ON PRODUCT DECOMPOSITIONS
OF COMPLEX SPACES
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INTRODUCTION

Every connected complex space \( U \) admits a maximal decomposition \( U \cong U_1 \times \ldots \times U_n \) with indecomposable \( U_i \neq \mathbb{C}^0 \) and it is natural to ask whether (or under which conditions) these factors are unique. When considering this question, one is of course tempted to copy the number-theoretic procedure, i.e., given two decompositions, to try at first to find a common factor and then to drop it. However, just this simplification turns out to be the real problem.

For general complex spaces, little can be said as to when \( X \times Y \cong X \times Z \) implies \( Y \cong Z \). Even in the compact case, no counterexamples were known until 1977. Then T. Shioda [12] and, some four years later, G. Parigi [10] presented various examples of compact complex manifolds \( Y \not\cong Z \) such that \( X \times Y \cong X \times Z \) for some torus \( X \). Shioda’s manifolds \( Y \) and \( Z \) are tori as well, and Parigi’s examples are total spaces of fibre bundles with finite structure group and torus basis; we shall denote this class of complex spaces by \( \mathcal{S} \).

Roughly during the same period, diverse criteria for cancellability in the category of reduced connected complex spaces have been proven ([1], [4], [13]). It turned out that in this situation, Shioda and Parigi had already more or less exhausted the scope of counterexamples:

As was shown in [5], \( X \times Y \cong X \times Z \) entails \( Y \cong Z \) if \( \{X, Y, Z\} \not\in \mathcal{F} \). Conversely, for every \( X \in \mathcal{F} \) there exist non-isomorphic \( Y, Z \) such that \( X \times Y \cong X \times Z \) (see [11]). The proof of the above cancellation result simultaneously yielded the uniqueness of the maximal decomposition for compact varieties that are not contained in \( \mathcal{F} \).

We shall now generalize the situation of [5] in several respects. Firstly, non-reduced complex spaces will be admitted, and the compactness condition will be weakened; for instance, in the cancellation problem, we only require one of the factors \( X, Y, Z \) to be compact. As one of the main results, we obtain that then the cancellation theorem of [5] carries over word for word (Theorem 5.2.1). Again, the proof brings about a (partial) answer to the decomposition problem: In any maximal decomposition of a connected complex space, the compact factors \( \not\in \mathcal{F} \) and the product of the other ones are unique (Theorem 5.3.2 and Theorem 5.3.4).

In the last chapter, we are concerned with a different type of generalization, which is inspired by the fact that in all counterexamples to the cancellation problem, the varieties \( Y \) and \( Z \) are still isogeneous, i.e., they can both be covered finitely by some common \( S \). Thus one is led to suspect that \( Y \) and \( Z \) are isogeneous, if so are \( X \times Y \) and \( X \times Z \). This is indeed the case, if at least one of the factors \( X \mid Y \) or \( Z \) is compact (Theorem 7.2.3). As a by-product of the proof, we obtain again a congenial decomposition result (Theorem 7.3.1).
It is easily seen that both the cancellation and the decomposition problem boil down to the following question: If \( X \times Y \cong U \times V \) is an isomorphism between connected complex spaces, what is the relation between the individual factors?

To cover also the non-reduced case, it is necessary to consider at first the corresponding local problem, where \( X, Y, U \) and \( V \) are replaced by germs of complex spaces with \( \dim X = 0 \). It is shown that \( X \times Y \) and \( U \times V \) admit a simultaneous subdecomposition, i.e. that there exist isomorphisms \( X \cong X_U \times X_V, Y \cong Y_U \times Y_V, U \cong X_U \times Y_U, V \cong X_V \times Y_V \) (compare Theorem 1.4.1). The same assertion holds, if \( X, Y, U \) and \( V \) are again complex spaces with \( \dim X = 0 \) (see Chapter 4). This latter result starts the induction on \( \dim X \) in the proof of Theorem 5.1.5, which states that \( X \times Y \) and \( U \times V \) admit a simultaneous subdecomposition, if e.g. \( \{X, Y, U, V\} \notin \mathcal{F} \). The induction step is brought about by a construction presented in Chapter 3, which assigns an isomorphism \( \overline{X} \times \overline{V} \cong \overline{U} \times \overline{V} \) to the given one, such that \( X \times Y \) and \( U \times V \) admit a simultaneous subdecomposition, if so do \( \overline{X} \times \overline{V} \) and \( \overline{U} \times \overline{V} \); it turns out that \( \dim \overline{X} < \dim X \) if \( \{X, Y, U, V\} \notin \mathcal{F} \).

The background material for this latter conclusion as well as for the construction of the isomorphism \( \overline{X} \times \overline{V} \cong \overline{U} \times \overline{V} \) is compiled in Chapter 2, the contents of which can be summed up as follows:

a) For every connected space \( S \), there exists a largest compact connected complex Lie group \( A(S) \) acting holomorphically and effectively on \( S \).

b) If there exists a holomorphic \( S \to A(S) \) that maps the orbit of some positive-dimensional closed complex subgroup \( T \) of \( A(S) \) onto \( T \), then \( S \in \mathcal{F} \).

Even for a reduced compact \( X \notin \mathcal{F} \), the unique indecomposable factors given by Theorem 5.3.4 are in general not unique as subspaces of \( X \), i.e. an automorphism of \( X \) need not be a product of isomorphisms between the indecomposable factors. The relation between \( \text{Aut} \) (\( X \)) and the automorphism groups of the factors is investigated in Chapter 6. It turns out that the situation simplifies considerably, if \( X \) is a projective variety.

Finally, when dealing with the isogeny situation, we start again with connected complex spaces \( X \times Y, U \times V \) which are now assumed to be isogenuous. Pursuing a similar line of reasoning as in Chapter 5, we show:

a) There exists a torus of maximal dimension which is a common isogeny factor of \( X, Y, U \) and \( V \).

b) If this torus is zero-dimensional, then there exist isogenies \( X' \sim X, Y' \sim Y, U' \sim U, V' \sim V \) such that \( X' \times Y' \cong U' \times V' \).

To this latter isomorphism, we can then apply the results of Chapter 5.

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0. PRELIMINARIES

0.1. Categories with (co-)products

Let \( \mathcal{A} \) be a category with a (co-)product \( \odot \). For \( A, B \in \mathcal{A} \) the canonical morphisms 
\( A \odot B \to A, A \odot B \to B \) will be denoted by \( p_1, p_2 \) (\( j_1, j_2 \)) or, if unambiguous, by \( p_A^1, p_B^2 \) (\( j_A^1, j_B^2 \)), or simply by \( p(j) \). By \( j_{A,B} \) or, if the meaning is clear from the context, by \( j \) we denote the natural isomorphism 
\( A \odot B \to B \odot A \), and we let \( j_A^1 = j \). Moreover, for every permutation \( \sigma \) of \( \{1, \ldots, n\} \) we let \( j_{\sigma}^\circ \) denote only one category at a time with exactly one fixed product (coproduct) that is not a coproduct (product).

\( A' \in \mathcal{A} \) is a factor of \( A \in \mathcal{A} \), if \( A \cong A' \odot A'' \) for some \( A'' \in \mathcal{A} \). If \( \mathcal{A} \) has a final (initial) object \( Z \), we shall say that \( A \in \mathcal{A} \) is indecomposable if every factor \( \neq Z \) of \( A \) is isomorphic to \( A \). A decomposition of \( A \in \mathcal{A} \) is an isomorphism \( A \to A_{\circ} \odot \cdots \odot A_{\circ} \) in \( \mathcal{A} \).

A final (initial) object \( Z \in \mathcal{A} \) is a semi-zero object, if \( \text{Mor}(Z, A) \neq \emptyset \) (\( \text{Mor}(A, Z) \neq \emptyset \)) for all \( A \in \mathcal{A} \). If \( \mathcal{A} \) has a semi-zero object \( Z \), then a morphism \( A \to B \) is called constant if it admits a factorization \( \phi = (A \to Z \to B) \).

0.2. Complex spaces and holomorphic mappings

0.2.1. Let \( U = (|U|, \partial_U), V = (|V|, S) \) be complex spaces and let \( f = (|f|, f) : U \to V \) be holomorphic. If \( U \) is reduced, we do not distinguish between \( U \) and \( |U| \) and between \( f \) and \( |f| \). We let \( d_0(U) := \min_{v \in U} \dim, U \).

A (closed or open) complex subspace \( U' \) of \( U \) will be indicated by the symbol \( U' \prec U \) (which also denotes the inclusion map); if \( U' \) is reduced, connected and compact, we sometimes write \( U' \prec \langle U \rangle \). If \( U \) is a complex Lie group and \( U' \prec \langle U \rangle \), we employ the symbol \( U' \triangleleft U \).

For \( V \prec V \) we denote by \( f^{-1}(V) \) the largest subspace \( S \) of \( U \) such that there exists a holomorphic factorization \( f|_S = (S \to V' \prec V) \). If \( U' \prec \langle U \rangle \) such that \( f|_U \) is proper, then there exists a smallest complex subspace \( S \) of \( V \) such that \( f|_{U'} \) admits a factorization through \( S \prec V \), and it will be denoted by \( f(U') \).

\( f \) is a quotient map, if it satisfies the following condition: For every open \( V' \subset V \) and every holomorphic \( g : f^{-1}(V') \to W \) that factors set-theoretically through \( f|_{f^{-1}(V')} \), there...
exists a unique holomorphic factorization of $g$ through $f|_{f^{-1}(V)}$.

If $f$ is proper with Stein factorization $(U \xrightarrow{\tau_f} S_f \xrightarrow{\tau} V)$, then $\tau_f$ is a quotient map.

If $f$ is a covering, if $U$ is connected and if $f$ is finite and locally biholomorphic. Coverings are quotient maps.

Let $\phi : |U| \to S$ be a map of sets. We shall say that the analytic quotient $\bar{\phi} : |U| \to S$ exists, if $S$ can be endowed with a complex structure such that $\bar{\phi} = |g|$ for some holomorphic quotient map $g : |U| \to S$.

Suppose that $f$ is finite and factors through $\phi$. If $\phi$ defines an analytic equivalence relation on $U$ (i.e., if $\{ (u, u') \in U \times U : \phi(u) = \phi(u') \}$ is analytic), then the analytic quotient $\bar{\phi} : |U| \to S$ exists (see [8], Proposition 49. A 13).

0.2.2. The cartesian product of complex spaces is a product in the category of all complex spaces, and $C^0$ is a semi-zero object. For $v \in V$ and every complex space $W$, we denote the constant holomorphic map $W \to C^0 \cong \{ v \} \hookrightarrow V$ by $[v]$. For $(u, v) \in U \times V$ we let $f_{j,v} := ([u], [v]) : U \to U \times V, j_v := ([u], id_V) : V \to U \times V$, if the meaning is clear from the context. If $g : U \times V \to W$ is holomorphic, then the partial maps $g \circ j_u, g \circ f_{j,v}$ will be denoted by $g(\cdot, u), g(u, \cdot)$, respectively. Let $g : U \times V \to A \times B$ be holomorphic, $(u, v) \in U \times V$. Then we let $l_g := p_A \circ g$ and $r_g := p_B \circ g$; moreover, when no ambiguity arises, we let $\bar{v} := r j_v, \bar{u} := r j_u, \bar{v} := l j_v, \bar{u} := l j_u$ and $\bar{u}, \bar{v}$ respectively.

**Lemma.** Let $g : U \times W \to V$ be a holomorphic map between connected complex spaces, and let $A \subseteq U \times W$ with $|p_U|(A) = |U|$ and $|p_W|(A) = |W|$.

If $f$ is constant on some open neighbourhood of $A$, then all partial maps $f(\cdot, w), f(u, \cdot)$ are constant.

**Proof.** For symmetry reasons, it suffices to consider the partial maps $f(\cdot, w)$; therefore we may assume that $W$ is reduced and irreducible. Given $(u, w) \in U \times W$, we have to show that $f(\cdot, w)$ is constant on every infinitesimal neighbourhood of $u$; thus we may assume $U_{\text{red}} = \{ u \}$. By assumption, there exists a non-empty open subset $W'$ of $W$ such that $f|_{U \times W'}$ is constant. Hence $f$ is constant, since $W$ is reduced and irreducible.

0.3. Products of groups

Although the groups considered in what follows need not be abelian, we denote the group composition by $a \cdot \text{sign}$. Then every group homomorphism $G \times H = G' \times H'$ is given by a matrix $\left( \begin{array}{cc} a & \beta \\ \gamma & \delta \end{array} \right)$ with $\alpha \in \text{Hom}(G, G')$ etc. Note that $\alpha + \beta = \beta + \alpha$ (when no ambiguity can arise, we do not distinguish between $\alpha$ and $\alpha \circ p_{G'}$ G and $j_G(G')$ etc.). The
composition of two such homomorphisms is given by the product \( \begin{pmatrix} \alpha' & \beta' \\ \gamma' & \delta' \end{pmatrix} \cdot \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} \alpha' \alpha + \beta' \gamma & \alpha' \beta' + \beta' \delta' + \alpha' \gamma + \beta' \delta \\ \gamma' \alpha + \delta' \gamma & \gamma' \beta + \delta' \delta + \gamma' \delta + \delta' \gamma \end{pmatrix} \) (where \( \alpha' \alpha + \beta' \gamma = \beta' \gamma + \alpha' \delta \) etc.).

0.3.1 Lemma. Let \( \varphi : G \times H \to G' \times H' \) be an isomorphism of groups given by \( \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \), and let \( \varphi^{-1} \) be given by \( \begin{pmatrix} \alpha' & \beta' \\ \gamma' & \delta' \end{pmatrix} \).

If \( \alpha \) or \( \alpha' \) is injective, then so is \( \delta \) or \( \delta' \), respectively; the same assertion holds if \( \langle \text{injective}\rangle \) is replaced by \( \langle \text{surjective}\rangle \).

Proof! Let \( \alpha \) be injective, and let \( h' \in \text{Ker} \delta' \). Then \( h' = \gamma \beta'(h') + \delta \delta'(h') = \gamma \beta'(h') \). If \( \delta'(h') = 0 \), then \( 0 = \alpha' \beta'(h') + \beta' \delta'(h') = \alpha' \beta'(h') \), whence \( \beta'(h') = 0 \) and therefore \( h' = \gamma \beta'(h') = 0 \). If \( \alpha' \) is injective, then the equation \( 0 = \alpha' \beta'(h') + \beta' \delta'(h') = \alpha' \beta'(h') \) again yields \( \beta'(h') = 0 \) whence \( h' = 0 \).

Let \( \alpha' \) be surjective and let \( h' \in H' \). Then \( \beta'(h') = \alpha'(g') \) for some \( g' \in G \) and therefore \( h' = \delta \delta'(h') + \gamma \beta'(h') = \delta \delta'(h') + \gamma \alpha'(g') = \delta \delta'(h') + \delta \gamma'(g') = \delta (\delta'(h') - \gamma'(g')) \) in \( \text{Im} \). If \( \alpha' \) is surjective, then \( \beta'(h') = \alpha'(g) \) for some \( g \in G \), and hence \( h' = \delta \delta'(h') + \gamma \beta'(h') + \gamma \alpha'(g) = \delta \delta'(h') + \gamma \alpha'(g) = \delta \delta'(h') - \delta \gamma'(g) \) in \( \text{Im} \).

\( \diamond \)

0.3.2 Lemma. Let \( \varphi \) be as in 0.3.1.

For all \( m, n \in \mathbb{N} \), the map \( G \times G \to G \), given by \( (g_1, g_2) \mapsto (\alpha' \alpha)^m(g_1) + (\beta' \gamma)^n(g_2) \), is a surjective homomorphism of groups.

If \( (\beta' \gamma)^n = 0 \) for some \( n \), then \( (\alpha' \alpha)^m \) is an isomorphism for all \( m \).

Proof! From \( \alpha' \alpha + \beta' \gamma = \text{id} \), we infer \( \alpha' \alpha \beta' \gamma = \beta' \gamma \alpha' \alpha \) and hence \( \text{id} = (\alpha' \alpha + \beta' \gamma)^{m \circ n} = ((\alpha' \alpha)^m) \circ ((\beta' \gamma)^n) = (\beta' \gamma)^n \circ (\alpha' \alpha)^m \). With suitable endomorphisms \( \chi, \psi \), that commute with \( \beta' \gamma \) and \( \alpha' \alpha \), this proves everything.

\( \diamond \)

0.3.2.a Corollary. Let \( g : V \times W \to V \times W \) be an endomorphism of \( K \)-vector spaces, and assume that \( g \big|_{\text{im} g} = \lambda \cdot \text{id}_V \) for some \( 0 \neq \lambda \in K \).

If \( (W \xrightarrow{g_2} V \times W \xrightarrow{P} W)^n \lambda = 0 \) for some \( n \), then the composition \( g \circ j_V \circ p_V \big|_{\text{im} g} \to \text{im} g \) is an isomorphism.

Proof! Let \( W' := \text{Im} g \), \( W' := \ker g \) and define \( \varphi : V' \times W' \to V \times W \) by \( \varphi((v'|w')) := v' + w' \). Then \( \varphi \) is an isomorphism the inverse of which is given by \( (v|w) \mapsto \begin{pmatrix} 1 \lambda \cdot g(v, w) \\ \lambda - \frac{1}{\lambda} \cdot g(v, w) \end{pmatrix} \). If \( \varphi \) is represented by the matrix \( \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \) with inverse
\[
\left( \begin{array}{c}
\alpha \\
\gamma
\end{array} \right) \left( \begin{array}{c}
\beta \\
\delta
\end{array} \right), \text{ then } \gamma \beta'(w) = 7 \left( \frac{1}{\lambda} \cdot g(0, w) \right) = \frac{1}{\lambda} \cdot p_\lambda(g(0, w)) \text{; thus } (\gamma \beta')^n = 0 \text{ for some } n. \text{ By 0.3.2.} \alpha' \alpha \text{ is an isomorphism, since } (\beta' \gamma)^{n+1} = \beta'(\gamma \beta')^n = 0. \text{ This proves the assertion, since } \alpha' \alpha(v') = \alpha'(p_\lambda(v')) = p_\lambda \left( \frac{1}{\lambda} \cdot g(p_\lambda(v'), 0) \right) = \frac{1}{\lambda} \cdot g(p_\lambda(v'), 0).
\]

0.33 Lemma. Let \( G_1 \times H_1 \supset G_2 \times H_2 \supset \ldots \supset G_n \times H_n \supset \ldots \) be a sequence of subgroups such that

\[
(*) \quad R_{n+2} = R_n \cap (G_{n+1} \times H_{n+1}) \text{ for } R \in \{G, H\}, n \in \mathbb{N}
\]

Denote by \( P_n \) the homomorphism

\[
G_{n+4} \xrightarrow{p_{G_{n+3}}} G_{n+3} \xrightarrow{p_{H_{n+2}}} G_{n+2} \xrightarrow{P_{G_{n+1}}} G_{n+1} \xrightarrow{P_{H_n}} G_n
\]

If \( P_n \circ P_{n+1} \circ \ldots \circ P_{n+4} = 0 \) for some \( n, k \in \mathbb{N} \), then \( G_m \times H_m = G_{m+1} \times H_{m+1} \) for all \( m \geq 0 \).

Proof. By assumption, the diagram

\[
\begin{array}{cccccccc}
G_{n+4} & \xrightarrow{p_{G_{n+3}}} & G_{n+3} & \xrightarrow{p_{H_{n+2}}} & H_{n+2} & \xrightarrow{P_{G_{n+1}}} & H_{n+1} & \xrightarrow{P_{H_n}} & G_n \\
\cap & \downarrow & \cap & \downarrow & \cap & \downarrow & \cap & \downarrow & \cap \\
G_{n+2} & \xrightarrow{p_{G_{n+1}}} & G_{n+1} & \xrightarrow{p_{H_n}} & H_n & \xrightarrow{P_{G_{n+1}}} & H_{n-1} & \xrightarrow{P_{H_n}} & G_{n-2}
\end{array}
\]

is commutative for all \( n \geq 2 \). Thus, if \( P_n \circ \ldots \circ P_{n+4} = 0 \) for some \( n, k \), then \( P_m \circ \ldots \circ P_{m+4} = 0 \) for all \( m \geq n \) with \( m - n \) even.

Furthermore, \( G_{l+3} \cap G_{l+2} = G_{l+1} \cap (G_{l+2} \times H_{l+2}) \cap G_{l+2} = G_{l+1} \cap G_{l+2} \cap H_{l+3} \cap H_{l+2} = H_{l+1} \cap H_{l+2} \cap (G_{l+2} \times H_{l+2}) \cap H_{l+2} = H_{l+1} \cap H_{l+2} \cap G_{2l+1} \cap H_{2l+1} \) for all \( l \geq 1 \). Therefore, we obtain a commutative diagram

\[
\begin{array}{cccccccc}
G_1 \times H_1 & \supset & G_2 \times H_2 & \supset & G_3 \times H_3 & \supset & \ldots \\
\downarrow \text{kan} \times \text{id} & \supset & \downarrow \text{id} \times \text{kan} & \supset & \downarrow \text{kan} \times \text{id} & \supset & \ldots \\
G_1 \times H_1 & \supset & G_2 \times H_2 & \supset & G_3 \times H_3 & \supset & \ldots \\
\end{array}
\]

where \( G_{2l+1} = G_{2l+1} \cap (G_1 \cap H_2), H_{2l+1} = H_{2l+1} \cap (H_1 \cap H_2) \) and \( H_{2l+1} = H_{2l+1} \cap (H_1 \cap H_2) \) (note that \( G_1 \cap G_2 \) and \( G_1 \cap H_2 \) commute).

Clearly, the lower line of this diagram again satisfies the condition \((*)\).

