nuclear, every equicontinuous basis of λ is absolute by a theorem of Dynin and Mitiagin (cf. [P] 10.2). Hence in our case we may omit «absolute». Let $\{a_1, a_2, \ldots\}$ be any equicontinuous basis of λ. Then $S(\lambda)$ is a perfect sequence space and by 2.(4) convergence-free. It follows from 5.(3) and 5.(4) that $S(\lambda)$ is a permutation of λ.

Let now $\{b_1, b_2, \ldots\}$ be a second equicontinuous basis of λ and T the isomorphism $T(\sum_{i=1}^{\infty} x_i b_i) = (x_1, x_2, \ldots) \in T(\lambda)$. Then $S(\lambda)$ is permutable in $T(\lambda)$. The unit element

Tb_i of $T(\lambda)$ is therefore equal to the unit element $Sa_{\pi(i)}$ of $S(\lambda)$, π a permutation of $\mathbb{N}$. Therefore $b_i = T^{-1}Sa_{\pi(i)}$ for all i, where $T^{-1}S$ is an automorphism A of λ. We have proved

(2) *If* $\{a_1, a_2, \ldots\}$ *and* $\{b_1, b_2, \ldots\}$ *are two equicontinuous bases of a space* $\lambda \in \mathscr{E}$, *then there exists an automorphism* A *of* λ *and a permutation* π *of* $\mathbb{N}$ *such that*

$$Ab_i = a_{\pi(i)}, \qquad i = 1, 2, \ldots .$$

In the terminology of Mitiagin [Mi] this means that every space of $\mathscr{E}$ has the quasiequivalence property.

I come back to problem 1. It has a positive solution in the simple case $H \cong \omega$:

(3) *Let* λ *be complete convergence-free,* H *a closed subspace isomorphic to* ω *(in the induced topology). Then* H *has a sectional complement in* λ.

Proof. Let A be the isomorphism of ω onto H, $a_i = Ae_i$, $i \in \mathbb{N}$. Then the set of all e_i is bounded in ω hence the set $B = \{a_i, i \in \mathbb{N}\}$ is bounded in λ. By 2.(1) B has a support W_1 which is a W-set of λ and $B \subset \lambda_W \cong \omega$. But in ω every closed subspace has a sectional complement: if $x = (0, \ldots, 0, x_n, x_{n+1}, \ldots) \in \omega$, $x_n \neq 0$, then we call n the length of x. The e_k with lengths k not occurring in H define a sectional complement S of H in λ_{W_1}. But then $\lambda = H \oplus S \oplus \lambda_{\mathbb{N} \sim W_1} = H \oplus F$, F a sectional subspace of λ.

This result solves problem 1 for $H \cong \omega$, but it says more: the complement can be chosen as a sectional subspace. That λ can be an arbitrary complete convergence-free space, not only a space of $\mathscr{E}$, should also be noted.

We say (cf. [K 3] § 38,3.) that a space $\nu \in \mathscr{E}$ is $\mathscr{E}$-detachable if every closed subspace $H \cong \nu$ of a $\lambda \in \mathscr{E}$ has a complement in λ, strictly $\mathscr{E}$-detachable if H has always a sectional complement in λ.

(1) says that ω is not only $\mathscr{E}$-detachable but also strictly $\mathscr{E}$-detachable and this answers, for $\nu = \omega$, the following

Problem 1'. Is every $\nu \in \mathscr{E}$ strictly $\mathscr{E}$-detachable?

A very natural question is stated in

Problem 2. Is every complemented subspace of a $\lambda \in \mathscr{E}$ isomorphic to a space of $\mathscr{E}$?

This is true for $\varphi, \omega, \varphi \oplus \omega, \varphi\omega, \omega\varphi$ (cf. [H]) but open for $\varphi\omega \oplus \omega\varphi$ and the spaces of higher degree.

But let us come back to problems 1 and 1'. We look at the dual situation. We say (cf. again [K 3], § 38,3.) that a space $\nu \in \mathscr{E}$ is $\mathscr{E}$-liftable if for every $\lambda \in \mathscr{E}$ and a quotient λ/H isomorphic to ν there exists a continuous projection of λ with kernel H.

We say further that a space $\nu \in \mathscr{E}$ is strictly $\mathscr{E}$-liftable if for every $\lambda \in \mathscr{E}$ and a quotient λ/H isomorphic to ν there exists a continuous projection P of λ with kernel H and $P(\lambda)$ a sectional complement of H with $P(\lambda) \cong \nu$.

Corresponding to the problems 1 and 1' we have

Problem 3. Is every $\nu \in \mathscr{E}$ $\mathscr{E}$-liftable?

Problem 3'. Is every $\nu \in \mathscr{E}$ strictly $\mathscr{E}$-liftable?

Again there is an example (the only one I know)

(4) *The space φ is strictly $\mathscr{E}$-liftable.*

Proof. Suppose $\lambda \in \mathscr{E}$ and H a closed subspace with $\lambda/H \cong \varphi$. Then $(\lambda/H)' = H^\perp \subset \lambda'$. Obviously $(\lambda/H)' \cong \omega$ in the topology $T_S(\varphi) = T_k(\varphi)$. If K is the canonical mapping of λ onto λ/H, then for M bounded in λ, $K(M)$ is a finite dimensional bounded set in λ/H and every bounded set in λ/H is of the form $K(M)$. This means that $T_b(\lambda)$ coincides on $H^\perp$ with $T_b(\lambda/H)$. Hence $H^\perp$ is a closed subspace of λ', $H^\perp \cong \omega$. Then by (3) $H^\perp$ has a sectional complement $S, \lambda' = H^\perp \oplus S$ and it follows $\lambda = H \oplus S^\perp$, $S^\perp \cong \varphi$.

The dual problem to problem 2 is

Problem 4. If a locally convex space E is liftable or strictly liftable in $\lambda \in \mathscr{E}$, is E isomorphic to a $\nu \in \mathscr{E}$?

Again this is true for $\varphi, \omega, \varphi \oplus \omega, \varphi\omega, \omega\varphi$, (this follows from the results of [H]) and unknown for $\varphi\omega \oplus \omega\varphi$.

Problem 5. Has every complemented subspace of a $\lambda \in \mathscr{E}$ a basis?

If problem 2 has a positive answer then problem 5 would also have a positive answer since every $\lambda \in \mathscr{E}$ has a basis.

Certainly more difficult is

Problem 6. Has every closed subspace of a $\lambda \in \mathscr{E}$ a basis?

Again this is true for $\varphi, \omega, \varphi \oplus \omega, \varphi\omega, \omega\varphi$ and unknown for $\varphi\omega \oplus \omega\varphi$.

I made some attempts to get a survey over the closed subspaces of $\varphi\omega \oplus \omega\varphi$ but failed. I found a closed subspace H which is not complemented, can be algebraically identified with φ, but the induced topology on H is strictly weaker than the normal topology on φ (see [H], [K 2], p. 304). In section 3 of [K 5] I constructed a closed subspace μ of $\varphi\omega \oplus \omega\varphi$ which is algebraically isomorphic to $\varphi\omega\varphi$. I stated there wrongly that μ is isomorphic to $\varphi\omega\varphi$, but the topology on μ induced by $\varphi\omega \oplus \omega\varphi$ is again weaker than the normal topology. Hence the statement in [K 5] that $\varphi\omega \oplus \omega\varphi$ and $\varphi\omega\varphi$ have the same classes of closed subspaces is not proved and probably false.

I hope the reader will feel like me, that a systematic study of the space $\varphi\omega \oplus \omega\varphi$ should be very interesting.

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