Let now \( n, k \in \mathbb{N} \) with \( P_n \circ \ldots \circ P_{n+4} = 0 \); obviously, we may assume that \( n \) is even.
For $1 \in \mathbb{N}$ with $2 \parallel \geq n$, consider the commutative diagram

\[
\begin{array}{cccccc}
G_{2l+4} & \overset{p}{\rightarrow} & G_{2l+3} & \rightarrow & H_{2l+2} & \rightarrow & H_{2l+1} & \rightarrow & G_{2l} \\
\parallel & & \downarrow \text{kan} & & \downarrow \text{kan} & & \downarrow \text{kan} & & \parallel \\
G_{2l+3} & \overset{p}{\rightarrow} & G_{2l+3}' & \rightarrow & H_{2l+2}' & \rightarrow & H_{2l+1}' & \rightarrow & G_{2l+1}'
\end{array}
\]

The arrow $H_{2l+1}' \rightarrow G_{2l}$ is injective, since $H_{1} \cap H_{2} = H_{2l+1} \cap H_{2l} = \text{Ker}(H_{2l+1} \sigma_{G_{2l+1}} G_{2l})$. Furthermore, the arrow $H_{2l+2}' \rightarrow H_{2l+1}'$ is injective since $p_{H_{2l+1}}(H_{1} \cap H_{2} + G_{1} \cap H_{2}) = p_{H_{2l+1}}(H_{2l} \cap H_{2l+2} + G_{2l} \cap H_{2l+2}) = H_{2l+1} \cap H_{2l+2} = \text{Ker}(H_{2l+1} \sigma_{G_{2l}} G_{2l})$

Now apply the same construction to the sequence $G_{1}' \times H_{1}' \supset G_{2} \times H_{2}' \supset G_{3} \times H_{3}' \supset \ldots$ with $G$ and $H$ interchanged; this yields a commutative diagram

\[
\begin{array}{cccccc}
G_{1}' \times H_{1}' & \supset & G_{2} \times H_{2}' & \supset & G_{3} \times H_{3}' & \supset \ldots \\
\downarrow \text{kan} & & \downarrow \text{kan} & & \downarrow \text{kan} & & \\
G_{1}'' \times H_{1}'' & \supset & G_{;} \times H_{2}'' & \supset & G_{3}'' \times H_{3}'' & \supset \ldots
\end{array}
\]

with the lower line again satisfying (*). Hence, for $n$ and $1$ as above, we obtain

\[
\begin{array}{cccccc}
G_{2l+4} & \overset{p}{\rightarrow} & G_{2l+3} & \rightarrow & H_{2l+2} & \rightarrow & H_{2l+1} & \rightarrow & G_{2l} \\
\parallel & & \downarrow \text{kan} & & \downarrow \text{kan} & & \downarrow \text{kan} & & \parallel \\
G_{2l+3} & \overset{p}{\rightarrow} & G_{2l+3}' & \rightarrow & H_{2l+2}' & \rightarrow & H_{2l+1}' & \rightarrow & G_{2l+1}' \\
\down & & \down & & \down & & 1 \\
G_{2l+4}'' & \rightarrow & G_{2l+3}'' & \rightarrow & H_{2l+2}'' & \rightarrow & H_{2l+1}'' & \rightarrow & G_{2l+1}''
\end{array}
\]

where the compositions $G_{2l+4} \rightarrow G_{2l+3} \rightarrow H_{2l+2}$ and $H_{2l+2} \rightarrow H_{2l+1} \rightarrow G_{2l}$ are injective.

We conclude that $P_{2l} \circ \ldots \circ P_{2l-4(k-1)} = 0$, if $P_{2l} : G_{2l+4} \rightarrow G_{2l+4}$ denotes the homomorphism given by the bottom line of the above diagram. Moreover, it is obvious from the construction, that $G_{2l+2} \times H_{2l+2} = G_{2l+1} \times H_{2l+1} = G_{2l} \times H_{2l+1}$ if $G_{2l+2} \times H_{2l+2} = G_{2l+1} \times H_{2l+1} = G_{2l} \times H_{2l+1}$.

Thus, if we proceed by induction on the minimal $k$ with $P_{n} \circ \ldots \circ P_{m+4(k-1)} = 0$, it remains to consider the case $k = 0$, i.e., the case $G_{0} = 0$. Then $G_{2l} = 0$, $G_{2l+1} \rightarrow H_{2l+1}$ and $H_{2l+1} \cap (G_{2l+1} \times H_{2l+1}) = G_{2l+1} \times H_{2l+1}$ and $R_{2l+3} = R_{2l+1} \cap H_{2l+2} = R_{2l+1} \cap (G_{2l+1} \times H_{2l+1}) = R_{2l+1}$ for $RE \{G, H\}$ and all $l$ with $2l \geq n$. $\Diamond$
1. LOCAL ALGEBRAS WITH ARTINIAN FACTORS

Let $\mathcal{K}$ be a field of characteristic zero with a complete valuation and denote by $\mathcal{S}_\mathcal{K}$ the category of local analytic K-algebras. The analytic tensor product is a coproduct in $\mathcal{S}_\mathcal{K}$ and $K$ is a zero-object in $\mathcal{S}_\mathcal{K}$.

For $A \in \mathcal{S}_\mathcal{K}$ with maximal ideal $m_A$, let $n_A \subset A$ be the nilradical of $A$. The canonical projection $A \rightarrow A/\mathfrak{n}_A =: A_{\text{red}}$ is denoted by red, or simply by $\text{red}$. The reduction of a homomorphism $f : A \rightarrow B$ in $\mathcal{S}_\mathcal{K}$ is indicated by $f_{\text{red}} : A_{\text{red}} \rightarrow B_{\text{red}}$, its Jacobian $m_A/m_A^2 \rightarrow m_B/m_B^2$ by $T_f : T_A \rightarrow T_B$.

For any local subalgebra $A' \subset A$ we let $A / A' := A / (A \cdot m_A)^{\ell}$.

1.1. A surjectivity criterion

Let $(f : A \rightarrow B) \in \mathcal{S}_\mathcal{K}$. It is well known that $f$ is surjective, if and only if so is its Jacobian $T_f$.

1.1.1 Lemma. Let $(g : A \otimes B \rightarrow A' \otimes B') \in \mathcal{S}_\mathcal{K}$ such that $\text{pr}_A \cdot g_{\mathfrak{J}_A}$ and $\text{pr}_B \cdot g_{\mathfrak{J}_B}$ are surjective.

If $\text{pr}_B \cdot g_{\mathfrak{J}_A}$ or $\text{pr}_A \cdot g_{\mathfrak{J}_B}$ is constant, then $g$ is surjective.

Proof. $T_g$ is given by a matrix that has the form

\[
\begin{pmatrix}
G_{11} & G_{12} \\
0 & G_{22}
\end{pmatrix}
\]

or

\[
\begin{pmatrix}
G_{11} & 0 \\
G_{21} & G_{22}
\end{pmatrix}
\]

with surjective $G_{11} : T_A \rightarrow T_{A'}$, $G_{22} : T_B \rightarrow T_{B'}$.

The following lemma provides the essential argument in the proof of the local cancellation theorem. Its assertion does no longer hold, if $\text{char}(K) > 0$.

1.1.2 Lemma. Let $(f : A \rightarrow B \otimes C) \in \mathcal{S}_\mathcal{K}$.

If $\text{pr}_B \cdot f|_{\mathfrak{n}_A}$ is injective with $\text{pr}_B \cdot f|_{\mathfrak{n}_A} : m_A^2 \cap m_B^2 \rightarrow m_B^2$, then $n_A \cap n_B^2 \subset \text{Ker} \text{pr}_C f$.

Proof. Let $a \in (m_A^2 \cap m_B^2) \cap n_A$, and let $m \in N$ be minimal with $a^{m+1} = 0$. If $\text{pr}_C f(a) \neq 0$, there exists $1 \in N$ with $\text{pr}_C f(a) \in m_A^2 \cap m_B^2$.

Let $B = N/m_B^{m+1} = C/m_C^{m+1}$, and let $\overline{f} = \text{kan} \circ f : A \rightarrow B \otimes C$. Then $\overline{f}(a) = b \otimes 1 + z \otimes c$ with $z \in m_B^{m+1} \cap m_C^{m+1}$, and hence $0 = \overline{f}(a)^{m+1} = (b \otimes 1 + 1 \otimes c)^{m+1} = (m + 1) \cdot (b \otimes 1 + z) \cdot (1 \otimes c) = (m + 1) \cdot b^m \otimes c$, a contradiction.

1.1.2 Corollary. (compare [9]) Let $f$ be as above, with $A$ artinian.

If $\text{pr}_B \cdot f$ and $T_{\text{pr}_A} f$ are injective, then $\text{pr}_C f$ is constant.
1.1.2-b Corollary. (compare [9].) Let \( f \) be as in 1.1.2 with \( A_{\text{red}} \) regular.

If \( p_{B} f |_{n_{A}} \) is injective with \( p_{B} f (n_{A} \setminus m_{A}^{2}) \subset m_{B} \setminus m_{B}^{2} \), then \( p_{C} f \) factors through \( \operatorname{red} A \).

Proof: It suffices to show that every minimal set of generators \( \{ n_{1}, \ldots, n_{s} \} \) of \( n_{A} \) is contained in \( m_{A} \setminus m_{A}^{2} \). Let \( n' \) be generated by \( \{ n_{1}, \ldots, n_{s} \} \cap (m_{A} \setminus m_{A}^{2}) \) and let \( A' := A/n' \). Then \( \dim A' = \dim A = \dim T_{A_{\text{red}}} = \dim T_{A'} \), whence \( A' \) is reduced, i.e. \( n' = n_{A} \).

1.1.2.c Corollary. Let \( (g : A \otimes B \to A' \otimes B') \in \mathcal{S}_{K} \) with \( A \) artinian.

If \( p_{A} g j_{A} \) is an isomorphism, and if \( p_{B} g j_{B} \) is surjective, then \( g \) is surjective.

Proof: Evident by 1.1.2.a and 1.1.1.

1.2. Isomorphisms between coproducts in \( \mathcal{S}_{K} \).

Let \( f : A \otimes B \to C \otimes D \) be an isomorphism in \( \mathcal{S}_{K} \), and let \( f_{A} := p_{A} f^{-1} j_{C} p_{C} f j_{A} \), \( f_{D} := p_{D} f_{D} \).

1.2.1 Lemma. Let \( f_{C} j_{A} \) or \( f_{D} \) be injective, then so is \( f_{A} \).

The same assertion holds if unjectives is replaced by «surjective».

Proof: Compare 0.3.1.

1.2.1.a Corollary. \( p_{C} f j_{A} \) or \( f_{A} \) is an isomorphism, if and only if \( p_{B} f^{-1} j_{D} \) or \( f_{D}^{-1} \) is, respectively.

Proof: Let \( p_{C} f j_{A} \) or \( f_{A} \) be an isomorphism. Note at first that it suffices to show that \( p_{B} f^{-1} j_{D} \) resp. \( f_{D}^{-1} \) is surjective: If \( p_{C} f j_{A} \) is bijective and \( p_{B} f^{-1} j_{D} \) is surjective, we obtain a sequence of surjective homomorphisms

\[
A \otimes B \to C \otimes D \to A \otimes D \to A \otimes B
\]

and we conclude that \( p_{B} f^{-1} j_{D} \) is also injective.

The assertion is now evident by 1.2.1.

1.2.1.b Corollary. Let \( A \) or \( C \) be artinian.

(i) If \( f_{C} j_{A} \) is an isomorphism, then so are \( p_{A} f^{-1} j_{C} \), \( p_{D} f j_{B} \), and \( p_{B} f^{-1} j_{D} \); moreover, \( p_{D} f j_{A} \) and \( p_{B} f^{-1} j_{C} \) are constant.

(ii) If \( f_{A} \) is surjective, then \( f_{A} \) and \( f_{D}^{-1} \) are isomorphisms, and \( p_{D} f j_{A} \) is constant.
Proof. In (i) as well as in (ii) \( A \) is artinian, if \( C \) is. Thus we may assume that \( A \) is artinian.

(i) If \( p_C f j_A \) is an isomorphism, then so is \( p_B f^{-1} j_D \) by 1.2.1.a and \( C \) is artinian. By 1.1.2, \( p_D f j_A \) is constant, whence \( p_D f j_A \) is surjective and therefore bijective, since \( B \cong D \) via \( p_B f^{-1} j_D \). Thus \( p_A f^{-1} j_C \) is an isomorphism, whence \( p_B f^{-1} j_D \) is constant.

(ii) If \( f_A \) is an isomorphism, then so is \( f_D^{-1} \) by 1.2.1.a. If \( T f_A \) and \( f_A \) are injective, then so are \( T p_C f j_A \) and \( p_C f j_A \), whence \( p_D f j_A \) is constant by 1.1.2.a.

1.2.2 Lemma. \( p_B f^{-1} j_C \) defines an isomorphism \( C/p_C f (A) \rightarrow B/p_B f^{-1}(D) \), whose inverse is given by \( p_C f j_A \).

Proof. \( p_B f^{-1} p_C f(m_A) \subseteq p_B f^{-1}(f(m_A) + C \otimes m_D) = p_B f^{-1}(C \otimes m_D) = B \cdot p_B f^{-1}(m_D) \), whence \( p_B f^{-1}(C \cdot p_C f(m_A)) \subseteq B \cdot p_B f^{-1}(m_D) \).

Furthermore, \( p_C f p_B f^{-1}(c) \subseteq p_C f(f^{-1}(c) + m_A \otimes B) = c + C \cdot p_C f(m_A) \) for all \( c \in m_C \), and the assertion follows for symmetry reasons.

1.2.3 Lemma. For all \( m, n \in \mathbb{N} \) the multiplication map \( \text{mult}: \text{Im} \ f_A^n \otimes \text{Im} \ f_A^n \rightarrow A \) is surjective.

Proof. By 0.3.2, the Jacobian of \( \text{mult} \) is surjective.

1.3. The structure of local algebras with artinian factors

Let \( f : A \otimes B \rightarrow C \otimes D \) be an isomorphism in \( \mathcal{S} \) and assume that \( A \) is artinian. Then \( A_C := \text{Im} \ f_A^m \otimes A \) is welldefined for \( m \gg 0 \) and \( \text{mult}: A_C \otimes A_D \rightarrow A \) is surjective by 1.2.3. Clearly, \( p_C f | A_C \), \( p_D f | A_D \) and their Jacobians are injective; thus \( p_D f | A_C \), \( p_C f | A_D \) are constant by 1.1.2.a. Therefore, \( A_C = f_A(A), A_D = f_A(D) \), and \( p_C f(A) = p_C f(A_C) := C_A, p_D f(A) = p_D f(A_D) := D_A \) are isomorphic to \( A_C, A_D \) via \( p_C f, p_D f \), respectively. Conversely, \( p_A f^{-1} \) induces isomorphisms \( C_A \rightarrow A_C, D_A \rightarrow A_D \) whence, again by 1.1.2.a, \( p_B f^{-1} | C_A \) and \( p_B f^{-1} | D_A \) are constant. Let \( B_C := p_B f^{-1}(C) \), \( B_D := p_B f^{-1}(D) \), \( B_C := p_C f(B_C), B_D := p_D f(B_D) \); then, by 1.2.3, the multiplications \( B_C \otimes B_D \rightarrow B, C_A \otimes C_B \rightarrow C, D_A \otimes D_B \rightarrow D \) are surjective.

13.1 Lemma. \( \text{mult}: A_C \otimes A_D \rightarrow A \) is an isomorphism.

Proof. Denote by \( \chi \) the composition

\[
A_C \otimes A_D \xrightarrow{\text{mult}} A \xrightarrow{j_A} C \otimes D \xrightarrow{p_A f^{-1} j_C \otimes p_D f^{-1} j_D} A \otimes A \xrightarrow{p_C f j_A \otimes p_D f j_A} C_A \otimes D_A.
\]

Then \( p_C f j_A | A \) = \( p_C f j_A p_A f^{-1} j_C p_C f j_A | A_D \) = \( p_C f j_A f_A | A_D \) and \( p_D f j_A | A_D \) are isomorphisms. By 1.1.2.c, \( \chi \) is surjective and thus bijective, whence \( \text{mult}: A_C \otimes A_D \rightarrow A \) is injective and hence an isomorphism.
1.3.2 Lemma. \((p_B f^{-1} p_D f \mid B_D \to B_D) = \text{id}_{B_D}, (p_D f p_B f^{-1} \mid D_B \to D_B) = \text{id}_{D_B}\) and \(p_B f^{-1} p_D f \mid B_D\) is constant; and the corresponding statements hold after interchanging \(C\) and \(D\). In particular, \(B_C\) and \(C_B\), as well as \(B_D\) and \(D_B\), are isomorphic via \(p_C f \mid p_D f\), with the respective inverse given by \(p_B f^{-1}\).

**Proof.** Let \(s \in m_{D_D}\); then \(p_B f^{-1} p_D f p_B f^{-1}(s) = p_B f^{-1} p_D f (f^{-1}(s) + (p_B f^{-1}(s) - f^{-1}(s))) = p_B f^{-1} p_D f(s)\), since \(p_D f(p_B f^{-1}(s) - f^{-1}(s)) \in p_D f(m_A \otimes B) = D \cdot m_{D_A}\), and \(p_B f^{-1} p_D f \mid D_A\) is constant. Thus \(p_B f^{-1} p_D f p_B f^{-1} \mid C = p_B f^{-1} p_J \mid C\) is constant, and \(p_B f^{-1} p_D f p_B f^{-1} \mid D = p_B f^{-1} \mid D\), and we conclude that \(p_B f^{-1} p_D f \mid B\) is constant and \(p_B f^{-1} p_D f \mid D = \text{id}_{B}\). Then also \(p_D f \mid p_B f \mid D = \text{id}_{D_B}\) since \(p_D f \mid D\) \(\to D\) is surjective by definition. \(\diamondsuit\)

1.3.2.a Corollary. The multiplications \(B_C \otimes B_D \to B, C_A \otimes C_B \to C, D_A \otimes D_B \to D\) are isomorphisms.

**Proof.** Let \(\chi := (C_A \otimes C_B \to C, f^{-1} j_C \otimes A \otimes B f_A \otimes p_B f^{-1} j_C p_C f \otimes \text{i}_B A_C \otimes B)\).

Then \(p_{A_C} \chi j_{C_A}^C = f_A \circ p_A f^{-1} j_{C_A} \) and \(p_{B_D} \chi j_{D_A}^D = p_B f^{-1} j_{C_B} \); and \(C_A \otimes C_B \otimes D_A \otimes D_B\) \(\otimes D_B\) is an isomorphism, and hence so are \(\chi\) and \(\text{mult}\) (see \(1.1.2.c\)).

Symmetrically, \(\text{mult} : D_A \otimes D_B \to D\) is an isomorphism.

Finally, \(\text{mult} \otimes \text{mult} : (A, \otimes A) \otimes (B, \otimes B) \to A \otimes B\) is surjective, \(\text{mult} \otimes \text{mult} : (C_A \otimes C_B) \otimes (D_A \otimes D_B) \to C \otimes D\) is an isomorphism, and \(A_C \otimes A_D \otimes B_C \otimes B_D \cong C_A \otimes C_B \otimes D_A \otimes D_B\). Thus \(\text{mult} : B_C \otimes B_D \to B\) is an isomorphism as well. \(\diamondsuit\)

In total, we have shown:

1.3.3 Theorem. Let \(f : A \otimes B \to C \otimes D\) be an isomorphism in \(\mathcal{L}_K\) with \(A\) artinian. Then there exists a commutative diagram of isomorphisms in \(\mathcal{L}_K\):

\[
\begin{array}{ccc}
(A_C \otimes A_D) \otimes (B_C \otimes B_D) & \xrightarrow{f} & (C_A \otimes C_B) \otimes (D_A \otimes D_B) \\
\downarrow \text{mult} \otimes \text{mult} & & \downarrow \text{mult} \otimes \text{mult} \\
A \otimes B & \xrightarrow{f} & C \otimes D \\
\end{array}
\]

where \(R_A = p_R f(A), R_B = (A_R) \mid B_R = p_B f^{-1}(R_B), R_B = p_R f(B_R), A_R = p_A f^{-1}(R_A)\) for \(R \in \{C, D\}\). In particular, \(R_S \cong S_R\) for \(R \in \{C, D\}\), \(S \in \{A, B\}\).

1.3.3.a Corollary. (Cancellation theorem, see \(16\)) Let \(R, R', S \in \mathcal{L}_K\) such that \(R, R'\) or \(S\) is artinian.

If \(R \otimes S \cong R' \otimes S\) then \(R \cong R'\).
Proof. We may assume that $S$ is indecomposable. Then either $R_S \cong S_R = \mathbb{K} = R'_S \cong S'_R$ or $R_S \cong S_R \cong S = S'_R \cong R'_S$. In the first case, $R \cong R_R \cong R'_R \cong R'$, and in the second one $R \cong R_R \otimes R_S \cong R_R \otimes R'_S \cong R'$.  

1.3.3.b Corollary (Decomposition theorem, see [6]) Every $S \in \mathcal{L}_R$ admits a unique decomposition (up to reordering) $S \cong S_1 \otimes \ldots \otimes S_n \otimes S''$ with indecomposable artinian $S_1, \ldots, S_n \in \mathcal{L}_R \setminus \{\mathbb{K}\}$ and with $S' \in \mathcal{L}_R$ having no artinian factor $\neq \mathbb{K}$.

Proof. Of course, we need only verify the uniqueness part. Using induction on $n$, it suffices to show: If $S \cong \tilde{S}_1 \otimes \ldots \otimes \tilde{S}_m \otimes S''$ is another decomposition of the same type, then there exists $1 \leq \nu' \leq n$ with $S_{\nu'} \cong \tilde{S}_1$ and $\otimes_{\nu' \leq n} S_{\nu} \otimes S' \cong \tilde{S}_2 \otimes \ldots \otimes \tilde{S}_m \otimes S''$.

The case $n = 0$ being trivial, we assume that the assertion is proven for some $n - 1 \geq 0$. Let $B := \otimes_{\nu \geq 2} S_\nu \otimes S'$, $D := \otimes_{\mu \geq 3} \tilde{S}_\mu \otimes S''$ and let $f : S_1 \otimes B \to \tilde{S}_1 \otimes D$ be some isomorphism. If $S_1 = (S_1)\tilde{S}_1$, then $\nu' := 1$. Otherwise, $S_2 \otimes \ldots \otimes S_n \otimes S' = B \cong \tilde{S}_1 \otimes B_D \cong \tilde{S}_1 \otimes D_B$ and $\tilde{S}_2 \otimes \ldots \otimes \tilde{S}_m \otimes S'' = D = S_1 \otimes D_B \cong S_1 \otimes B_D$. From the induction hypothesis, we infer that $\tilde{S}_1 \cong S_{\nu'} \otimes \otimes_{\nu' \geq 1} S_{\nu} \otimes S' \cong \tilde{B}_D$ for some $2 \leq \nu' \leq n$, and hence $\otimes_{\nu' \leq n} S_{\nu} \otimes S' = S$, $\otimes B_D \cong D = \otimes_{\mu \geq 2} \tilde{S}_\mu \otimes S''$. 

In view of the applications we have in mind, it is advisable to reformulate 1.3.3 in terms of quotient algebras.

13.4 Theorem. Let $f : A \otimes B \to C \otimes D$ be an isomorphism in $\mathcal{L}_R$ with $A$ artinian, and let $R_S, S_R$ be as in 1.3.3, where $R \in \{A, B\}, S \in \{C, D\}$. Let $\{R, R'\} = \{A, B\}, \{S, S'\} = \{C, D\}$. Then

(i) $\text{kan} \circ f|_{R'} : R \to C/\!\!/C_R \otimes D/\!\!/D_R$ and $\text{kan} \circ f^{-1} f_S : S \to A/\!\!/A_S \otimes B/\!\!/B_S$ are isomorphisms.

(ii) The composition $R^\to \to R_S \to S/\!\!/S_R$ factors through $R \to R/\!\!/R_S$ with an isomorphism $R/\!\!/R_S \to S/\!\!/S_R$, and $S \xrightarrow{f_S^{-1}} B \xrightarrow{\text{kan}} B/\!\!/B_S$ factors through $S \to S/\!\!/S_A \to B/\!\!/B_S$. The homomorphism $f_A : A \to A$ resp. $f_A' : A \to A$ factors through $\text{kan} : A \to A/\!\!/A_D$ resp. $\text{kan} : A \to A/\!\!/A_C$ and the resulting composition $A/\!\!/A_D \to A \to A/\!\!/A_D$ resp. $A/\!\!/A_C \to A \to A/\!\!/A_C$ is an isomorphism.

Proof. $\text{kan}|_{R_S} : R/\!\!/R_S \to S/\!\!/S_R$, $\text{kan}|_{S_R} : S/\!\!/S_R \to S/\!\!/S_R$ are isomorphisms, since so are mult: $R_S \otimes R_S \to R_S \to R$, mult $S_R \otimes S_R \to S$.

(i) Let $\phi_R := (R_C \otimes R_D \to R \xrightarrow{f_S} C \otimes D \xrightarrow{\text{kan}} C/\!\!/C_R \otimes D/\!\!/D_R)$

and $\psi_S := (S_A \otimes S_R \xrightarrow{\text{mult}} S \xrightarrow{f^{-1} f_S} A \otimes B \xrightarrow{\text{kan}} A/\!\!/A_S \otimes B/\!\!/B_S)$. 

\[ \phi_R := (R_C \otimes R_D \xrightarrow{\text{mult}} R \xrightarrow{f_S} C \otimes D \xrightarrow{\text{kan}} C/\!\!/C_R \otimes D/\!\!/D_R) \]
Then $p_{S|R} \phi_R j_{R_S} : R_S \to S//S_R$ and $p_{R|R_S} \phi_S j_{S_R} : S_R \to R//R_S$ are isomorphisms, whence so are $\phi_A | \psi_C | \psi_D$ by 1.1.2.c.

From the commutative diagram

$$
\begin{array}{ccc}
B_D & \xrightarrow{\text{kan}} & B & \xrightarrow{\text{kan}} & C \\
\downarrow & & \downarrow & & \downarrow \\
K & \xrightarrow{\text{kan}} & B//B_D & \xrightarrow{\gamma} & C//C_A
\end{array}
$$

(compare 1.2.2 and the definition of $C_A | B_D$), we infer that $p_{C|C_A} \phi_B j_{B_D}$ is constant; symmetrically so is $p_{D|D_A} \phi_B j_{B_D}$. In particular, the Jacobian of $\phi_B$ is surjective, and hence so is $\phi_B$.

(ii) The case $R = B$ follows from 1.2.2 and the definition of $B_C | B_B, C_A | D_A$. The case $R = A$ follows from $S_A = p_S f(A)$ and $p_S f(A, S) = K$ for $\{S, S'\} = \{C, D\}$. The morphism $f_A$ is constant on $A_D$ and hence factors through $A_A = A_B f_A$, and $A_A \to A_B$ is an isomorphism. The corresponding statement for $f_A$ follows symmetrically.

1.4. Germs of complex spaces with zero-dimensional factors

1.4.1 Theorem. Let $\phi : X \times Y \to U \times V$ be an isomorphism between germs of complex spaces, and assume that $X$ is zero-dimensional. Let $\{S, S'\} = \{U, V\}$, $\{R, R'\} = \{X, Y\}$, and denote by

$$
X_S | \quad X \to S' \to X \\
Y_S \quad \text{the fibre of} \\
S_X \quad S \to Y \to S \\
S_Y \quad S \to X
$$

(where each arrow denotes the corresponding partial map given by $\phi$ or $\phi^{-1}$).

Then

(i) $p_S | X_S \times Y_S \to S$ and $p_R | U_R \times V_R \to R$ are isomorphisms.

(ii) The partial map $S \to R$ defines an isomorphism $S_R \to R_S$, the partial map $Y \to S$ defines an isomorphism $Y_S \to S_Y$, and the composition of partial maps $X \to S \to X$ factors through the inclusion $X_S \leftarrow X$, inducing an isomorphism $X_S \to X_S$.

The proof is evident by 1.3.4.

1.4.1.a Corollary. Let $X, Y, Z$ be germs of complex spaces such that at least one of them is zero-dimensional.

If $X \times Z \cong Y \times Z$, then $X \cong Y$. 

1.4.2-b Corollary. Every germ $U$ of a complex space admits a unique decomposition (up to reordering) $U \cong U_1 \times \ldots \times U_n \times U'$ with zero-dimensional indecomposable $U_\nu \not\cong \mathbb{C}^0$ and with $U'$ having no zero-dimensional factor $\not\cong \mathbb{C}^0$.

2. FAMILIES OF HOLOMORPHIC MAPPINGS

When considering the complex analytic cancellation problem, one is faced immediately with various families of holomorphic mappings — eight at first sight, but actually a lot more. In this chapter, we prepare the way for dealing with them.

2.1. The simultaneous Stein factorization

Let $\phi : W \times U \to V$ be a holomorphic map between connected complex spaces. Then $W$ and $U$ can be interpreted as parameter spaces of holomorphic maps from $U$ or $W$ into $V$ with evaluation map $\phi|$. In general, we consider $U$ to be the common domain of the maps parameterized by $W$. Sometimes, however, it is advisable to interchange the roles of the two factors, and it will be done without further comment.

Mostly, we shall not distinguish between $w \in W$ and the partial map $\phi(., u)$ (or between $u$ and $\phi(., u)$).

2.1.1 Lemma and Definition. Assume that $\Phi := (p_W, \phi) : W \times U \to W \times V$ is proper.

Then the partial maps $w \in W$ admit a simultaneous Stein factorization, i.e. the Stein factorization $\Phi_\phi = (W \times U \xrightarrow{\tau_\phi} S_\phi \xrightarrow{\phi} W \times V)$ satisfies $\tau_\phi = id_W \times \tau_w, \Phi(w, .) = (p_W, w)$ for all $w \in W$, where $w = (U \xrightarrow{\tau_w} S_w \xrightarrow{\phi} V)$ is the Stein factorization of $w \in W$.

Proof. By ([5], 4.3), the assertion is true for reduced $U|$. Thus there exists for every $w \in W$ a commutative diagram

$\begin{array}{ccc}
wxu & \xrightarrow{\tau_w} & S_\phi \\
\downarrow id \times \tau_w & \nearrow h_w & \\
w \times S_w & & \\
\end{array}$

with some homeomorphism $h_w$. The mapping $h_w$ is biholomorphic, since both $id \times \tau_w$ and $\tau_\phi$ are quotient maps. \hfill \Box

For the remainder of this work, we let therefore $\pi_\phi := \tau_\phi : U \to U_\phi := S_w$ for $w \in W$ arbitrary, and $\phi_\phi := p_V \circ \Phi : W \times U_\phi \to V$.

2.2. Effectively parametrized families

Let $\phi : W \times U \to V$ be a holomorphic map between connected complex spaces, and let $\text{Hol}(U, V) := \{ \alpha : U \to V : \alpha \text{ holomorphic} \}$. $\text{Hol}(U) := \text{Hol}(U, U)|\text{Aut}(U) := \{ \alpha \in \text{Hol}(U) \}$.
Hol (U) : \( \alpha \) biholomorphic. The evaluation map \( \text{Hol} (U, V) \times U \rightarrow V \) will be denoted by \( E_{U,V} \) or by \( E_U \), if \( U = V \) (or by \( E \) if the meaning is clear from the context). For \( \mathbf{u} \in U \cup U' \subset U \cap \text{Hol} (U, V) \), we denote by \( \cdot \mathbf{u} \) the composition \( E_{U,V} \circ \mathbf{u} : \text{Hol} (U, V) \rightarrow \text{Hol} (U, V) \times U \rightarrow V \), and we let \( H U' := E_{U,V} (U \times U') \), and \( H U := H \{ \mathbf{u} \} \). We shall say that \( W \) is (almost) effectively parametrized if the natural map \( \rho : W \times W \rightarrow V \) is injective (or, respectively, has discrete fibres).

Let \( \Phi : W \times V \rightarrow V \) be another holomorphic map between connected complex spaces. When no ambiguity can arise, we denote by \( W \times V \) the image of \( W \times V \) under \( \rho \). For compact \( U \), \( \text{Hol} (U, V) \) carries the compact-open topology (see [2]). For compact \( U \), we henceforth tacitly assume \( \text{Hol} (U, V) \) to be endowed with this complex structure. Note that then, according to 2.2.1, \( \text{dim} W \) is finite (resp. surjective), then every \( U \in V \) is finite (resp. surjective).

Proof. We may assume that \( U, V, W \) are reduced; furthermore, a trivial argument shows that \( W \) can be assumed irreducible. Applying 2.1.1 to the family \( \{ \phi (\cdot, \mathbf{u}) \}_{\mathbf{u} \in U} \) yields (ii) and the first part of (i). Let \( V' \) be an irreducible component of \( V \). If \( w \in W \) is surjective, there exists an irreducible component \( U' \) of \( U \) with \( w (U') \subset V' \). Then \( \phi (U' \times W) \subset V' \), whence \( \text{dim} W \leq \text{dim} V' \) by the first part of (i).

2.2.2 Lemma and Notation. If \( W \) is compact, then the analytic quotient \( W \rightarrow \rho (W) \) exists. \( \rho (W) \) together with this complex structure will be denoted by \( \rho (W) \). The evaluation map \( E_{U,V} : \rho (W) \rightarrow V \) is holomorphic.

Proof. Let \( \phi = (W \times U \times U \times U) \rightarrow V \) be the simultaneous Stein factorization of the partial maps \( u : W \rightarrow V \). Obviously, \( \rho \) factors through \( \pi \) and \( \rho \) defines an analytic equivalence relation on \( W \). If \( \rho (W) \) is endowed with the quotient topology, then the natural map \( W \rightarrow \rho (W) \) and the orbit maps \( \cdot \mathbf{u} : \rho (W) \rightarrow V \) are finite; hence, by ([8], 49.A...
the analytic quotient $W_\phi \to \rho_\phi(W)$ exists. Denote the corresponding complex space by $\rho_\phi(W)$; then $\rho_\phi : W \to \rho_\phi(W)$ and $\rho_\phi \times \text{id}_U$ are quotient maps, since $\mathcal{G}_{W_\phi} = (\pi_\phi)_* \mathcal{G}_W$. In particular, the evaluation map is holomorphic.

Note that if $U$ is compact, $\rho_\phi(W)$ need not coincide with the complex subspace $\rho_\phi(W) \hookrightarrow \text{Hol}(U, V)$; in general, the latter structure is a substructure of that on $\rho_\phi(W)$. Nevertheless, for non-compact $U$, when no such rivalry can occur, we shall introduce the notion of a reduced connected compact complex subspace of $\text{Hol}(U, V)$:

2.23 Definition. Assume that $U$ is non-compact. If $W$ is compact, reduced and weakly normal, and if $W$ is effectively parametrized, then $W$ is called a reduced connected compact complex subspace of $\text{Hol}(U, V)$, expressed by the symbol $W \hookrightarrow \text{Hol}(U, V)$.

2.2.3 Remarks:

(i) If $W$ is reduced and compact, then the weak normalization of $\rho_\phi(W)$ is a reduced compact complex subspace of $\text{Hol}(U, V)$.

(ii) Let $W_1 \hookrightarrow \text{Hol}(U, V), W_2 \hookrightarrow \text{Hol}(V, V_1)$. Then $W_2 \circ W_1$ carries a unique structure of a reduced connected compact complex subspace of $\text{Hol}(U, V_1)$, with which we shall always assume it to be endowed. Note that, in contrast to the case $U$ compact, the inclusion $W_2 \circ W_1$ need not be an embedding; it is, though, if it is bijective.

2.3. Action of compact complex Lie groups

Let $U$ be a connected complex space.

23.1 Lemma and Notation. There exists $A(U) \hookrightarrow \text{Hol}(U)$ with $\text{id}_U \in A(U)$ such that the following condition holds: If $\phi : W \times U \to U$ is holomorphic with reduced compact connected $W$, such that $\text{id}_U \in \rho_\phi(W)$, then $\rho_\phi(W) \circ A(U)$ and $\rho_\phi : W \to A(U)$ is holomorphic.

In particular, $A(U)$ admits no proper complex substructure, with respect to which the evaluation map $E_U$ remains holomorphic.

$A(U)$ is a compact complex Lie group and $A(U)$ is a normal subgroup of $\text{Aut}(U)$; if $U$ is compact, then $A(U)$ is central in the identity component $\text{Aut}_0(U)$ of $\text{Aut}(U)$.

Proof: By 2.2.2 and 2.2.1(i), there exists an irreducible $A(U) \hookrightarrow \text{Hol}(U)$ of maximal dimension with $\text{id}_U \in A(U)$. Then $\text{id}_U \in A(U) \circ A(U) \hookrightarrow \text{Hol}(U)$, whence the inclusion $\alpha \circ A(U) \to A(U) \circ A(U)$ is bijective and therefore biholomorphic for all $\alpha \in A(U)$. Thus $A(U) \circ \text{Aut}(U)$ and $A(U)$ is a compact complex Lie group.
Let \( W \subseteq \text{Hol}(U) \) with \( \text{id}_U \in W \) and let \( W' \) be an irreducible component of \( W \) that meets \( A(U) \). Then the composition \( A(U) \overset{\alpha}{\longrightarrow} W_0 \circ A(U) \rightarrow W' \circ A(U) \) is bijective and hence biholomorphic for all \( W_0 \in W' \cap A(U) \). Thus \( |W'| \circ A(U) \) and hence \( |W| \circ A(U) \).

On the other hand, the composition \( W \overset{\alpha}{\longrightarrow} W \circ \text{id} \rightarrow W \circ A(U) = A(U) \) is injective and holomorphic, whence \( (W, \mathcal{G}_W') \subseteq A(U) \) for a suitable complex substructure \( \mathcal{G}_W' \) of \( W \). By 2.1.1, the orbit maps \( A(U) \overset{\alpha}{\longrightarrow} A(U) \circ \text{id} \rightarrow U \) are finite and hence locally biholomorphic. Thus no reduced subspace of \( A(U) \) can admit a proper complex substructure with respect to which the evaluation map remains holomorphic. We conclude that if \( W \times U \rightarrow U \) is as postulated, then \( \rho^U_\phi(W) \circ A(U) \) and the inclusion \( \rho^U_\phi(W) \rightarrow A(U) \) is holomorphic, and hence so is \( \rho^U_\phi(W) \rightarrow A(U) \).

\[ A(U) \text{ is normal in } Aut(U), \text{ since } \alpha \circ A(U) \circ \alpha^{-1} \subseteq Aut(U) \text{ for every } \alpha \in Aut(U). \]

If \( U \) is compact, then \( A(U) \) is a compact connected complex subgroup of the connected complex Lie group \( Aut(U) \) and hence is central.

In the last chapter, we shall make use of the following generalization of the above result:

\[ 2.3.2 \textbf{Lemma: Let } \ldots \rightarrow U_{n+1}^{\alpha_{n+1}} \rightarrow \ldots \rightarrow U_1^{\alpha_1} \rightarrow U_0 \text{ be a sequence of coverings, and let } \]
\[ W_n \subseteq \text{Hol}(U_n) \text{ with } \alpha_n \in W_n, n \in \mathbb{N}. \]
\[ \text{Then } |W_n| \circ A(U_n) \circ \alpha_n \text{ for } n \gg 0, \text{ and the inclusion is holomorphic.} \]
\[ \text{In particular, any } W \subseteq \text{Hol}(U) \text{ containing a covering } \alpha \text{ lies in } A(U) \circ \alpha. \]

\[ \textbf{Proof:} \text{We may assume that all } W_n \text{ are irreducible. It suffices to show that } W_n \circ \ldots \circ W_{n+k} \subseteq A(U_n) \circ \alpha_n \circ \ldots \circ \alpha_{n+k} \text{ for some } k \geq 1, \text{ since the } \alpha_n \text{ are surjective and locally biholomorphic.} \]

On the other hand, by 2.2.1(i), \( W_n \circ \ldots \circ W_{n+k} \circ \ldots \circ W_{n+k+1} = W_n \circ \ldots \circ W_{n+k} \circ \alpha_{n+k+1} \circ \ldots \circ \alpha_{n+k+1} \text{ for all } n \geq 1, \text{ and for } k \text{ sufficiently large (depending on } n). \text{ Thus, after suitably condensing the given sequence, we may assume that } W_n \circ \alpha_{n+k} \rightarrow W_n \circ \alpha_{n+k+1} \text{ for all } n, \text{ whence, in particular, } \dim W_n \leq \dim W_n + 1 \text{. Cutting off a sufficiently long initial sequence, we can assume that } \dim W_n = \dim W_{n+1} \text{ for all } n. \text{ The inclusion } W_n \circ \alpha_{n+1} \rightarrow W_n \circ \alpha_{n+2} \text{ is bijective and hence biholomorphic, and, utilizing its inverse, we obtain a holomorphic map} \]
\[ \phi := (W_n \times W_{n+1} \rightarrow W_n \circ W_{n+1} \rightarrow W_n \circ \alpha_{n+1} \rightarrow W_n) \]

with \( \phi((\alpha_{n+1}, \ldots) = \text{id}_{W_n} \). Thus \( \rho^U_\phi(W_{n+1}) \subseteq A(W_n) \) and \( \rho^U_\phi \) is finite, since so is \( W_{n+1} \rightarrow \alpha_n \circ W_{n+1} \). From \( \dim W_n \geq \dim A(W_n) \geq \dim W_{n+1} = \dim W_n \), we infer that \( W_n \subseteq A(W_n) \) is a torus. Denote by \( g_n : C^k \rightarrow W_n \) the universal covering and assume that \( g_n(0) = \alpha_n \). Let \( E_{n+1} := E_{n+1}(U_n \circ (g_{n+1} \times \text{id}_{U_{n+1}}) : C^k \times U_{n+1} \rightarrow U_n \) and denote by \( h_n : C^k \rightarrow C^k \)
the linear lifting of \( \phi(\alpha, \cdot) : W_{n+1} \to W_n \). Then there exists a unique \( E'_{n+1} : C^k \times U_{n+1} \to U_{n+1} \) with \( \alpha_n \circ E'_{n+1} = E'_{n+1} \circ (h \times \alpha_{n+1})(0, \cdot) \), and \( E'_{n+1}(0, \cdot) = \text{id}_{U_{n+1}} \).

The simple-arrow part of the diagram

\[
\begin{array}{ccc}
C^k \times U_{n+2} & \xrightarrow{E_{n+2}} & U_{n+1} \\
\downarrow h \times \alpha_{n+1} & & \downarrow \text{id} \\
C^k \times U_{n+1} & \xrightarrow{E_{n+1}} & U_{n+1}
\end{array}
\]

is commutative, and from \( E_{n+2}(0, \cdot) = \alpha_{n+1} = E'_{n+1} \circ (h \times \alpha_{n+1})(0, \cdot) \), we infer that the entire diagram is commutative, since \( \alpha_n \) is a covering. Thus \( \rho_{E_n}(C^k) \circ \alpha_n = W_n \) (as subsets of \( \text{Hol}(U_{n+1}, U_n) \)), and we can endow \( V_n := \rho_{E_n}(C^k) \) with the complex structure given by the bijection \( V_n \to U_{n+1} \circ \alpha_n = W_n \).

The diagram

\[
\begin{array}{ccc}
V_{n+1} \times U_{n+1} & \xrightarrow{E_{n+1}} & U_{n+1} \\
\downarrow & & \downarrow \\
W_n \times U_{n+1} & \xrightarrow{E_{n+1}} & U_{n+1}
\end{array}
\]

is commutative with locally biholomorphic vertical arrows; thus \( E_{U_{n+1}} \) is holomorphic, whence \( V_n \to \text{Hol}(U_{n+1}) \). As \( U_n \subseteq V_n \) the assertion follows.

2.3.2 Remarks. Assume that \( U \) and \( W \) are compact and that some \( w_0 \in W \) is a covering \( U \to V \).

If \( \rho_\phi(W) \circ \text{Hol}(V) \circ w_0 \) then the corresponding map \( \overline{\phi} : W \to \text{Hol}(V) \) is holomorphic with image in \( \text{Aut}(V) \).

**Proof.** Evidently, \( \rho_\phi = (W \xrightarrow{\overline{\phi}} \text{Hol}(V) \to \text{Hol}(V) \circ w_0 \in \text{Hol}(U, V)) \) and \( \text{Hol}(V) \to \text{Hol}(V) \circ w_0 \) is biholomorphic, since \( w_0 \) is surjective and locally biholomorphic. Furthermore, \( |\overline{\phi}|(W) \in A(V) \), and \( \text{Aut}(V) \) is open in \( \text{Hol}(V) \).

2.3.3 Definitions. Let \( g : U \to V \) be a holomorphic map between connected complex spaces, and let \( T \sqsubseteq A(U) \).

\( g \) is \( T \)-equivariant if there exists a map \( g_\alpha : T \to A(V) \) with \( g_\alpha(O) = 0 \) and \( g_\alpha(\alpha) \circ g = g \circ \alpha \) for all \( \alpha \in T \).

\( g \) is \( T - T' \)-equivariant if \( g \) is \( T \)-equivariant with \( g_\alpha(T) \subseteq T' \sqsubseteq A(V) \).

2.3.3 Remarks. Let \( g \) be a \( T \)-equivariant.

(i) \( g_\alpha \) is uniquely determined and is a homomorphism of complex Lie groups.

(ii) If \( f_\alpha \neq 0 \), then \( g_\alpha \) is finite.
Proof. Let \( u_0 \in U \) and consider the commutative diagram

\[
\begin{array}{ccc}
T & \xrightarrow{g} & A(V) \\
\downarrow & & \downarrow \\
1 \cdot u_0 & \xrightarrow{1 \cdot g(u_0)} & 1 \cdot g(u_0) \\
T \cdot u_0 & \xrightarrow{g} & A(V) \cdot g(u_0)
\end{array}
\]

The mapping \( g( u_0) \) is locally biholomorphic and \( g_\ast(0) = 0 \); thus \( g_\ast \) is a homomorphism of complex Lie groups. In particular, \( g_\ast \) is uniquely determined by the equation \( g_\ast(u_0) = g \circ u_0 \).

If \( g \) is finite in \( u_0 \), then \( g_\ast \) is finite in 0 and hence everywhere.

\[\diamond\]

2.3.4 Lemma. Let \( g : U \to V \) be a holomorphic map between connected complex spaces.

(i) Let \( T \sqsubset A(U) \), \( T'' \sqsubset A(V) \) such that \( g(T \cdot u_0) \subseteq T'' \cdot g(u_0) \) for some \( u_0 \in U \). If \( \bigcap_{u \in U} T'' \cdot g(u) = 0 \) (e.g., if \( g \) is surjective), then \( g \) is \( T - T'' \)-equivariant.

(ii) If \( g \) is proper with \( g_\ast \mathcal{O}_U = \mathcal{O}_V \), then \( g \) is \( A(U) \)-equivariant.

(iii) Let \( T'' \sqsubset A(V) \). If \( g \) is a covering, then there exists a unique \( T \sqsubset A(U) \) such that \( g \) is \( T - T'' \)-equivariant. In particular, \( \dim A(U) \geq \dim A(V) \).

(iv) Let \( h : V \to V'' \) be a covering, and let \( T \sqsubset A(U) \). If \( g \) is surjective and \( h \circ g \) is \( T \)-equivariant, then \( g \) is \( T \)-equivariant.

Proof (i) We may assume \( g(T \cdot u_0) = T'' \cdot g(u_0) \). Then \( T'' \circ g \circ T \rightarrow \text{Hol}(U, V) \) with \( (g \circ T) \cdot u_0 = (T'' \circ g \circ T) \cdot u_0 = (T'' \circ g) \cdot u_0 \); thus \( T'' \circ g \circ T = T'' \circ g = g \circ T \) by 2.2.1(i), and we conclude that \( g(T \cdot u) = T'' \cdot g(u) \) for all \( u \in U \). The maps \( g_\ast : A(T \cdot u) \to A(T'' \cdot g(u)) = T'' / T'' \cdot g(u) \) are \( A(T \cdot u) \)-equivariant with \( (g_\ast)_\ast : A(T \cdot u) \to A(T'' \cdot g(u)) = T'' / T'' \cdot g(u) \). As every \( T'' \cdot g(u) \) is finite and \( \bigcap_{u \in U} T'' \cdot g(u) = 0 \), there exists a holomorphic homomorphism \( g_\ast \cdot T \rightarrow T'' \) such that every composition \( T \xrightarrow{\text{kan}} A(T \cdot u) \xrightarrow{(g_\ast)_\ast} T'' / T'' \cdot g(u) \) factors through \( g_\ast \cdot T \). By construction, \( g_\ast(\alpha) \circ g = g \circ \alpha \) for all \( \alpha \in T \).

(ii) Let \( \phi := g \circ E_U = (T \times U \xrightarrow{id \times \phi} T \times U \xrightarrow{\phi \cdot U} V) \) be the simultaneous Stein factorization. Then every \( \phi_{st}(t \cdot \cdot) \) is biholomorphic, since \( \mathcal{O}_V = g_\ast \mathcal{O}_U = g_\ast t_\cdot \mathcal{O}_U = g_{st}(t) \cdot \mathcal{O}_U = g_{st}(t) \cdot \mathcal{O}_U \). By 2.3.1, the assertion follows with \( g_\ast := \rho g_{st} \).

Assertion (iii) follows from (i) by applying Lemma 2.3.2 to the sequence of coverings

\[
\ldots \rightarrow U \xrightarrow{id} U \rightarrow \ldots \rightarrow U \xrightarrow{g} V \]

with \( W := T'' \circ g \cdot U \).

Assertion (iv) is evident by (i) and (iii).
2.3.4.a Corollary. \( A(U \times V) = A(U) \times A(V). \)

**Proof.** The inclusion \( A(U) \times A(V) \subseteq A(U \times V) \) is obvious. To show the converse, let \( \phi := p_U \circ E = (A(U) \times V) \times (U \times V) \xrightarrow{\pi_1 \times \id} A_0 \times (U \times V) \xrightarrow{\phi_A} U \) be the simultaneous Stein factorization, and let \( \psi := \phi_\alpha \circ (\id_{A_0} \times j_{V_0}) : A_0 \times U \to A_0 \times (U \times V) \to U \) for some fixed \( v_0 \in V \). Then \( E \subseteq \{ \phi, \psi \} \) whence \( \rho_\phi(A_0) \subseteq A(U) \) by 2.3.1, and therefore \( p_U (A(U \times V)(u, v_0)) = (\rho_\psi(A_0))u = c(A(U)u) \) for all \( u \in U \). By Lemma 2.3.4(i), \( p_U \) is \( A(U \times V) \)-equivariant, and, symmetrally, so is \( p_V \). Evidently, \( \{ (p_U)_*, (p_V)_* \} : A(U \times V) \to A(U) \times A(V) \) is injective, and the assertion follows. \( \Box \)

2.3.4.b Corollary. Let \( T', T \) be tori and let \( \phi: T' \times U \to T \) be a holomorphic. If some \( \phi(t_0, \cdot) : U \to T \) is constant, then \( \phi \) factors through \( p_{T'} \).

**Proof.** We may assume \( t_0 = 0, \phi(t_0, \cdot) = [0] \cdot T' \) acts effectively on \( T' \times U \) via addition in the first factor. By Lemma 2.3.4(i), \( \phi \) is \( T' \)-equivariant, thus \( [0] = \phi(0, \cdot) = (\phi \circ (-t)) (t_0, \cdot) = \phi_\ast(-t) \circ \phi(t_0, \cdot) \), i.e., \( \phi(t, \cdot) = \phi_\ast(t) \).

Let \( T \subseteq A(U) \). By ([?], Satz IV.10.1), there exists a holomorphic structure on \( |U|/T \) such that the quotient map \( g \) becomes holomorphic. Replacing this structure by \( \sigma_{\mathcal{O}_U} \), we conclude that the analytic quotient \( U \to |U|/T \) exists; it will be denoted by \( q_{T} : U \to |U|/T \).

We shall employ the following notation: \( (Q_U : U \to U_\infty) := (q_{A(U)} : U \to U/A(U)). \)

2.3.4.c Corollary.

(i) \( Q_{U \times V} = Q_U \times Q_V \).

(ii) The mapping \( U \to U_\infty \) is functorial with respect to proper holomorphic mappings that satisfy \( \sigma_{\mathcal{O}_U} = 8. \).

(iii) There exists a covering \( U' \to U \) such that every covering \( g : U \to U' \) is \( A(U_1) \)-equivariant. In particular, there exists a covering \( g_\infty : (U_1)_\infty \to (U')_\infty \) with \( g_\infty \circ Q_{U_1} = Q_U \circ g \).

**Proof.** (i) follows from 2.3.4.a, (ii) from 2.3.4(ii); to prove (iii), note that by 2.2.1, every covering \( U' \to U \) satisfies \( \dim A(U') \leq d_\delta(U') = d_\delta(U) \). Thus, if \( U' \to U \) is a covering with \( \dim A(U') \) maximal, then every covering \( U_1 \to U' \) is \( A(U_1) \)-equivariant by 2.3.4(iii) and 2.3.4(i).

2.4. Torsion bundles over tori

Let \( U \) be a connected complex space.

2.4.1. Definition. Let \( \pi : U \to T \) be holomorphic, \( T \) a k-dimensional torus. We shall say that \( \pi \) is a torsion bundle over \( T \) with fibre \( U_\delta \), if \( \pi \) is a \( U_\delta \)-bundle with finite structure group such that the total space of the associated principal bundle is connected.
Notation. \((\pi : U \to T) \in \mathcal{F}_k\) with fibre \(U_0\). Sometimes we also say \(U \in \mathcal{F}_k\), if there exists \((\pi : U \to T) \in \mathcal{F}_k\) with some fibre. With this convention we let \(\mathcal{F} := \bigcup_{k \geq 1} \mathcal{F}_k\).

\[2.4.1.\] Remarks, examples, and notations

(i) Every connected complex space lies in \(\mathcal{F}_0\). If \((\pi : U \to T) \in \mathcal{F}_k\) with fibre \(U_0\), and \((\tau : V \to T') \in \mathcal{F}_l\) with fibre \(V_0\), then \(\tau \times \pi \in \mathcal{F}_{k+l}\) with fibre \(U_0 \times V_0\). In particular, \(U \times V \in \mathcal{F}\) if \(U \in \mathcal{F}\) or \(V \in \mathcal{F}\). We shall see later on that the converse holds, too.

(ii) Let \(T := \mathbb{C}/\mathbb{Z} + i\mathbb{Z}\) and let \(\pi_j : T \to T_j := \mathbb{Z}_j\) be the \(\mathbb{Z}_j\)-principal bundle given by the \(\mathbb{Z}_j\)-action \(\mathbb{Z}_j \times T \ni (n, t) \mapsto t + \frac{n_j}{5} \in T\), where \(1 \leq j \leq 4\). For every complex space \(U_0\) with non-trivial \(\mathbb{Z}_j\)-action, the \(U_0\)-bundle \(\pi_j(U_0)\) associated to \(\pi_j\) is a torsion bundle over \(T_j := \mathbb{Z}_j\) with fibre \(U_0\). The bundles \(\pi_j(U_0)\) and \(\pi_{5-j}(U_0)\) are isomorphic via \(t \mapsto -t\), whereas \(\pi_j(U_0)\) and \(\pi_k(U_0)\) are not isomorphic for \(k \neq 5, 5 - j\). The associated fibre spaces, however, and, a fortiori, their total spaces, may be isomorphic. For instance, if \(U_0 = \mathcal{V}^3\) for some \(V\), where \(\mathbb{Z}_3\) acts by cyclic permutation of the coordinates, then the fibre spaces associated to the \(\pi_j(U_0)\) are all isomorphic. On the other hand, if \(U_0 = P\), with \(\mathbb{Z}_5\)-action \((n, (x_0 : x_1)) \mapsto (x_0 : \varepsilon^nx_1)\) where \(\varepsilon = \exp\left(\frac{2\pi i}{5}\right)\), then not even the total spaces of \(\pi_1(P)\) and \(\pi_2(P)\) are isomorphic (see [5], 6.2).

(iii) Let \((\pi : U \to T) \in \mathcal{F}_k\) with fibre \(U_0\), and let \(\pi : T' \to T\) be the associated principal bundle. Then \(T'\) is a \(k\)-dimensional torus and \(\pi\) is a covering. We may assume that \(\pi'\) is a homomorphism and identify the structure group \(\Gamma\) of \(\pi\) with \(\text{Ker} \pi'\). Assume that the \(\Gamma\)-action on \(T'\) is given by \((t, v) \mapsto t + \chi(v)\) with some \(\chi \in \text{Aut}(\Gamma)\) and define \(\chi : \Gamma \to \text{Aut}(U_0)\) by \(\chi(v) := \chi(v)^{-1}\) (7) where we consider \(\Gamma\) as a subgroup of \(\text{Aut}(U_0)\). Then the natural map \(T' \times U_0 \to U\) given by \((t, v) \sim (t + \chi(v), (-\gamma)(v))\) coincides with the quotient map \(q : T' \times U_0 \to (\text{graph} \chi)\). Consider the cartesian square

\[
\begin{array}{ccc}
T' \times U_0 & \xrightarrow{q} & U \\
\downarrow \pi & & \downarrow \pi \\
T' & \xrightarrow{\pi} & T
\end{array}
\]

and let \(T'\) act on \(T' \times U_0\) via addition in the first factor. Applying Lemma 2.3.4(iv) to \(T' \times U_0 \to q(T' \times U_0)\), we infer from 2.3.4(i) that \(q\) is \(T'\)-equivariant. Moreover, \(q_*\) is injective, since \(\text{Ker} q_* \subset \Gamma\) and \(q_*(\gamma, u) \neq q_*(\gamma', u)\) for all \(\gamma, \gamma' \in \Gamma\).

Therefore, we shall from now on consider \(T' \sqcup A(U)\) in this sense.
(iv) Let $S$ be a $k$-dimensional torus, and let $\chi : \Gamma \to \text{Aut}(U_0)$ be a monomorphism from a finite subgroup $\Gamma$ of $S$ into the automorphism group of some complex space $U_0$. Then, evidently, the map $(S \times U_0) / \text{graph}(\chi) \to S / \Gamma$, given by $(s, u) \mapsto s + \Gamma$, is a torsion bundle over $S / \Gamma$ with fibre $U_0$.

Conversely, by (iii), every $\pi \in \mathcal{F}_k$ arises in this way.

(v) Let $(\pi : U \to T) \in \mathcal{F}_k$ with fibre $U_0$, and let $V$ be a connected component of $\pi^{-1}(0) \cong U_0$. Denote by $A \subset T \sqcup A(U)$ the isotropy group of $V$; then $A \subset \Gamma := \text{Ker} \pi|_- (\pi_0^\Gamma$ as in (iii)), and $A$ stabilizes every connected component of $\pi^{-1}(0)$. In particular, $A$ contains every isotropy group $T_u$ for $u \in \pi^{-1}(0)$. Thus $\pi_\delta = \pi'_\delta = (\pi|_- \cong T'/\Gamma_\delta \cong T')$ with some homomorphism $\lambda|_- \text{ and the restrictions } \pi : T'u \to T$, $u \in \pi^{-1}(0)$, all factor through $\lambda|_- \text{ As } U \text{ is the disjoint union of the } T'_u, u \in \pi^{-1}(0), \text{ we obtain a map (of sets) } \pi_\delta : U \to T'/\Delta\text{ with } \pi = \lambda \circ \pi_\delta$ is holomorphic, since $\lambda$ is locally biholomorphic.

The commutative diagram

$$
\begin{array}{ccc}
T' \times U_0 & \xrightarrow{q} & U \\
\downarrow \Pr & & \downarrow \Pr \\
T' / \Delta & \xrightarrow{\lambda} & T
\end{array}
$$

immediately yields that $\pi_\delta \in \mathcal{F}_k$ with fibre $V$ and structure group $A$. If $U = (T' \times U_0) / \text{graph}(\chi) \text{ (according to (iv))}$, then $U = (T' \times V) / \text{graph}(\psi), \text{ where } \psi := (\chi|_{\Delta} \to \text{Aut}(V))_\downarrow$.

Note that for compact reduced $U$, the equation $\pi = \lambda \circ \pi_\delta$ is just the Stein factorization of $\pi|_-$. The following characterization of $\mathcal{F}_k$ is one of the essential ingredients of the investigations performed in Chapter 5:

2.4.2 Lemma. Let $\pi : U \to T$ be a holomorphic map into a $k$-dimensional torus $T$ and assume that there exists a $k$-dimensional $T' \sqcup A(U)$ with $\pi(\pi_0^T) = T$ for some $u_0 \in U$.

Then $\pi \in \mathcal{F}_k$ with fibre $\pi^{-1}(\pi(u_0)) = U_0$.

Proof. By Lemma 2.3.4(i), the map $\pi$ is $T'$-equivariant with $\pi_*(T') = T$; in particular, $\pi$ is locally trivial, and the diagram

$$
\begin{array}{ccc}
T' \times U_0 & \xrightarrow{E_U} & U \\
\downarrow \Pr & & \downarrow \Pr \\
T' & \xrightarrow{\lambda} & T \\
\downarrow u_0 & & \downarrow \star \\
T' u_0
\end{array}
$$
commutes. Again by 2.3.4(i), \(E_U\) is \(T'\)-equivariant (with respect to the addition in the first factor) whence \(E_U\) is a covering. Thus every \(t \in T'\) defines a map \(\chi_t : \Gamma = \text{Ker } \pi_\ast \to \text{Aut } (U_0)\) such that \(E_U^{-1}(E_U(t, u)) = \chi_t(\Gamma)u\) for all \(u \in U\). Now \(t(E_U(\gamma, \chi_0(\gamma)(u))) = t(E_U(0, u)) = E_U(t, 0) = E_U(t + \gamma, \chi_t(\gamma)(u)) = t(E_U(\gamma, \chi_t(\gamma)(u)))\) for all \(t \in T', u \in U\), whence \(\chi_t = \chi_0\) for all \(t \in T'\); in particular, \(\chi := \chi_0\) is a homomorphism, since \(\chi(\gamma + 7') = \chi_0(\gamma') \circ \chi_0(\gamma) = \chi(\gamma') \circ \chi(\gamma)\) for all \(\gamma, \gamma'\). Evidently, \(E_U : T' \times U_0 \to U\) factors through \(T' \times U_0 \xrightarrow{\text{kan} \times \text{id}} (T/\text{Ker } \chi) \times U_0\), and we conclude that \(\text{Ker } \chi = 0\). Therefore \(\pi_1\) can be represented as in 2.4.1.a(iv).

2.4.2.a Corollary. Let \(\alpha : U \to \text{A}(U)\) be holomorphic. Fix some \(u_0 \in U\) and define \(\alpha_n : U \to U\) by \(\alpha_n := (U \xrightarrow{\alpha} \text{A}(U) \xrightarrow{u_0} U)^n\) for all \(n \in \mathbb{N}\).

There exists \(k \in \mathbb{N}\) such that \((\alpha \circ \alpha_n : U \to \alpha(\alpha_{n-1}(\text{A}(U)u_0)))) \in \mathcal{S}_k\) for all \(n \geq 0\).

Proof. We may assume \(\alpha(u_0) = 0\). Then \(T_n := \alpha(\alpha_n(\text{A}(U)u_0)) \subseteq \text{A}(U)\) and \(T_{n+1} \subseteq T_n\), whence \(T_n = T_{n+1}\) for \(n \geq 0\). Letting \(k := \dim T_n\) for \(n \geq 0\), the assertion follows from Lemma 2.4.2, since \(\alpha_n : U \to U\) factors through \(u_0 : T_{n-1} \to U\).

2.4.2.b Corollary. Let \(U \times V \in \mathcal{S}_k\) \(\not\subset \mathcal{S}_k\) then \(U \in \mathcal{S}_k\).

Proof. Let \((\pi : U \times V \to T) \in \mathcal{S}_k\). Composing \(\pi\) with some covering \(T \to T'\), we may assume \(T \subseteq \text{A}(U \times V)\) (compare 2.4.1.a(iii)), and that \(\pi_\ast : T \to T\) is homothetic.

Fix some \((u_0, v_0) \in U \times V\) and considering \(g := (A(U) \times A(V) = A(U) \times A(V) \xrightarrow{(u_0, v_0)} A(U) \times V \xrightarrow{\pi_\ast} T)\); evidently, \(g(T) = T\). For \(S \in \{U, V\}\) let \(d_S := \lim \dim \lim \text{Im } (T) \xrightarrow{P_{A(S)}} A(S) \xrightarrow{\pi_\ast} A(U) \times A(V) \xrightarrow{g} T)^n\); then \(S \in \mathcal{S}_{d_S}\) by 2.4.2.a whence \(d_S = 0\). Thus the lifting \(\overline{g} : A(U) \times A(V) \to \overline{T}\) to the universal coverings with \(\overline{g}(0, 0) = 0\) satisfies the condition of 0.3.2.a, and we conclude that \(T \xrightarrow{\text{P}_{A(U)}} A(U) \xrightarrow{\overline{g}} A(U) \times V \xrightarrow{g} T\) is surjective, whence \(d_U = k\).

2.4.3 Definition. Let \((\pi_j : U_j \to T_j) \in \mathcal{S}_k\) with fibre \(V_j, j = 1, 2\) \(A\) holomorphic map \(f : U_1 \to U_2\) is a \(\mathcal{S}\)-morphism, if it is \(T_1 \to T_2\)-equivariant and fibre-preserving with respect to \(\pi_1, \pi_2\).

For brevity of expression, we employ the notation: \(f : \pi_1 \to \pi_2\).
2.4.3.a Remarks

(i) If \( f: \pi_1 \to \pi_2 \) there exists a commutative diagram of holomorphic mappings:

\[
\begin{array}{ccc}
T'_1 \times V_1 & \overset{q_1}{\to} & U_1 \\
\downarrow f \times f_0 & & \downarrow f \\
T'_2 \times V_2 & \overset{q_2}{\to} & U_2 \\
\end{array}
\]

(ii) \( f: U_1 \to U_2 \) is fibre-preserving, if \( f \) maps at least one fibre of \( \pi_1 \) into one of \( \pi_2 \).

(iii) A surjective holomorphic \( f: U_1 \to U_2 \) is a \( \mathcal{S} \)-morphism if and only if it maps some fibre of \( \pi_1 \) into one of \( \pi_2 \), and some orbit of \( T'_1 \) into one of \( T'_2 \).

**Proof.** (i) The existence of the righthand rectangle is obvious. Let \( f_0 := f|\pi_1^{-1}(0) \to \pi_2^{-1}(0) \). Then the lefthand rectangle commutes, since \( q_2(f_*(t)|f_*(u)) = q_2(f_*(t)(0, f_0(u))) = f_*(t)(q_1(0, f_0(u))) = f(t(q_1(0, u))) = f(q_1(t, u)). \)

(ii) and (iii) follow from 2.3.4.b, 2.4.1.a(v), and from 2.3.4(i). \( \square \)

3. PRELIMINARIES ON ISOMORPHISMS BETWEEN PRODUCTS

Let \( f = (lf, \tau f): X \times Y \to U \times V \) be a biholomorphic map between connected complex spaces. This is the starting position for both the cancellation and the decomposition problem. We shall now develop some techniques for reducing the situation to a simpler one.

3.1. Relations between the \textbf{partial} maps

3.1.1 Lemma. Let \((x|y) \in X \times Y\) \((u|v) := f(x|y)\).

(i) \([\overline{x}] = \tau f(\overline{y}) \) induces a biholomorphic map \( \text{im} F := \overline{\overline{x}^{-1}(u)} \) onto \( \overline{\overline{y}^{-1}(y)} \), whose inverse is given by \( \overline{u} = \overline{lf}^{-1}(u, \ldots) \).

(ii) If \( \overline{\overline{y}} \) is biholomorphic in \( x \), then \( \overline{u} \) is biholomorphic in \( v \).

(iii) If every \( \overline{\overline{y}} \) where \( y \in Y \), is biholomorphic, then so is every \( \overline{\overline{u}} \) where \( u \in U \).

**Proof.** (i) From \( id_{p \times \{v\}} = \overline{f^{-1}} \circ f|_{p \times \{v\}} = \overline{id} \circ (f|_{p \times \{v\}}) \circ f|_{p \times \{v\}} \) we infer \( \overline{u} \overline{y}|_F = id \), and \( \overline{u} \overline{y}|_F = \| y \|_p \), and the assertion follows with a symmetry argument.

Assertion (ii) is evident by 1.2.1.a.

(iii) The \textbf{partial} maps \( \overline{f} \) resp. \( \overline{u} \) are all biholomorphic, if and only if \((lf, pf_y): X \times Y \to U \times Y \) resp. \((pf_u, \tau f^{-1}): U \times V \to U \times Y \) is biholomorphic. Thus the assertion follows from \((pf_u,\tau f^{-1}) \circ f = (lf, pf_y). \)
3.1.1.a Corollary. If all \( \overrightarrow{y} \) are biholomorphic, then \( Y \cong V \).

3.1.1.b Remark. Let
\[
\begin{array}{ccc}
X \times Y & \xrightarrow{f} & U \times V \\
1 \times q_1 & \downarrow & 1 \times (p_2 \times q_2) \\
X' \times Y' & \xrightarrow{f'} & U' \times V'
\end{array}
\]
be commutative and let \( y' := q_1(y) \).

Then \( \overrightarrow{y}' \circ p_1 = p_2 \circ f' \circ (p_1 \times q_1)(\cdot, y) = p_2 \circ (p_2 \times q_2) \circ f(\cdot, y) = p_2 \circ \overrightarrow{y} \). In particular, if \( p_1 \) is surjective, then \( \overrightarrow{y} \) is constant, if \( \overrightarrow{y}' \) is.

3.2. Degenerating Isomorphisms

3.2.1 Definition. Let \( (x, y) \in X \times Y \), \((u, v) = f(x, y)\). \( f \) degenerates \( X \times Y \rightarrow U \times V \) if the reduction of the map \((u, x, y, v)^n\) is constant for \( n \gg 0 \). We say that \( f \) degenerates if \( f \) degenerates with respect to some \((x, y)\).

3.2.1.a Examples.

(i) If \( f \) is a product of isomorphisms \( X \rightarrow U \), \( Y \rightarrow V \), then e.g. every \( \overrightarrow{u} \), \( u \in V \), is constant, whence \( f \) degenerates with respect to every \((x, y)\).

(ii) Let \( X = Y = U = V \) be a one-dimensional torus, and let \( f \) be given by \( f(x, y) = (2x + y, x + y) \). Then \((\overrightarrow{u}, \overrightarrow{x}, \overrightarrow{y}, \overrightarrow{v})(x') = 2x' + 3x + 4y \) for all \((x, x') \in X, y \in Y, (u, v) = f(x, y) \). Thus \( f \) does not degenerate.

3.2.1.b Remarks.

(i) If \( f \) degenerates with respect to \( (x, y) \), then \( J \circ f \circ J \) degenerates with respect to \( (y, x) \), and \( J \circ f^{-1} \) degenerates with respect to \((u, v) = f(x, y)\). It is not clear whether e.g. \( f^{-1} \) degenerates.

(ii) Let \( X \) be compact, \((x_0, y, u_0, v_0) \in X \times Y \times U \times V \). If \(|(u_0, x_0, v_0, y_0)^n|\) is constant for some fixed \( n \in \mathbb{N} \), then so is every \(|(u, x, y, v)^n|\) (compare Lemma 2.1.1). In particular, if \( f \) degenerates with respect to some \((x_0, y_0)\), then \( f \) degenerates with respect to every \((x, y)\), and the minimal \( n \) from the definition does not depend on \((x, y)\).

(iii) Let
\[
\begin{array}{ccc}
X \times Y & \xrightarrow{f} & U \times V \\
1 \times q_1 & \downarrow & 1 \times (p_2 \times q_2) \\
X' \times Y' & \xrightarrow{f'} & U' \times V'
\end{array}
\]
be commutative with surjective \( p_1 \) and biholomorphic \( f' \), and assume that \( f \) degenerates with respect to some \((x, y)\). Then, by 3.1.1.b, \( f' \) degenerates with respect to \( (p_1 \times q_1)(x, y) \).
3.3. Simultaneous subdecompositions

33.1 Lemma and Notation. Let $Y_1 := Y \times V, y_1 := (y, v) \in Y_1$, and let $S_1 f : x \times Y_1 \rightarrow X \times Y_1$ be given by $S, f := (f^{-1} \times \text{id}) \circ J \circ (f \times \text{id})$. For $n \geq 1$ let $Y_{n+1} := Y_n \times Y_n$. Then

(i) $S_{n+1} f \circ S_{n+1} f = \text{id}_{X \times Y_{n+1}}$ and

(ii) $p_X \circ S_{n+1} f((., (y, v))) = (\overrightarrow{v}, \overrightarrow{y})^{2n}$

for all $n \geq 0$.

Proof: It suffices to consider the case $n = 0$. Then (i) is evident from the definition of $S_1 f$, and (ii) follows from $S_1 f((., (y, v))) = (f^{-1} \times \text{id}) \circ J \circ (f(., y), [v]) = (f^{-1} \times \text{id}) \circ (lf(., y), [v], rf(., y)) = (\overrightarrow{v}, \overrightarrow{y}, \overrightarrow{v}, \overrightarrow{y})$. \hfill \Box

3.3.2 Definition. Let $(x | y) \in X \times Y, (u | v) = f (x | y)$. We shall say that $(x | y)$ decomposes $f$, if the following conditions are fulfilled:

(i) For $\{(A, B), (C, D)\} = \{(X, Y), (U, V)\}$ with $a, b, c, d \in \{x, y, u, v\}$ accordingly, the systems of complex subspaces

$$\{((\overrightarrow{d} b)^n)^{-1}(a) : n \in \mathbb{N}\}, \quad \{((\overrightarrow{c} b)^n)^{-1}(a) : n \in \mathbb{N}\}$$

$$\{((\overrightarrow{d} a)^n)^{-1}(b) : n \in \mathbb{N}\}, \quad \{((\overrightarrow{c} a)^n)^{-1}(b) : n \in \mathbb{N}\}$$

have maximal elements $A_D, A_n, B_D, B_n$, respectively.

(ii) For $\{(A, B), (C, D)\} = \{(X, Y), (U, V)\}$ with $A \in \{X, U\}$, the maps $A_C \times B_C \rightarrow C$ given by $p_C \circ f$ (if $A = X$) or by $p_C \circ f^{-1}$ (if $A = U$) are biholomorphic.

(iii) The isomorphism $f : U_X \times V_X \times U_Y \times V_Y \rightarrow X_U \times X_U \times Y_V \times Y_V$ induced by $f$ via (ii) satisfies:

Each of the partial maps $R_S, S_R \rightarrow R_S, S_R \rightarrow S_R$ given by $f, f^{-1}$ and $x, y, u, v$ (where $R \in \{U, V\}, S \in \{X, Y\}$) is biholomorphic (i.e. the composition $U_X \rightarrow U_X \times S_R \times U_U \times S_U$).

$f$ induces a simultaneous subdecomposition, if some $(x, y)$ decomposes $f$.

3.3.2a Remarks:

(i) The condition 3.3.2(iii) is well-defined, since by construction $s \in S_R \cap S_R$ for all possible combinations (i.e. $\overrightarrow{v} \overrightarrow{y}(x) = l f^{-1}(l f(x, y), v) = l f^{-1}(f(x, y)) = x \text{ etc.}$).

(ii) $f$ as in 3.2.1.a(i) induces a simultaneous subdecomposition, $f$ as in 3.2.1.a(ii) does not.
(iii) If \((x, y)\) decomposes \(f\), then \((x, y)\) also decomposes \(f \circ f\) and \(f(x, y)\) decomposes \(f^{-1}\).

(iv) Assume that \(f\) induces a simultaneous subdecomposition and that \(X \cong U\) is indecomposable; then \(Y \cong V\).

In fact, if \(X \not\cong C^0\), then either \(X = X_U\), and hence \(U = U_X\), or \(X = X_V\) and \(U = U_Y\).

In the first case, we conclude \(V = V_Y \cong Y_V = Y\); in the second one, \(Y \cong U_Y \times V_Y = X_Y \times Y_V \cong V\).

### 3.3.2 Example

With the notations of 2.4.1.a, let \(X = U = T\), and \(Y\) be the total space of \(\pi_1(P_1), V\) that of \(\pi_2(P_1)\). Then \(X \cong U\) is indecomposable. \(X \times Y\) is isomorphic to \(U \times V\) via the map induced by \(T \times T \times P, \exists (s, t, x) \mapsto (3s + 5t, s + 2t, x) \in T \times T \times P_1\), but \(Y\) is not isomorphic to \(V\).

### 3.3.3 Lemma

Let

\[
\begin{array}{c}
\begin{array}{c}
X \times Y \xrightarrow{f} U \times V \\
p_1 \times p_2 \downarrow \quad \downarrow q_1 \times q_2 \\
X' \times Y' \xrightarrow{f'} U' \times V'
\end{array}
\end{array}
\]

be a commutative diagram of holomorphic maps between connected complex spaces with \(f'\) biholomorphic. Assume that \(p_1 = \text{id}\), or \(p_2 = \text{id}\), and that \(q_1 = \text{id}_U\) or \(q_2 = \text{id}_V\).

If \(\pi_1(x', y') = (p_1(x), p_2(y))\) decomposes \(f\), then \((x, y)\) decomposes \(f'\).

**Proof:** By 3.3.2.a(iii), we need only consider the case \(p_2 = \text{id}_U, q_2 = \text{id}_V\). Then \(X_U\) and \(Y_U\) exist and are equal to \(V_X\) resp. \(Y_U\). From the commutative diagram

\[
\begin{array}{ccc}
X & \xrightarrow{\bar{y}} & V \\
\downarrow p_1 \quad \quad \quad \quad \quad \quad \quad \quad \quad \downarrow p_1 \\
X' & \xrightarrow{\bar{y}'} & V'
\end{array}
\]

(compare 3.1.1.b), we infer that \(V_Y\) exists and is equal to \(V_{U_Y}\), and that \(X_U\) exists and is equal to \(p_1^{-1}(X_{U_Y})\).

Symmetrically: \(X_U\) and \(Y_U\) exist and are equal to \(X_{V_U}\) and \(U_X\) exists and is equal to \(q_1^{-1}(U_{X_U})\).

From the commutative diagram

\[
\begin{array}{ccc}
\bar{y}^{-1}(u) & \xrightarrow{\bar{y}_U} & u^{-1}(y) \\
\downarrow p_1 \quad \quad \quad \quad \quad \quad \quad \quad \quad \downarrow \text{id}_U \\
\bar{y}'^{-1}(u') & \xrightarrow{\bar{y}_U'} & u'^{-1}(y')
\end{array}
\]
(compare 3.1.1.(ii)), we infer that \( p_1 \downarrow y^{-1} (u) \rightarrow y' \downarrow (u') \) is well-defined and biholomorphic.

Let now \( S_n f \mid S_n f' \) be as in 3.3.1. By construction, the diagram

\[
\begin{array}{ccc}
X' \times Y' & \xrightarrow{S_n f} & X \times Y \\
\downarrow p_1 \times \text{id} & & \downarrow q_1 \times \text{id} \\
S_n f & & \\
\end{array}
\]

is well-defined and commutative.

Applying the above remark to \( S_n \mid \hat{f} \), we conclude that

\[
p_1 \downarrow \tilde{y}_n^{-1} (x) \rightarrow \tilde{y}_n' (x') \text{ is well-defined and biholomorphic (where } \tilde{y}_n = \hat{f} S_n (u, y) : X \rightarrow X)\). Thus, by 3.3.1.(ii), \( X_{Y'} \) exists and \( p_1 \downarrow X_{Y'} \rightarrow X_{Y''} \) is well-defined and biholomorphic. Symmetrically: \( U_{Y'} \) exists and \( q_1 \downarrow U_{Y'} \rightarrow U_{Y''} \) is well-defined and biholomorphic.

From the commutative diagram

\[
\begin{array}{ccc}
X_{Y'} \times Y_{Y'} & \xrightarrow{\tau f} & V' \\
\cong \downarrow p_1 \times \text{id} & & \downarrow \ \\
X_{Y'}' \times Y_{Y'}' & \equiv & V'
\end{array}
\]

we infer that \( \tau f \downarrow X_{Y'} \times Y_{Y'} \rightarrow V \) is biholomorphic. Symmetrically: \( \tau f^{-1} \downarrow U_{Y'} \times Y_{Y'} \rightarrow X \) is biholomorphic.

The commutative diagram

\[
\begin{array}{ccc}
X_{U'} \times Y_{U'} & \xrightarrow{\tau f'} & U' \\
\cong \downarrow p_1 \times \text{id} & & \downarrow q_1 \\
X_{U'}' \times Y_{U'}' & \equiv & U'
\end{array}
\]

yields that \( \tau f' \downarrow X_{U'} \times Y_{U} \rightarrow U \) is biholomorphic, since \( X_{U'} \times Y_{U} = (p_1 \times \text{id})^{-1} (X_{U'} \times Y_{U'}) \downarrow U = q_1^{-1} (U') \downarrow U \). Symmetrically, \( \tau f^{-1} \downarrow U_{U'} \times V_{U'} \rightarrow X \) is biholomorphic.

To verify condition 3.3.2.(iii), let \( R \in \{ U, V \} \), \( S \in \{ X, Y \} \), and denote by \( j : R_S \rightarrow U_X \times V_X \times U_Y \times V_Y \) the natural embedding given by \( (u, v) \) (i.e. \( U_X \rightarrow U_X \times \{ (v, u, v) \} \) etc.), with corresponding \( j' : R_{S'} \rightarrow U_X' \times V_X' \times U_Y' \times V_Y' \). Consider the commutative
diagram

\[
\begin{array}{ccc}
R_S & \longrightarrow & R'_S \\
\downarrow R & & \downarrow R' \\
U_x \times V_x \times U_y \times V_y & \overset{P}{\longrightarrow} & U'_x \times V'_x \times U'_y \times V'_y \\
\downarrow \text{id} \times \text{id} & & \downarrow \text{id} \times \text{id} \\
X' \times Y' & \overset{f}{\longrightarrow} & U' \times V' \\
\downarrow f & & \downarrow f' \\
x \times Y & \overset{q \times \text{id}}{\longrightarrow} & U' \times V' \\
\downarrow \text{id} \times f & & \downarrow \text{id} \times f' \\
X_U \times Y_u \times X_v \times Y_v & \longrightarrow & X'_u \times Y'_u \times X'_v \times Y'_v \\
\downarrow p & & \downarrow p \\
S_R & \longrightarrow & S'_R
\end{array}
\]

with \( P := q, \times \text{id}, x_q, \times \text{id}, Q := p, \times \text{id}, x_p, \times \text{id}. \)

If \( R_S \neq U_X \), then \( P \mid Q \) define isomorphisms \( j(R_S) \rightarrow j'(R'_S) \), resp. \( j(S_R) \rightarrow j'(S'_R) \), whence \( p \circ \bar{f} \circ j : R_S \rightarrow S_R \) is biholomorphic.

Let now \( R_S = U_X \). The diagram

\[
\begin{array}{ccc}
X_U \times Y_u \times X_v \times Y_v & \overset{P}{\longrightarrow} & X_u \\
\downarrow q & & \downarrow p \\
X'_u \times Y'_u \times X'_v \times Y'_v & \overset{P}{\longrightarrow} & X'_u
\end{array}
\]

is clearly cartesian and from \( \bar{f}(j(U_X)) = Q^{-1}(\bar{f} \circ j(U_X)) \) we infer that \( p \mid \bar{f}(j(U_X)) \rightarrow X_u \) is biholomorphic, since \( p \mid \bar{f}(j(U_X)) \rightarrow X'_u \) is.

### 3.4. Dimension-decreasing constructions

#### 3.4.1. Consider at first the double-arrow part of the diagram

\[
\begin{array}{ccc}
X \times Y & \overset{f}{\longrightarrow} & U \times V \\
\downarrow \text{id} \times f & & \downarrow \text{id} \times f \\
X' \times Y & \overset{f'}{\longrightarrow} & U' \times V' \\
\downarrow \text{id} \times f & & \downarrow \text{id} \times f \\
X \times Y & \overset{\text{id}}{\longrightarrow} & U \times Y \\
\downarrow \text{id} & & \downarrow \text{id} \\
U \times Y & \overset{\text{id}}{\longrightarrow} & U \times Y
\end{array}
\]

and

\[
\begin{array}{ccc}
X \times Y & \overset{f}{\longrightarrow} & U \times V \\
\downarrow \text{id} \times f & & \downarrow \text{id} \times f \\
X' \times Y & \overset{f'}{\longrightarrow} & U' \times V' \\
\downarrow \text{id} \times f & & \downarrow \text{id} \times f \\
X \times Y & \overset{\text{id}}{\longrightarrow} & U \times Y \\
\downarrow \text{id} & & \downarrow \text{id} \\
U \times Y & \overset{\text{id}}{\longrightarrow} & U \times Y
\end{array}
\]
which is clear commutative. In particular \((lf,pr)\) is proper if and only if so is \((pr,rf)\).

Assume now that \((lf,p)\) and \((pr,rf)\) are proper and let their Stein factorizations be given by the simple arrows (compare 2.1.1). As \(p'\) \(q'\) are quotient maps, the above diagram can be commutatively enlarged by uniquely determined holomorphic arrows \(f' : X' \times Y \rightarrow U' \times V'\) \((f')' : U' \times V' \rightarrow X' \times Y\) which are obviously inverse to each other.

Interchanging \(U\) and \(V\), if allowed (i.e. if the corresponding arrows are proper), we obtain

\[
\begin{array}{c}
X \times Y \\
\downarrow \left( pf, x+y \right) \\
Y \times V \\
\end{array}
\]

and again we can insert unique holomorphic maps \(f'' : X'' \times Y \rightarrow U'' \times V, (f')'' : U'' \times V \rightarrow X'' \times Y\).

3.4.1.a Remark. Let \((x, y) \in X \times Y\) and let \((x', y') = (p'(x) | y), (x'' | y'') = (p''(z) | y)\).

(i) If \(f\) degenerates with respect to \((x, y)\), then \(f'\) degenerates with respect to \((x', y')\), and \(f''\) degenerates with respect to \((x'', y'')\).

(ii) If \((x', y')\) decomposes \(f'\), or if \((x'', y'')\) decomposes \(f''\), then \((x, y)\) decomposes \(f\).

Assume now that \(X\) is compact, i.e. that both constructions can be performed. We shall see that they commute (in the obvious sense). Applying the "construction to \(f'\) yields just as above

\[
\begin{array}{c}
X' \times Y \\
\downarrow \left( pf, x+y \right) \\
Y \times V' \\
\end{array}
\]

since the Stein factorization of \((rf''^{-1}, id)\) is evidently given by the corresponding simple
arrows. Symmetrically, we obtain:

\[
X'' \times Y \xrightarrow{\mathcal{L}} U'' \times V \xrightarrow{\mathcal{L}^{-1}} X'' \times Y
\]

\[
\downarrow (p'') \times \text{id} \quad \downarrow \text{id} \times \mathcal{L} \quad \downarrow (p') \times \text{id}
\]

\[
\downarrow (f'' \cdot p) \quad (X'')' \times Y \quad \downarrow (p'' \cdot f''^{-1}) \quad U'' \times V' \quad \downarrow (f' \cdot p) \quad (X'')' \times Y
\]

\[
\therefore \quad U'' \times Y \quad \xrightarrow{\text{id}} \quad U'' \times Y \quad \xrightarrow{\text{id}} \quad U'' \times Y
\]

and we conclude that \((f'')'' = (f'')'\).

Let now \(\mathcal{F} := (f'')'' : [X] \times |Y| \rightarrow [U] \times |V|\) (although \(Y = |Y|\) it is convenient to mark each entry with the same symbol), and let \(P := ([p] \times \text{id}) := ((p'')'' \times \text{id}) : X \times Y \rightarrow [X] \times [Y]|_Q := (q'' \times q') : U \times V \rightarrow [U] \times |V|

3.4.1.b Remark. Let \((x, y) \in X \times Y\), and let \(|(x, y)| = P(x, y)\).

(i) If \(f\) degenerates with respect to \((x, y)\), then \(f\) degenerates with respect to \(|(x, y)|\).

(ii) If \(|(x, y)|\) decomposes \(|f|\) then \((x, y)\) decomposes \(f\).

(iii) The Stein factorization of every \(\overrightarrow{u} \overrightarrow{v} : X \rightarrow U \rightarrow Y\) and every \(\overrightarrow{u} \overrightarrow{v} : X \rightarrow V \rightarrow Y\) (with arbitrary \((u, v) \in U \times V\)) has the form \(X \xrightarrow{P} [X] \rightarrow |Y|

Proof. (i) and (ii) follow again from 3.2.1.b(iii) and 3.3.

(iii) By construction, all partial maps \(U'' \rightarrow Y, V' \rightarrow Y, (X'')' \rightarrow U'', (X'')'' \rightarrow V'\) are finite, and hence so are the compositions \([X] \rightarrow [U] \rightarrow [Y] = Y, [X] \rightarrow [V] \rightarrow [Y] = Y\). On the other hand, \(|p|\) is a quotient map with connected fibres.

3.4.2. We shall now present a similar construction that will take care of the non-compact factors.

Let \((x, y) \in X \times Y, (u, v) = f(x, y)\) denote by \(X_0, Y_0, U_0, V_0\) the orbits \(A(X) x, A(Y) y, A(U) u, A(V) v\) and let \(f_0 := f|X_0 \times Y_0 \rightarrow U_0 \times V_0\) (compare 2.3.4.a). Applying the \(\cdot\)-construction (3.4.1) to \(f_0 \circ J\) we obtain a commutative diagram

\[
\begin{align*}
X_0 \times Y_0 & \xrightarrow{f_0} U_0 \times V_0 \\
\downarrow \text{id} \times p_0 & \downarrow \text{id} \times q_0 \\
X_0 \times Y_0 & \xrightarrow{f_0'} U_0 \times V_0 \\
\end{align*}
\]

As \(Y_0, V_0\) are orbits of \(A(Y), A(V)\), respectively, there exist connected compact complex subgroups \(A' \sqsubset A(Y), B' \sqsubset A(V)\) such that \(p_0 = (q_A : Y_0 \rightarrow Y_0/A')\) and \(q_0 = (q_B : Y_0/B')\)
Applying 2.1.1 to the composition $(X \times Y) \times A(Y) \overset{\text{id} \times E}{\rightarrow} X \times Y \overset{f}{\rightarrow} U$ and to $(U \times V) \times A(V) \overset{\text{id} \times E}{\rightarrow} U \times V \overset{f}{\rightarrow} X$, we see that $A'$, $B'$ do not depend on the choice of $(x, y)$. Moreover, by 2.3.4(i), $f$ is $A' \rightarrow B'$-equivariant. Thus, denoting by $'p$, $'q$ the quotient maps $Y \rightarrow Y/A'$, $V \rightarrow V/B'$ respectively, we arrive at a commutative diagram

$$
\begin{array}{ccc}
X \times Y & \xrightarrow{f} & U \times V \\
\downarrow \text{id} \times 'p & & \downarrow \text{id} \times 'q \\
X \times Y/A' & \xrightarrow{f} & U \times V/B'
\end{array}
$$

where $'f$ and $(f^{-1})'$ are holomorphic and inverse to each other.

Again, we may interchange $U$ and $V$ to obtain

$$
\begin{array}{ccc}
X \times Y & \xrightarrow{f} & U \times V \\
\downarrow \text{id} \times 'p & & \downarrow \text{id} \times 'q \\
X \times Y/A' & \xrightarrow{f} & U \times V/B'
\end{array}
$$

Finally, we can construct $"f$ and $("f)$, which again coincide, and will be denoted by $f|: X| \times Y| \rightarrow U| \times V|$. The quotient maps $X \times Y \rightarrow X| \times Y|, U \times V \rightarrow U| \times V|$ will be indicated by $P| = (\text{id}, x, p|), Q| = ("q x', q)$ respectively.

### 3.4.2. Remarks

Let $(x, y) \in X \times Y$, and let $(s, y) = (\text{id}, x, \pi)$. Then $(s, y)$ decomposes $f|$ if and only if $(x, y)$ decomposes $f|$. Moreover, $f|$ factors through $p|: Y \rightarrow X$ such that the corresponding map $X| \rightarrow X$ is finite on the images $p|((A(Y) y) \rightarrow X)$.

### 3.4.3

Let now $X$ be compact and let $\bar{f} := [(f)]: \overline{X} \times \overline{Y} \rightarrow \overline{U} \times \overline{V}$ (which does not coincide with $[f]|!$); moreover, let $\bar{P} := [(P)|: \overline{X} \times \overline{Y} \rightarrow \overline{U} \times \overline{V}$ and $\overline{Q} := [(Q)|: \overline{U} \times \overline{V} \rightarrow \overline{X} \times \overline{Y}$. 

Summing up, we arrive at the commutative diagram

\[
\begin{array}{ccc}
X \times Y & \xrightarrow{\overline{p}} & \overline{X} \times \overline{Y} \\
\downarrow \overline{id} \times p | & & \downarrow | p \times \overline{id} \\
X| \times Y| & & \overline{X}| \times \overline{Y}| \\
\downarrow f & & \downarrow \overline{f} \\
U| \times V| & & \overline{U} \times \overline{V}| \\
\uparrow q & & \uparrow \\
u \times \nu & & u \times \overline{v}
\end{array}
\]

3.4.3.a Remark. Let \((x, y) \in X \times Y\), and let \((\overline{x}, \overline{y}) = \overline{p}(x, y)\).

(i) If \(f\) degenerates with respect to \((x, y)\), then \(\overline{f}\) degenerates with respect \(\overline{t}_0(\overline{x}, \overline{y})\).

(ii) If \((\overline{x}, \overline{y})\) decomposes \(\overline{f}\), then \((x, y)\) decomposes \(f\).

4. COMPLEX SPACES WITH ZERO-DIMENSIONAL FACTORS

This chapter provides the connecting link between the local and the global situation.

Let \(f: X \times Y \to U \times V\) be a biholomorphic map between connected complex spaces, and assume that \(X_{\text{red}} = \{x\}\). For \(y \in Y\) let \(\overline{y} = (x, y)\).

**Theorem.** \(f\) induces a simultaneous subdecomposition.

More explicitly, we have:

(i) Every \((x, y) \in X \times Y\) decomposes \(f\).

(ii) Let \(\overline{y} = (x, y) \in X \times Y\), \((u, v) = f(\overline{y})\). For \(R \in \{X, Y\}, S \in \{U, V\}\) denote by \(R_S(\overline{y}), S_R(\overline{y})\) the subfactors given by \(\overline{y}\) according to 3.3.2. then

\[
U_X(\overline{y}) = (x \overline{v})^{-1}(u), \quad X_U(\overline{y}) = (\overline{u} \overline{y})^{-1}(x),
\]

and \(\overline{\overline{u}}\) induces an isomorphism \(U_X(\overline{y}) \to X_U(\overline{y})\);

\[
V_X(\overline{y}) = (\overline{x} \overline{v})^{-1}(v), \quad X_V(\overline{y}) = (\overline{v} \overline{y})^{-1}(x),
\]

and \(\overline{\overline{u}}\) induces an isomorphism \(V_X(\overline{y}) \to X_V(\overline{y})\);

\[
U_Y(\overline{y}) = \overline{\overline{v}}^{-1}(x), \quad Y_U(\overline{y}) = \overline{\overline{x}}^{-1}(v),
\]

and \(\overline{\overline{u}}\) induces an isomorphism \(V_Y(\overline{y}) \to X_V(\overline{y})\).
and \( \overrightarrow{v} \) induces an isomorphism \( U_{Y'}(\overrightarrow{y}) \to Y_{U}(\overrightarrow{y}) \) whose inverse is given by \( \overrightarrow{x} \).

\[
V_{Y}(\overrightarrow{y}) = \overrightarrow{u}^{-1}(x), \quad Y_{Y}(\overrightarrow{y}) = \overrightarrow{x}^{-1}(u),
\]

and \( \overrightarrow{w} \) induces an isomorphism \( V_{Y}(\overrightarrow{y}) \to Y_{V}(\overrightarrow{y}) \) whose inverse is given by \( \overrightarrow{x} \).

(iii) Let \( y' \in Y, S \in \{U, V\} \). Then \( X_{S}(\overrightarrow{y}) = X_{S}(\overrightarrow{y}') =: X_{S} \) and \( Y_{S}(\overrightarrow{y}) = Y_{S}(\overrightarrow{y}') =: Y_{S} \).

**Proof.** Let \( y \in Y \) be fixed, and let \( S \in \{U, V\} \). By Theorem 1.4.1, \( X_{S}(\overrightarrow{y}) \) and \( Y_{S}(\overrightarrow{y}) \) exist, and

1. the relations postulated in (ii) are satisfied.
2. \( f^{-1} \) defines an isomorphism \( U_{X}(\overrightarrow{y}) \times V_{X}(\overrightarrow{y}) \to X \),

3. the compositions \( X_{U}(\overrightarrow{y}) \leftarrow X \times Y_{U}(\overrightarrow{y}) \) and \( X_{V}(\overrightarrow{y}) \leftarrow X \times Y_{V}(\overrightarrow{y}) \) are well-defined and biholomorphic.

Let now \( Y' \) be the irreducible component of \( Y \) that contains \( y \) and assume from now on that \( y \) satisfies the following condition:

(*) For every \( y' \) in some neighbourhood of \( y \), any embedding \( X_{U}(\overrightarrow{y}') \leftarrow X_{U}(\overrightarrow{y}) \) is an isomorphism.

Such points \( y \) exist, since \( \dim X = 0 \).

Let \( \phi := (U \times V) \circ f^{-1} \circ X(\overrightarrow{y}(y)) \). Then \( \phi(\overrightarrow{y}, \overrightarrow{u})|U_{X}(\overrightarrow{y}) \to X_{U}(\overrightarrow{y}) \) is biholomorphic by (1) and (3); therefore \( \phi(\overrightarrow{u}, \overrightarrow{v})|U_{X}(\overrightarrow{y}') \to X_{U}(\overrightarrow{y}) \) is an embedding and hence, by (1) and (*), an isomorphism for \( (\overrightarrow{y}', \overrightarrow{y}) \) sufficiently close to \( (\overrightarrow{y}, \overrightarrow{y}) \). Using 1.1.2.a, we conclude that \( \phi(\overrightarrow{u}', \overrightarrow{v}) \) is constant on \( V_{X}(\overrightarrow{y}') \) for \( (\overrightarrow{u}', \overrightarrow{y}) \) sufficiently close to \( (\overrightarrow{u}, \overrightarrow{y}) \); in particular, if \( y' \) is close to \( y \) and \( (\overrightarrow{u}', \overrightarrow{v}) = f(\overrightarrow{x}, \overrightarrow{y}) \), then \( X_{V}(\overrightarrow{y}) = \overrightarrow{u}(V_{X}(\overrightarrow{y}')) \circ X(\overrightarrow{y}) \), and as shown above, \( X_{U}(\overrightarrow{y}') \approx X_{U}(\overrightarrow{y}) \). On the other hand, by (1) and (2), \( X \) is isomorphic to every \( X_{U}(\overrightarrow{y}') \times X_{V}(\overrightarrow{y}'), \overrightarrow{v} \in Y \); thus \( X_{V}(\overrightarrow{y}) = X_{V}(\overrightarrow{y}) \) for \( \overrightarrow{y} \) close to \( y \). This means (see (1)) that \( \overrightarrow{u}' \overrightarrow{y}' \) is constant on \( X_{V}(\overrightarrow{y}) \) for \( \overrightarrow{y} \) close to \( y \) and hence for all \( \overrightarrow{y} \in Y' \).

Thus, if \( y \) is chosen according to (*), then \( X_{U}(\overrightarrow{y}) \) is contained in every \( X_{V}(\overrightarrow{y}) \) for \( \overrightarrow{y} \) close to \( y \) or \( \overrightarrow{y} \in Y' \); in particular, any embedding \( X_{V}(\overrightarrow{y}) \leftarrow X_{V}(\overrightarrow{y}) \) is an isomorphism for \( \overrightarrow{y} \) close to \( y \). We can therefore interchange \( U \) and \( V \) in the above considerations and obtain that every \( X_{U}(\overrightarrow{y}) \) contains \( X_{U}(\overrightarrow{y}) \) for \( \overrightarrow{y} \in Y' \). Using again (2), we conclude that \( X_{U}(\overrightarrow{y}) = X_{U}(\overrightarrow{y}) \), \( X_{V}(\overrightarrow{y}) = X_{V}(\overrightarrow{y}) \) for all \( \overrightarrow{y} \in Y' \), and hence, as \( Y \) is connected:

4. \( X_{U}(\overrightarrow{y}) = X_{U}(\overrightarrow{y}) =: X_{U} \) and \( X_{V}(\overrightarrow{y}) = X_{V}(\overrightarrow{y}) =: X_{V} \) for all \( \overrightarrow{y} \in Y' \).

Let \( \psi := I_{f} \circ f \circ (x, y) \) : \( X \times Y \to X \); then \( \psi(\overrightarrow{x}, (y, y)) = \text{id} \), for all \( y \). Using 1.1.2.a, we see that \( \psi(\overrightarrow{x}, \overrightarrow{y}) \) is constant on some neighbourhood of the diagonal in \( X \times Y \), and from the lemma in 0.2.2 we infer that each partial map \( \psi(\overrightarrow{x}, (y, y))) \) is constant. Thus \( f(\overrightarrow{x}, \overrightarrow{y}) \) defines an embedding
On product decompositions of complex spaces

\[ Y \to \bigcap_{v \in V} \overrightarrow{v}^{-1}(x) \times \bigcap_{u \in U} \overrightarrow{u}^{-1}(x), \text{ since every } \overrightarrow{v} : Y \to X, \overrightarrow{u} : Y \to X \text{ is constant.} \]

On the other hand, by 3.1.1, the maps \( \overrightarrow{v}, \overrightarrow{u} \) induce isomorphisms \( \overrightarrow{v}^{-1}(v) \to \overrightarrow{v}^{-1}(x), \overrightarrow{u}^{-1}(u) \to \overrightarrow{u}^{-1}(x) \), respectively, for all \((u, v) \in U \times V\). We conclude that \( \overrightarrow{v}^{-1}(x), \overrightarrow{u}^{-1}(x) \) do not depend on \((u, v)\), and that \( f(\overrightarrow{x}) \) defines an isomorphism \( Y \to \overrightarrow{v}^{-1}(s) \times \overrightarrow{u}^{-1}(x) \) with inverse \( \tau f^{-1} \lvert \overrightarrow{v}^{-1}(x) \times \overrightarrow{u}^{-1}(x) \to Y \). This yields

\[
\begin{align*}
\overrightarrow{v}^{-1}(v) &= (((\overrightarrow{v} \overrightarrow{x})^n)^{-1}(y), \\
\overrightarrow{u}^{-1}(u) &= (((\overrightarrow{u} x)^n)^{-1}(v)), \\
\overrightarrow{v}^{-1}(x) &= (((\overrightarrow{y} \overrightarrow{v})^n)^{-1}(u), \\
\overrightarrow{u}^{-1}(x) &= (((\overrightarrow{y} \overrightarrow{u})^n)^{-1}(v)),
\end{align*}
\]

for all \( n \geq 1 \). Thus

(5) \( S_Y(\overrightarrow{v}) \) and \( Y_S(\overrightarrow{v}) \) exist for \( S \in \{U, V\} \) and satisfy the relations postulated in (ii).

Furthermore,

(6) \( S_Y(\overrightarrow{v}) = S_Y(\overrightarrow{v}) \) does not depend on \( y \), and \( f(\overrightarrow{x}) \) defines an isomorphism \( Y \to U \times V \) whose inverse is given by \( \tau f^{-1} \).

(5) and (6) immediately yield:

(7) \( Y_S(\overrightarrow{v}) = Y_S(\overrightarrow{v}) \) for all \( y' \in Y_S(\overrightarrow{v}) \)

By 1.4.1, the restriction \( \lfloor f \rfloor \lvert X_U(\overrightarrow{v}) \times Y_U(\overrightarrow{v}) \to U \) is biholomorphic in \((x, y)\), and hence, by (4) and (7), is biholomorphic in every \((x, y')\) with \( y' \in Y_U(\overrightarrow{v}) \). On the other hand, the reduction \( \lfloor (\lfloor f \rfloor)_{red} \lvert (X_U(\overrightarrow{v}) \times Y_U(\overrightarrow{v}))_{red} \to U_{red} \) is biholomorphic by (5). Thus:

(8) \( \lfloor f \rfloor \lvert X_U \times Y_U(\overrightarrow{v}) \to U \) is biholomorphic, and, symmetrically, so is \( \tau f \lvert X_U \times Y_U(\overrightarrow{v}) \to V \).

Collecting what we have shown up to now, we observe that

(ii) is proven by (1) and (5),

(iii) is proven by (4) and (6), and,

by (ii), (iii), (2), (6) and (8), every \((x, y)\) satisfies the conditions 3.3.2.(i) and 3.3.2.(ii).

To complete the proof, it remains to verify the condition 3.3.2.(iii). Consider the commutative diagram of biholomorphic mappings

\[
\begin{array}{ccc}
(U_X(\overrightarrow{v}) \times V_X(\overrightarrow{v})) \times (U_Y \times V_Y) & \xrightarrow{f} & (X_U(\overrightarrow{v}) \times Y_U(\overrightarrow{v})) \times (X_Y \times Y_Y) \\
\downarrow \lvert f \lvert^{-1} \times \lfloor f \rfloor & & \downarrow \lvert f \lvert^{-1} \times \lfloor f \rfloor \\
X \times Y & \xrightarrow{f} & U \times V
\end{array}
\]

and denote every partial embedding \( R_S(\overrightarrow{v}) \leftarrow U_X(\overrightarrow{v}) \times V_X(\overrightarrow{v}) \times U_Y \times V_Y, S_R(\overrightarrow{v}) \leftarrow X_U \times Y_U(\overrightarrow{v}) \times X_Y \times Y_Y \) by \( j \) (Le. \( U_X(\overrightarrow{v}) \to U_X(\overrightarrow{v}) \times \{ (v, u, v) \} \) etc.).
By (ii), there exists a commutative diagram

$$U_X(\overline{y}) \times \{v\} \xrightarrow{\bar{f}(\cdot, (u, v))} X_U \times Y_U = \leftarrow X_U \times \{v\}$$

with biholomorphic vertical arrows. Thus \( \bar{f} \circ j \) maps \( U_X(\overline{y}) \) biholomorphically onto \( X_U \times \{v\} \).

All we have used to derive this diagram from the preceding one was the fact that \( \overline{y} \) induces an isomorphism \( U_X(\overline{y}) \rightarrow X_U \). Hence, by (ii), the same type of diagram exists, mutatis mutandis, for \( V_X(\overline{y}), U_Y, V_Y, Y_U(\overline{y}), Y_V(\overline{y}) \), and we conclude:

\[
\begin{align*}
\tau f \circ j & \quad V_X(\overline{y}) \quad X_U \times \{y\} \\
\bar{f} \circ j & \quad U_Y(\overline{y}) \quad \{x\} \times Y_U(\overline{y}) \\
\tau f^{-1} \circ j & \quad V_Y(\overline{y}) \quad \text{maps} \quad Y_U(\overline{y}) \quad \text{biholomorphically onto} \quad \{x\} \times Y_V(\overline{y}) \\
\tau f^{-1} \circ j & \quad Y_U(\overline{y}) \quad U_V \times \{v\} \\
\tau f^{-1} \circ j & \quad Y_V(\overline{y}) \quad \{u\} \times V_Y
\end{align*}
\]

Finally, there exists a commutative diagram

\[
\begin{array}{c}
X_U \times \{y\} \\
\downarrow \bar{v} \circ f \\
\downarrow f \\
\downarrow f^{-1} \quad \downarrow f^{-1} \quad \downarrow f^{-1} \\
U \quad \{v\} \quad X \quad \{u\} \quad V \quad \{v\} \\
\end{array}
\]

(compare (3) for the diagonal in the lefthand rectangle), and we conclude that \( \bar{f}^{-1} \circ j \) maps \( X_U \) biholomorphically onto \( U_X(\overline{y}) \times \{v\} \). Symmetrically: \( \tau f^{-1} \circ j \) maps \( X_V \) biholomorphically onto \( \{u\} \times V_Y(\overline{y}) \).

Thus, an even stronger condition than 3.3.2.(iii) is fulfilled.

\[\diamond\]

5. COMPLEX SPACES WITH COMPACT FACTORS

Generalizing the situation of the preceding chapter, we consider now biholomorphic mappings \( f : X \times Y \rightarrow U \times V \) with compact \( X \). As demonstrated by Example 3.2.1.a(ii) (see also 3.3.2.a(ii)), \( f \) need no longer induce a simultaneous subdecomposition; it will, however, if \( \{X, Y, U, V\} \not\in \mathcal{F}_k \) for all \( k \geq 1 \) – a condition that is of course fulfilled, if \( \dim X = 0 \). This result is the basis for the subsequent investigations concerning cancellability and decomposability.
5.1. The structure induced by two decompositions

Let \( f : X \times Y \to U \times V \) be a biholomorphic map between connected complex spaces, and assume that \( X \) is compact. Fix some \((x_0, y_0) \in X \times Y\) let \((u_0, v_0) := f(x, y)\), and consider the sequence of holomorphic maps

\[
(*) \ldots \to X \xrightarrow{y_0} U \xrightarrow{\xi_0} V \xrightarrow{v_0} X \xrightarrow{y_0} U \to \ldots
\]

To simplify the notations, we denote by \( S \overset{(*)}{\to} S' \) the map \( S \to S' \) given by a subsequence of \( (*) \) that starts at \( S \), consists of \( n \) arrows, and ends at \( S' \) (where \( \{ S, S' \} \subset \{ X, Y, U, V \} \)); furthermore, we let \( (S \overset{(*)}{\to} S') := \text{id}, \) and we say that \( S \overset{(*)}{\to} S' \) contains \( S_1 \overset{(*)}{\to} S'_1 \) if \( (S \overset{(*)}{\to} S') = (S \overset{(*)}{\to} S'_1 \overset{(*)}{\to} S'_2 \overset{(*)}{\to} \ldots \overset{(*)}{\to} S_m) \) with suitable \( k \mid n \).

5.1.1 Lemma. Let \( \{ S, S' \} \subset \{ X, Y, U, V \} \) with corresponding \( s_0 \), \( s'_0 \in \{ x_0, y_0, u_0, v_0 \} \); \( l \geq 2 \), then \( S \overset{(*)}{\to} S' \) factors through \( s_0 : \text{Hol}(S') \to S' \) with \( s_0 \mapsto \text{id} \).

5.1.2 Proposition. Let \( l \overset{(*)}{\to} \lim_{n \to \infty} \text{dimIm}(X) \overset{(*)}{\to} X \).

For every \( S \in \{ X, Y, U, V \} \) there exists \( \pi_S : S \to T_S \in \mathcal{S}_l \) with some connected fibre \( F \).

In particular, if \( \{ X, Y, U, V \} \not\subset \mathcal{S}_l \) for all \( k \geq 0 \), then \( f \) degenerates with respect to \((x_0, u_0)\).

Proof. By 5.1.1 and 2.4.2.a and 2.4.1.a.(v), the map \( S \overset{(*)}{\to} S \) gives rise to some \( \pi_S : \text{Hol}(S) \to T_S \) \( \in \mathcal{S}_l \) with connected fibre \( F \). As \( S \overset{(*)}{\to} S \) contains \( X \), we conclude that \( \text{dim}(S) = \text{dim}(X) = L \) for all \( S \in \{ X, Y, U, V \} \).

5.1.2.a Remark. Let \( S \in \{ X, Y, U, V \} \) with corresponding \( s_0 \in \{ x_0, y_0, u_0, v_0 \} \) and let \( \pi_S : S \to T_S \) be as in 5.1.2 with corresponding \( T_S \supset \text{A}(S) \) (compare 2.4.1.a.(iii)). By construction, \( S \overset{(*)}{\to} S \) factors through the inclusion \( T_S \overset{(*)}{\to} S \) of the orbit \( T_S \overset{(*)}{\to} S \) for \( \eta \in 0 \). Furthermore, \( \overline{y}_0(T_X \overline{x}_0) = T_U \overline{u}_0, \overline{v}_0(T_U \overline{u}_0) = T_Y \overline{v}_0, \overline{v}_0(T_Y \overline{v}_0) = T_V \overline{v}_0, \overline{u}_0(T_V \overline{u}_0) = T_V \overline{u}_0 \).
5.1.2. Corollary. If \(|f| \dim \text{Im}(X \rightarrow X) = \dim X\), then \((\pi_X)_\text{red} : X_{\text{red}} \rightarrow T_X\) is biholomorphic.

Proof. By 2.2.1, \(\dim X \geq d_0(X) \geq \dim A(X),\) and by 5.1.1.(i) and 2.3.1, \(\dim A(X) \geq \dim X\). Thus \(X_{\text{red}} = A(X) z_d\) and \(\dim X = |f|\) and we conclude that \((\pi_X)_\text{red}\) is locally biholomorphic, and hence biholomorphic, since \(F_x\) is connected.

Recall now the diagram

\[
\begin{array}{cccc}
X \times Y & \xrightarrow{\varphi} & \overline{X} \times \overline{Y} \\
\downarrow \text{id} \times |p| & & & \uparrow |p| \times \text{id} \\
X \times Y & \downarrow \varphi & & \downarrow \overline{\varphi} \\
\downarrow \overline{|p|} & & & \downarrow \overline{|p|} \\
U \times V & \xrightarrow{\overline{\varphi}} & \overline{U} \times \overline{V} \\
\end{array}
\]

that was constructed in 3.4.3.

5.1.3 Lemma. There exist finite holomorphic maps \(g, h : \overline{X} \rightarrow X\) such that

(i) the Stein factorization of \(X \rightarrow X\) is given by \((X \rightarrow X) = g \circ |p|\) and

(ii) the Stein factorization of \(\overline{x_0} \overline{x_0} \overline{u_0} \overline{y_0} : X \rightarrow X\) is given by \(\overline{x_0} \overline{x_0} \overline{u_0} \overline{y_0} = h \circ |p|\)

Proof. Consider the following commutative diagrams derived from the above one:

\[
\begin{array}{ccc}
X & \xrightarrow{(\cdot, 2)} & Y \\
\| & & \| \\
X & \rightarrow & Y \\
\downarrow |p| & \rightarrow & \downarrow |p| \\
\overline{X} & \rightarrow & \overline{Y}
\end{array}
\]
5.1.3 Corollary. If \( f \) degenerates, then so do \( J \circ f \) and \( f^{-1} \) (compare 3.2.1.b(i) and (ii)).

Proof: Evident by 5.1.3 and 3.2.1.b.

5.1.3.c Corollary. \( f \) is \( T_X' \times T_Y' - T_U' \times T_V' \)-equivariant.

Proof: The group isomorphism \( f_*: A(X) \times A(Y) \rightarrow A(U) \times A(V) \) is given by a matrix \[
\begin{pmatrix}
\alpha & \beta \\
\gamma & \delta
\end{pmatrix}
\]
with inverse \[
\begin{pmatrix}
\alpha' & \beta' \\
\gamma' & \delta'
\end{pmatrix}
\]. From 5.1.2.a we infer \( \alpha(T_X') = T_U' \) and \( \delta(T_Y') = T_V' \).

By 3.4.1.b(iii), the diagonal arrows \( \overline{X} \rightarrow Y \) are finite for both diagrams; moreover, by 3.4.2.a(iii), the diagonal arrows \( Y \rightarrow X \) are finite whenever \( (p_1)(A(Y)y) \). Now, by 5.1.1(i) and 2.3.1, the image of \( X \) under \((\ast,2)\) is contained (set-theoretically) in the orbit \( A(Y)y_0 \), and, for symmetry reasons, \( (\overline{u_0}, \overline{x_0}) \subset A(Y)y_0 \) as well. Thus, if we denote by \( g \) resp. \( h \) the composite of the diagonal arrows in the corresponding diagram, the assertion is proven, since \( p_r \) is a quotient map with connected fibres.

In general, however, \( f \) need not be a \( \mathcal{S} \)-morphism between \( \pi_X \times \pi_Y \) and \( \pi_U \times \pi_V \).
5.1.4. Example. Let $T$ be a one-dimensional torus, and let $X = Y = U = V = T \times P_2$, where $P_2 \hookrightarrow \mathbb{C}$ denotes the double point. Defining $f : X \times Y \rightarrow U \times V$ by $f((s, z), (t, y)) := ((2s + t + xy, x), (s + t + xy, y))$, one easily checks that $(\pi_R \downarrow R \rightarrow T_R) = (p_X : T \times P_2 \rightarrow T)$ for all factors $R$. Thus $f$ is not fibre-preserving with respect to $\pi_X \times \pi_Y, \pi_U \times \pi_V$.

Fortunately, nothing of this kind can happen in the reduced case:

5.1.4. Lemma. $f$ is a $\mathcal{F}$-morphism $\pi_X \times \pi_Y \rightarrow \pi_U \times \pi_V$, if one of the following conditions is fulfilled:

(i) $\overline{f} : \pi_X \times \pi_Y \rightarrow \pi_U \times \pi_V$

(ii) $\pi_X$ is biholomorphic.

(iii) $X$ is reduced.

Proof. By 5.1.3.c, we need only show that $f$ is fibre-preserving.

(i) Let $\lambda_S : S \rightarrow \overline{S}$ be the canonical projection (i.e. $\lambda_X = \overline{p}$ etc.). As $S \rightarrow S$, the construction of $\pi_S$ immediately yields $\pi_S = \pi_S \circ \lambda_S$. Thus, if $(\pi_U \times \pi_V) \circ \overline{f} = \overline{f} \circ (\pi_X \times \pi_Y)$ with a suitable holomorphic $\overline{f} : T_X \times T_Y \rightarrow T_U \times T_V$, then $(\pi_U \times \pi_V) \circ f = (\pi_U \times \pi_V) \circ (\lambda_X \times \lambda_Y) \circ f = (\pi_U \times \pi_V) \circ \overline{f} \circ (\lambda_X \times \lambda_Y) = \overline{f} \circ (\pi_X \times \pi_Y)$, i.e. $f$ is fibre-preserving.

(ii) Let $S \in \{U, V\}$. Every composition $X \rightarrow S \rightarrow Y$ of partial maps is an immersion of the form $T_X \rightarrow T_Y$ with suitable $y$; therefore every $Y \rightarrow S \rightarrow X$ factors through $\pi_Y : Y \rightarrow T_Y$. We conclude that $\overline{\lambda}$ resp. $\overline{\lambda}$ maps every fibre of $\pi_S$ into one of $\pi_U$ resp. $\pi_V$; in other words, $lf((\pi_X^{-1} \pi_X(x)) \times (\pi_Y^{-1} \pi_Y(y))) = lhf((x) \times (\pi_Y^{-1} \pi_Y(y))) \subset \pi_U^{-1} \pi_U(lhf(x, y)))$ and $\tau f((\pi_X^{-1} \pi_X(x)) \times (\pi_Y^{-1} \pi_Y(y))) \subset \pi_Y^{-1} \pi_Y(\tau f(x, y)))$.

Assertion (ii) follows from 5.1.3.a and from (i) and (ii) by induction on dim $X - \dim T_X$, since $T_X = T_X$.

5.1.5. Theorem. Let $f : X \times Y \rightarrow U \times V$ be a biholomorphic map between connected complex spaces with $X$ compact.

If $f$ degenerates, then every $(x, y) \in X \times Y$ decomposes $f$.

In particular, $f$ induces a simultaneous subdecomposition, if $\{X, Y, U, V\} \not\in \mathcal{F}_k$ for all $k \geq 1$.

Proof. We proceed by induction on dim $X$. noting that the case dim $X = 0$ has been settled in Chapter 4.

Let dim $X \geq 1$. Then $\dim X < \dim X$ by Corollary 5.1.3.a and $\overline{f}$ degenerates by 3.4.3.(i). Thus, by induction hypothesis, every $(x, \overline{y}) \in X \times \overline{Y}$ decomposes $\overline{f}$, and the assertion follows from 3.4.3.(ii).
5.1.5.a Corollary. If \( f \) is a \( \mathcal{F}\)-morphism \( \pi_X \times \pi_Y \to \pi_U \times \pi_V \) (e.g. if \( X \) is reduced), then the corresponding isomorphism \( F_X \times F_Y \to F_U \times F_V \) between the fibres induces a simultaneous subdecomposition.

Proof. Clearly, \( F_X \times F_Y \to F_U \times F_V \) degenerates.

Note that in Example 5.1.4.a, there still exists some isomorphism \( F_X \times F_Y \to F_U \times F_V \) that induces a simultaneous subdecomposition. It would be (mildly) interesting, whether at least this statement remains true in general.

5.2. Cancellation

5.2.1 Theorem. Let \( g : X \times Y \to X \times Z \) be a biholomorphic map between connected complex spaces, and assume that \( X, Y \) or \( Z \) is compact.

If \( (X, Y, Z) \notin \mathcal{F}_k \) for all \( k \geq 1 \), then \( Y \cong Z \).

Proof. We may assume that \( X \) is indecomposable. Then the assertion follows from 5.1.3.b, 5.1.5 and 3.3.2.a(iv).

5.2.1.a Examples. \( X \) cancels in the sense of 5.2.1, if \( \dim X = 0 \) if \( X \) has vanishing first Betti number or non-vanishing Euler characteristic, if \( \dim A(X) = 0 \) (in particular, if \( X \) admits at most countably many holomorphic automorphisms), if \( X \) is Stein, etc. Further examples (with \( X \) compact and reduced) can be found in ([5], 1.3).

Conversely, G. Parigi has shown that for any \( X \in \mathcal{F} \) there exist non-isomorphic \( Y, Z \) with \( X \times Y \cong X \times Z \) (see [11]); he states this fact for compact reduced \( X \) only, but his proof is easily seen to work for general \( X \) as well.

An interesting question arising in this context is the following: If \( X \times Y \cong X \times Z \), what is the relation between \( Y \) and \( Z \)?

In view of Example 5.1.4.a, it seems reasonable to restrict one’s attention at first to the reduced case, where one can find at least some structural similarity. By 5.1.2, 5.1.4 and 5.1.5.a, we obtain then commutative diagrams:

\[
\begin{array}{ccc}
X \times Y & \overset{\cong}{\longrightarrow} & X \times Z \\
\downarrow \pi_X \times \pi_Y & & \downarrow \pi_X \times \pi_Z \\
T_X \times T_Y & \overset{\cong}{\longrightarrow} & T_X \times T_Z \\
(F \times F_1) \times (F'' \times F_1') & \overset{\cong}{\longrightarrow} & (F \times F''_1) \times (F'' \times F_1') \\
\downarrow \cong & & \downarrow \cong \\
\pi_X^{-1} \pi_X(x) \times \pi_Y^{-1} \pi_Y(y) & \overset{\cong}{\longrightarrow} & \pi_X^{-1} \pi_X(x') \times \pi_Z^{-1} \pi_Z(z)
\end{array}
\]
However, there is no reason for \( \pi_X^{-1} \pi_X(x) \) and \( \tau_X^{-1} \tau_X(x') \) to be isomorphic. Thus we are faced with a much more difficult question than the decomposition problem, namely:

Given \( (\pi_j : X \to T_j) \in \mathcal{F}_k \) with fibre \( X_j, j = 1, 2 \) such that there exists a \( \mathcal{F} \)-morphism \( h : \pi_1 \to \pi_2 \) with \( h_*(T'_j) = T'_2 \), what is the relation between \( X \) and \( X_2' \)?

In Chapter 7, at least a necessary condition for \( Y, Z \) to satisfy \( X \times Y \cong X \times Z \) with suitable \( X \) will be given.

A more restricted version of the cancellation problem is the question of whether \( X \times X \cong X \times Y \) implies \( X \cong Y \). No counterexample with compact \( X, Y \) seems to be known. Shioda proved that no counterexample with tori \( X, Y \) can exist ([12]). Parigi's varieties \( Y \not\cong Z \) with \( X \times Y \cong X \times Z \) satisfy by construction \( Y \not\cong X \not\cong Z \).

5.3. Decomposition with respect to \( \mathcal{P} \)-categories

Denote by \( \mathcal{C} \) the category of all compact connected complex spaces.

5.3.1 Definition. A subcategory \( \mathcal{K} \subset \mathcal{C} \) is a \( \mathcal{P} \)-category if it has the following property: \( X \times Y \in \mathcal{K} \) if and only if \( X, Y \in \mathcal{K} \).

5.3.1.a Remarks and Examples.

(i) \( \mathcal{C}^0 \) lies in every non-empty \( \mathcal{P} \)-category. The intersection of \( \mathcal{P} \)-categories is a \( \mathcal{P} \)-category.

(ii) Each of the following is a \( \mathcal{P} \)-category: \( \mathcal{C}^1, \mathcal{C} \text{ red} := \{ X \in \mathcal{C} : X = X_1 \}, \mathcal{C}^0_0 := \{ X \in \mathcal{C} : \dim X = 0 \}, \mathcal{C} \setminus \mathcal{F} \) (see 2.4.2.b), \( \{ X \in \mathcal{C} : X \text{ projective} \}, \{ X \in \mathcal{C} : X \text{ Moishezon} \}, \{ X \in \mathcal{C} : \text{trdeg} \mathcal{C}(X) = 0 \}, \{ \text{tori} \} \).

5.3.2 Theorem. Let \( U \) be a connected complex space, and let \( \mathcal{K} \subset \mathcal{C} \setminus \mathcal{F} \) be a \( \mathcal{P} \)-category.

There exists a unique decomposition \( U \cong U_\mathcal{K} \times U' \) with \( U_\mathcal{K} \in \mathcal{K} \) such that \( U' \) has no factor in \( \mathcal{K} \setminus \{ \mathcal{C}^0 \} \).

If \( f = (f, \tau f) : U_\mathcal{K} \times U' \to U_\mathcal{K} \times U' \) is biholomorphic, then every partial map \( \tau f(\cdot, u'), \tau f'(\cdot, u) \) (where \( j = \pm 1 \)) is biholomorphic, and every composition \( (\tau f(\cdot, u'))^n \) is constant for \( n \) sufficiently large.

Proof. Let \( f : U_\mathcal{K} \times U' \to U_\mathcal{K}' \times U'_1 \) be biholomorphic, where \( U_\mathcal{K}, U_1 \in \mathcal{K} \) such that \( U', U_1' \) have no factor in \( \mathcal{K} \setminus \{ \mathcal{C}^0 \} \). \( f \) degenerates with respect to every \( (u, u') \in U_\mathcal{K} \times U' \), since \( \mathcal{K} \subset \mathcal{C} \setminus \mathcal{F} \). Therefore, every \( (u, u') \) decomposes \( f \), and hence gives rise to a commutative diagram

\[
\begin{array}{ccc}
(F_1^1 \times F_1^{-1}) \times (F_1^{-1} \times F_1^{-1}) & \rightarrow & (F_1^1 \times F_1^{-1}) \times (F_1^{-1} \times F_1^{-1}) \\
\downarrow & & \downarrow \\
U \times U' & \rightarrow & U \times U'
\end{array}
\]
according to 3.3.2 (we choose this new notation, in order to avoid e.g. $U^{1}_{\mathcal{C}}$ appearing as an index; moreover, we do not distinguish between the subfactors that are biholomorphically correlated by 3.3.2.(iii)).

If $\ell f(\cdot, u')$ were not biholomorphic, then $F^{-1}_{1} \in \mathcal{A} \setminus \{C^{0}\}$ or $F^{-1}_{1} \in \mathcal{A} \setminus \{C^{0}\}$, whence $U'$ or $U'$ would admit a factor in $\mathcal{A} \setminus \{C^{0}\}$.

Thus all $\ell f(\cdot, u')$ and, symmetrically, all $\ell f^{-1}(\cdot, u')$ are biholomorphic, whence, by Lemma 3.1.1.(iii), so are all $\tau f^{-1}(\cdot, u), \tau f(\cdot, u)$.

The theorem is now completely proven, since, in particular, $U^{1}_{\mathcal{C}} \cong F^{-1}_{1} \cong U^{1}_{\mathcal{C}}$ and $U' \cong F^{-1}_{1} \cong U'$. (compare 3.3.2.(i)).

53.3 Lemma. Let $f : X \times Y \to U \times V$ be an isomorphism in $\mathcal{A}$, and assume that $X \not\subseteq C^{0}$ is indecomposable and not contained in $\mathcal{A}$.

There exists a unique $S \in \{U, V\}$ with $S \cong X \times S'$ such that the resulting isomorphism $\bar{f} : X \times Y \to X \times (S \times S')$ (where $S, S' = \{U, V\}$) satisfies: Every partial map $\bar{f}^{j}(\cdot, b), \tau f^{j}(\cdot, b)$ is biholomorphic, and every $(\bar{f}^{j}(\cdot, b))^{n}$ is constant for $n > 0$. (where $b \in Y$ or $b \in S \times S'$, according as $j = 1$ or $j = -1$).

Proof: Fix some $(x_{0}, y_{0}) \in X \times Y$ and consider the diagram corresponding to the simultaneous subdecomposition given by $(x_{0}, y_{0})$ (note that $f$ degenerates):

$$
\begin{array}{ccc}
(X_{U} \times X_{V}) \times (Y_{U} \times Y_{V}) & \to & (U_{X} \times U_{Y}) \times (V_{X} \times V_{Y}) \\
\downarrow & & \downarrow \\
X \times Y & \to & u \times v
\end{array}
$$

We may assume that $X = X_{U}$, since $X$ is indecomposable; denote by $\bar{f}$ the resulting isomorphism $X \times Y = X_{U} \times Y \to U_{X} \times (U_{Y} \times V_{Y}) \cong X \times (U_{Y} \times V_{Y})$. Then $\bar{f}(\cdot, y_{0})$ and $\bar{f}^{-1}(\cdot, \tau f(\cdot, x_{0}, y_{0}))$ are biholomorphic by 3.3.2.(ii). As $\text{Aut}(X)$ is open in $\text{Hol}(X)$, the holomorphic maps $Y \ni y \mapsto \bar{f}(\cdot, y), U_{Y} \ni (u, v) \mapsto \bar{f}^{-1}(\cdot, (u, v))$ both have their image in $\text{Aut}(X)$. Thus all $\bar{f}(\cdot, y), \bar{f}^{-1}(\cdot, u, v)$ are biholomorphic, whence, by 3.1.1, so are all $\tau f^{-1}(\cdot, y), \tau f(\cdot, x, u, v)$.

Now $X$ is indecomposable and $(x_{0}, y_{0})$ decomposes $f$; therefore (compare 3.3.2.(i), (ii)) all $(\bar{f}^{j}(\cdot, x, u, v))^{n}$ become constant for $n$ sufficiently large.

Assume now that in addition $V = V_{U} \cong X \times V_{0}$ with all the postulated properties for the resulting isomorphism $\bar{f} : X \times Y \to X \times (U \times V_{0}) = X \times (X \times U_{Y} \times V_{0})$. Fix some $(u, v) \in U_{Y} \times V_{0}$ and let $\phi := \bar{f}^{-1}(\cdot, (u, v)) : X \times X \to X$. By construction, both
\(\phi(x, \cdot)\) and \(\phi(\cdot, x)\) are contained in \(A(X)\) for all \(x \in X\), which can only happen, if every orbit map \(\phi : A(X) \to X_{\text{red}}\) is biholomorphic and if \(X\) is reduced in every smooth point of \(X_{\text{red}}\) (see 1.1.2.b). Thus \(C^0 \neq X \cong A(X)\) in contradiction to \(X \notin \mathcal{F}\). \(\diamondsuit\)

5.3.4 Theorem. Let \(X \in \mathcal{E} \setminus \mathcal{F}\).

Then \(X\) admits a unique decomposition (up to reordering) \(X \cong X_1^1 \times \ldots \times X_l^1\) such that \(X_\lambda^1 \cong Y_\lambda^n\) with \(n_\lambda \geq 1\) and \(X_\lambda^1 \neq C^0\) indecomposable and pairwise non-isomorphic for \(1 \leq \lambda \leq l\).

If \(f \in \text{Aut}(X)\), then every partial map \(X_\lambda^1 \to X_\lambda^1\) given by \(f\) or \(f^{-1}\) is biholomorphic, and every composition of partial maps \(\left(X_\lambda^1 \to \prod_{\lambda \neq \lambda'} X_\lambda^1 \to X_\lambda^1\right)^n\) is constant for \(n \gg 0\).

Moreover, there exist permutations \(\sigma_\lambda\) of \(\{1, \ldots, n_\lambda\}\) such that \(f := (J_{\sigma_1} \times \ldots \times J_{\sigma_1})\) of \(X_1^1 \times \ldots \times X_{l_1}^1 \times \ldots \times X_{l_{|U|}}^1 \to X_1^1 \times \ldots \times X_{l_{|U|}}^1\)

where \(X_\kappa \neq X_i\) satisfies: All partial maps \(X_\lambda^1 \to X_\lambda^1\) given by \(\overline{f}\) or \(\overline{f^{-1}}\) are biholomorphic, and all compositions \(\left(X_\lambda^1 \to \prod_{(\lambda, \nu) \neq (\lambda', \nu')} X_\lambda^1 \to X_\lambda^1\right)^n\) are constant for \(n \gg 0\).

Proof: Evident by Lemma 5.3.3. \(\diamondsuit\)

5.4. Some Examples

Let \(p, q\) with \(p \neq q\) be primes, and let \(A, B\) be connected complex spaces such that \(\mathbb{Z}_p\) acts non-trivially on \(A\) and \(\mathbb{Z}_q\) acts non-trivially on \(B\). Fix some generators \(\alpha \in \mathbb{Z}_p\), \(\beta \in \mathbb{Z}_q\), and let \(T := \mathbb{C}/\mathbb{Z} + \mathbb{Z}_p\).

For \(1 \leq r \leq p = 1, 1 \leq s \leq q - 1\) define \(\alpha_r \in \text{Aut}(T \times A)\) by \(\alpha_r(t, a) := (t + \frac{1}{p}, \alpha^r(a))\), \(\beta_s \in \text{Aut}(T \times B)\) by \(\beta_s(t, b) := (s + \frac{1}{q}, \beta^s(b))\), and let \(\gamma \in Aut(T \times A \times B)\) be given by \(\gamma(t, a, b) := \left(t + \frac{1}{pq}, \alpha(a), \beta(b)\right)\). Then the quotients \(A_{1/\alpha_r, B} := (T \times B)/\beta_s, \mathbb{A}B := (T \times A \times B)/\gamma\) are total spaces of torsion bundles over \(T/\left(\frac{1}{p}\right), T/\left(\frac{1}{q}\right), T/\left(\frac{1}{pq}\right)\), respectively. Evidently, \(A_r \cong A_{p-r}\) via \(t \mapsto -t\).
5.4.1 Lemma.

(i) \( T \times A_r \cong T \times A_{r'} \) for all \( r, r' \in \{1; \ldots, p-1\} \).

(ii) \( A_r \times B_s \cong A_{r'} \times B_{s'} \) for all \( r \in \{1, \ldots, p-1\} \), and \( s, s' \in \{1, \ldots, q-1\} \).

(iii) \( A_r \times A_r \cong A_{r'} \times A_{r'} \) if \( r^2 \equiv \pm r'^2 \pmod{p} \).

(iv) Assume that \( p = 2 \), \( q = 3 \), and let \( C, D \) be connected complex spaces with non-trivial \( \mathbb{Z}_2 \)-resp. \( \mathbb{Z}_3 \)-action. Then \( A \times B \times C \cong A \times D \times C \).

Proof. Let \( \Phi : T \times T \rightarrow T \times T \) be given by the matrix

\[
\begin{pmatrix}
\lambda & p \\
\lambda' & p + \rho
\end{pmatrix}, \text{ where } \lambda' \equiv r \pmod{p}, \text{ and } (\lambda' - \lambda)p + \lambda \rho = 1,
\]

\[
\begin{pmatrix}
\lambda p + 1 & \mu q \\
\lambda p & \mu q + \rho
\end{pmatrix}, \text{ where } s' \equiv s \pmod{q}, \text{ and } \lambda p^2 + \mu q = 1 - \rho,
\]

\[
\begin{pmatrix}
\rho & \lambda p \\
\mu q & \rho
\end{pmatrix}, \text{ where } r' = \rho \equiv r \pmod{p} \text{, and } \mu p^2 = \lambda p^2 \pm 1,
\]

\[
\begin{pmatrix}
3 & 4 \\
16 & 21
\end{pmatrix}.
\]

Then

\[
(i) \Phi \times \text{id}, \quad (ii) \Phi \times \text{id},
\]

\[
(iii) \Phi \times \text{id}, \quad (iv) \Phi \times \text{id} \text{ induces an isomorphism as postulated.}
\]

5.4.2 Lemma.

(i) Every \( A \) is indecomposable.

(ii) \( A \times B \) has no non-trivial compact factor. If \( A \) or \( B \) is compact, then \( A \times B \) is indecomposable.

(iii) \( A_r \cong A_{r'} \) if and only if there exists \( \gamma \in \text{Aut}(A) \) with \( \gamma \alpha^n = \alpha^{r'} \). In particular, if \( \mathbb{Z}_p \) is central in \( \text{Aut}(A) \), then \( A_r \not\cong A_{r'} \) for \( n \not\equiv \pm r' \pmod{p} \).
Proof: Let \( S \in (A, AB) \) with \( S \cong S_1 \times S_2 \). By 2.4.2.b, we may assume that there exists \((S_1^a \rightarrow T^a_1) \in \mathcal{F} \) with some fibre \( S_1^a \) and with \( T^a_1 \) isogenous to \( T \). By (2) and (3), every isomorphism \( S \rightarrow S_1^a \times S_2^a \) is a \( \mathcal{F} \)-morphism \( \pi \rightarrow \pi_1 \circ p_{S_1} \) (where \( \pi \) denotes the given torsion bundle \( A_\pi \rightarrow T/\mathbb{Z}_p \) resp. \( AB \rightarrow T/\mathbb{Z}_p \)).

(i) Clearly, \( A_\pi \) cannot be \( \mathcal{F} \)-isomorphic to \( T \times A \). Thus \( S_1^a \neq C^0 \), and we conclude that \( S_1^a = C^0 \), since \( A \cong S_1^a \times S_2^a \) is indecomposable.

(ii) Again, \( AB \) is not \( \mathcal{F} \)-isomorphic to \( T \times A \times B \), whence \( S_1^a \neq C^0 \). By (4), the isomorphism \( A \times B \rightarrow S_1^a \times S_2^a \) between the fibres degenerates and therefore induces a simultaneous subdecomposition, if \( A, B, S_1 \) or \( S_2 \) is compact (see 5.1.5). Denote this isomorphism by \( f \), and assume that \( S_1 \) or \( S_2 \) is compact with \( S_2 \neq C^0 \). Then either all partial maps \( \tau_f(a_1, a_2, \ldots, a_n) : B \rightarrow S_2 \) or all \( \tau_f(a_1, a_2, \ldots, a_n) : A \rightarrow S_2 \) are biholomorphic by (1). On the other hand, it is evident that \( \tau_f(a_1, a_2, \ldots, a_n) : B \rightarrow S_2 \) is indecomposable. By (2) and (3), the \( \tau_f(a_1, a_2, \ldots, a_n) : A \rightarrow S_2 \) is compact with \( S_2 \neq C^0 \).

Assertion (iii) is obvious, since every \( A_\pi \rightarrow A_\pi \) is a \( \mathcal{F} \)-morphism.}

For \( k \geq 2 \) let \( \varepsilon_k := \exp \left( \frac{2\pi i}{k} \right) \) and let \( Z_k \) act on \( P_2 \) via \( (\kappa, x) \mapsto (\varepsilon_k^\kappa x, x) \). If we want to indicate this action, we let \( Z(k) := P_2 \) in what follows. By blowing up \( x \in P_2 \) \( l+1 \) times, where \( l \geq 1 \), we mean: blow up \( x \) times and then blow up \( x \) any point in the exceptional curve.

Let \( X(k) \) be the manifold that arises from \( P_2 \) by blowing up (once) every \( (\varepsilon_k^\kappa x, x) \), \( 1 \leq \kappa \leq k \), by blowing up \( l+2 \) times the points \( (0 : 1 : 1) \) for \( 0 \leq l \leq 2 \) and by blowing up five times the point \( (1 : 0 : 0) \). The \( Z_k \)-action on \( P_2 \) lifts to \( X(k) \) and also restricts to the complement \( U(k) \subset X(k) \) of the inverse image of \( (1 : 0 : 0) \). It is easy to see that \( Z_k = \mathrm{Aut}(X(k)) \) and \( Z_k = \mathrm{Aut}(U(k)) \). Thus, by 5.4.2.(iii), \( A(p) \cong A(p) \), (where \( A \in \{X, U\} \)), if and only if \( r \equiv r \mod p \).

Clearly, every pair \( (A(p), B(q)) \) with \( A, B \in \{U, X, Z\} \) satisfies the conditions (1) - (4).

5.43. Examples.

a) There exist indecomposable connected complex spaces \( X, U, U' \) with \( X \) compact, and with \( U \neq U' \) having no compact factor \( C^0 \) such that \( U \cong U' \) and \( X \times U \cong X \times U' : X := T, U := U(q) \), \( U' := U(q) \) with \( q \geq 5 \) (see 5.4.1(i), 5.4.2(i)).

b) There exist indecomposable connected complex spaces \( X, U \) with \( X' \) compact, and with \( U \) having non compact factor \( C^0 \) such that \( X \neq X' \) and \( X \times U \cong X' \times U : U := U(q), X := X(p) \), \( X' := X(p) \) with \( p \geq 5 \) (see 5.4.1(ii), 5.4.2(ii)).

c) There exist indecomposable connected complex spaces \( X, X', U, U' \) with \( X, X' \) compact, and with \( U \neq U' \) having no compact factor \( C^0 \) such that \( U \cong U' \) and \( X \times U \cong X \times U' : U := U(q), X := X(p) \), \( X' := X(p) \) with \( p \geq 5 \) (see 5.4.1(ii), 5.4.2(iii)).
and with $U \neq U'$ admitting no compact factor $\not\subset C^0$, such that $X \not\cong X'$, $U \not\cong U'$, and $X \times X \cong X' \times X'$, $U \times U \cong U' \times U'$.

$$X := X(5)_1, X' := X(5)_2, U := U(5)_1, U' := U(5)_2 \text{ (see 5.4.1.(iii)), 5.4.2.(i)).}$$

d) There exist connected complex spaces $X, U \neq V, \mathcal{W}$, with $X \not\subset C^0$ indecomposable and compact, and with $U \neq V, \mathcal{W}$, having no compact factor $\not\subset C^0$, such that $X \times U \cong V \times \mathcal{W}$:

$$X := X(2)Z(3), U := U(2)U(3), V := X(2)U(3), \mathcal{W} := U(2)Z(3) \text{ (see 5.4.1.(iv), 5.4.2.(ii)).}$$

e) There exist $X, Y \neq U \neq V$ with $X \not\subset C^0 \neq Y$ compact, and with $U, V$ admitting no compact factor $\not\subset C^0$, such that $\dim X \neq \dim Y$ and $X \times U \cong Y \times V$.

$$X := X(2)Z(3), Y := X(2)X(3), U := U(2)X(3), V := U(2)Z(3) \text{ (see 5.4.1.(iv), 5.4.2.(ii)).}$$

In particular, we see that for general $U$, there is no possibility of introducing a reasonable notion of a unique maximal compact factor.

Choosing $A, B \not\subset C$ appropriately, one can show in a similar way that a general $X \in \mathcal{G}$ does not admit a unique maximal factor in any of the $\mathcal{P}$-categories listed in 5.3.1.a(ii) other than $\mathcal{G}_0$ or $\mathcal{G} \setminus \mathcal{P}$.

6. AUTOMORPHISMS OF PRODUCTS

Let $U$ be a connected complex space with standard decomposition $U \cong U_1 \times U' \cong X_1 \times \ldots \times X_k \times U'$ (compare 5.3.4.a), and let $\varphi \in \text{Aut}(U)$. By 5.3.2 and 5.3.4, every partial map $U_1 \rightarrow U_1, X_1^i \rightarrow X_1^i, U' \rightarrow U'$, given by $\varphi$ or $\varphi^{-1}$ is biholomorphic. In general, however, $\varphi$ need not be a product of isomorphisms between the individual factors. For every $\varphi$ to be a product of automorphisms of $U_1$ and $U'$, it is necessary that there exist no non-constant holomorphic mappings $U \rightarrow \text{Aut}(U_1), U \rightarrow \text{Aut}(U')$. In the reduced case, this condition is easily seen to be sufficient as well; in general, it is not.

If $U$ is reduced and compact with $A(U) = 0$, then evidently all $\varphi \in \text{Aut}(U)$ are products of isomorphisms between the indecomposable factors of $U$. This assertion does no longer hold for non-reduced $U$; for instance, the automorphism of $P^2 \times P^2 (P^2 \rightarrow C$ the double point) given by $(x, y) \mapsto (x + xy, y + sy)$ is not a product.

In view of these difficulties, we henceforth restrict our attention to the compact reduced case.

6.1. Decomposition-preserving automorphisms

Let $X$ be a reduced compact complex space with a decomposition $f : X \rightarrow Y_1 \times \ldots Y_n$.

6.1.1. Definition. An automorphism $\varphi$ of $X$ preserves the decomposition $f$, if all partial maps $Y_\nu \rightarrow Y_\nu (1 \leq \nu \leq n)$, given by $\varphi$ and $\varphi^{-1}$ are biholomorphic. We let $\text{Aut}_f(X) := \{\varphi \in \text{Aut}(X) : \varphi \text{ preserves } f\}$. 

6.1.1. Remarks. From 5.3.2 and 5.3.4, we infer:

(i) If $Y_\mu$ and $Y_\nu$ have no positive-dimensional common factor for all $1 \leq \mu, \nu \leq n, \mu \neq \nu$, and if at most one $Y_\mu$ is contained in $\mathcal{P}(n)$, then $\operatorname{Aut}(X) = \operatorname{Aut}_f(X) \cdot \sigma \in S(n)$.

(ii) If $Y_1, \ldots, Y_n \notin \mathcal{P}$ are indecomposable, then $\operatorname{Aut}_\text{norma}(X, Y) \cdot \sigma \in S(n)$.

We shall now demonstrate how to construct $\operatorname{Aut}_f(X)$ from $\operatorname{Aut}(Y_1 \times \operatorname{Aut}(Y_2))$. Then, using the above remarks, one can build up successively $\operatorname{Aut}(X)$ from $\prod_{\lambda=1}^n \operatorname{Aut}(X_\lambda) \times \operatorname{Aut}(X')$, where $X \cong \left( \prod_{\lambda=1}^n X_\lambda^n \right) \times X'$ is a standard decomposition of $X$.

To simplify the notation, we consider reduced compact complex spaces $Y \times Z$ with $Y \notin \mathcal{P}$, and we let $\operatorname{Aut}(Y \times Z) = \operatorname{Aut}(\mathcal{Y}) \times \operatorname{Aut}(\mathcal{Z})$. Then, by 5.1.2, every $\phi \in \operatorname{Aut}(Y \times Z)$ degenerates.

Let $\phi \in \operatorname{Aut}(Y \times Z)$, and fix some $(y_0, z_0) \in Y \times Z$. By Theorem 5.3.2, there exist $(\alpha, \delta) \in \operatorname{Aut}(Y) \times \operatorname{Aut}(Z)$ and $\beta \in \operatorname{Hol}(Y, A(Y)))$, $\gamma' \in \operatorname{Hol}(Y, A(Z)))$ with $\beta(z_0) = \text{id}_Y$, $\gamma'(y_0) = \text{id}_Z$, such that $\phi(y, z) = (\beta'(z)(\alpha(y)), \gamma'(y)(\delta(z)))$ for all $(y, z) \in Y \times Z$. As $A(Y \times Z)$ is normal in $\operatorname{Aut}(Y \times Z)$, there exist $\beta \in \operatorname{Hol}(Z, A(Y)))$ with $\beta(z_0) = \text{id}_Y$ and $\gamma \in \operatorname{Hol}(Y, A(Z)))$ with $\gamma(y_0) = \text{id}_Z$ such that $\alpha \circ \beta(z) = \beta'(z) \circ \alpha$ and $\delta \circ \gamma(y) = \gamma'(y) \circ \delta$ for all $y \in Y$, $z \in Z$. Evidently, the quadruple $(\alpha, \beta, \gamma, \delta)$ is uniquely determined by these properties.

We shall now derive a necessary and sufficient criterion for such a quadruple $(\alpha, \beta, \gamma, \delta)$ to define $\phi \in \operatorname{Aut}_f(Y \times Z)$ in the way described above. For $(\beta, \gamma) \in \operatorname{Hol}(Y, A(Y))) \times \operatorname{Hol}(Y, A(Z)))$, define $(\beta, \gamma) : Y \times Z \rightarrow Y \times Z$ by $(y, z) \mapsto (\beta(z)(y), \gamma(y)(z))$. Evidently, it suffices to find out under which conditions $(\beta, \gamma) \in \operatorname{Aut}(Y \times Z)$.

To begin with, we reduce the situation to the case where $Y \times Z$ are tori.

6.1.2. Lemma and Definition. The functor $\mathcal{B}_\text{red} \rightarrow \mathcal{B}_\text{ns}$, $Z \mapsto \bigcup \{ \operatorname{Hol}(Z, T) : T \text{ a torus} \}$, is represented by $\operatorname{alb}^0 : Z \mapsto (\operatorname{alb}_Z^0 : Z \rightarrow \operatorname{Alb}^0(Z))$.

$\operatorname{alb}^0_Z$ is called the weak Albanese map of $Z$.

The proof can be copied word for word from the corresponding one for smooth varieties. Note that $\operatorname{alb}^0_Z = \operatorname{alb}_Z$ if $Z$ is smooth.

Let $(x_0, y_0) \in Y \times Z$ with $\operatorname{alb}^0(x_0, y_0) = 0$ and let $(\beta, \gamma) \in \operatorname{Hol}(Z, A(Y))) \times \operatorname{Hol}(Y, A(Z)))$ with $\beta(z_0) = \text{id}_Y, \gamma(y_0) = \text{id}_Z$. Then $\operatorname{alb}^0((\beta, \gamma)) : \operatorname{Alb}^0(Y \times Z) \rightarrow \operatorname{Alb}^0(Z)$ is a holomorphic homomorphism. Moreover, if we let $\overline{\beta}$ be the composition $\left( \operatorname{Alb}^0(Z) \stackrel{\operatorname{alb}^0}{\rightarrow} \operatorname{Alb}^0(A(Y))) = A(Y) \stackrel{\operatorname{alb}^0}{\rightarrow} A(\operatorname{Alb}^0(Y)) = \operatorname{Alb}^0(Y) \right.$ and
\( \overline{\gamma} : \text{Alb}^0(Y) \to \text{Alb}^0(Z) \) accordingly, then \( \text{alb}^0(\langle \beta, \gamma \rangle) = \langle \overline{\beta}, \overline{\gamma} \rangle \).

6.1.3 Lemma. The map \( \langle \beta, \gamma \rangle \) is biholomorphic, if and only if so is \( \langle \overline{\beta}, \overline{\gamma} \rangle \).

Proof. Let \( \langle \beta, \gamma \rangle \) be biholomorphic (the other implication is trivial). It suffices to show that \( \beta \) is injective; for this, in turn, we need only show that \( \langle \beta, \gamma \rangle \) is injective on every fibre of \( \text{alb}^0_{Y \times Z} \).

Let \( Y_0 \times Z_0 \) be some fibre of \( \text{alb}^0_{Y \times Z} = \text{alb}^0_Y \times \text{alb}^0_Z \). Then \( \beta|_{Z_0} = [\beta(z_1)] \), \( \gamma|_{Y_0} = [\gamma(y_1)] \) for any \( z \in Z_0, y \in Y_0 \), and therefore \( \langle \beta, \gamma \rangle|_{Y_0 \times Z_0} = \beta(z_1) \times \gamma(y_1) \) is injective.

Let now \( \overline{Y} := \text{Alb}^0(Y) \), \( \overline{Z} := \text{Alb}^0(Z) \).

6.1.4 Lemma. \( \langle \beta, \gamma \rangle \) is biholomorphic, if and only if \( (\beta \circ \overline{\gamma})^\ast = 0 \) for \( n \gg 0 \).

Proof. Let \( \sigma := \overline{\beta} \gamma|_\sigma := \overline{\beta} \gamma \); then \( \sigma \) is nilpotent, if and only if so is \( \tau \). The homomorphism \( \langle \beta, \gamma \rangle \) is an isomorphism, if and only if there exists an endomorphism of \( \overline{Y} \times \overline{Z} \) given by a matrix \( \begin{pmatrix} \alpha' & \beta' \\ \gamma' & \delta' \end{pmatrix} \) such that \( \begin{pmatrix} \alpha' + \beta' \gamma & \alpha' \beta + \beta' \\ \gamma' + \delta' \gamma & \gamma \beta + \delta' \end{pmatrix} = \begin{pmatrix} \text{id} & 0 \\ 0 & \text{id} \end{pmatrix} \). If \( \sigma \) is nilpotent, then \( \text{id} - \sigma \) and \( \text{id} - \tau \) are invertible, and a simple computation shows that the matrix \( \begin{pmatrix} \alpha' & \beta' \\ \gamma' & \delta' \end{pmatrix} \), given by \( \alpha' = (\text{id} - \sigma)^{-1}, \beta' = (\text{id} - \tau)^{-1}, \gamma' = -\delta', \gamma = -\delta' \gamma \) defines an inverse of \( \langle \beta, \gamma \rangle \).

Conversely, if \( \langle \beta, \gamma \rangle \) is invertible, then \( \sigma \) is \( \langle \beta, \gamma \rangle \) and \( \langle \beta, \gamma \rangle \) degenerates, since \( U \notin \mathcal{J} \). By 3.1, 3.2, \( \langle \beta, \gamma \rangle \) degenerates as well, i.e., if \( \langle \beta, \gamma \rangle \) is given by \( \begin{pmatrix} \alpha' & \beta' \\ 7' & 6' \end{pmatrix} \), then \( \overline{\langle Y \times Y \rangle} \to \overline{Z} \to \overline{\overline{Y}} \to \overline{Z} \to \overline{\overline{Y}} \to \overline{\overline{\overline{Y}}} \) for \( n \gg 0 \). Now \( \beta' = -\alpha' \beta = -\overline{\beta} \gamma, \gamma' = -\delta' \gamma = -\overline{\gamma} \alpha' \), and we conclude that \( \alpha' \alpha' \overline{\beta} \gamma = \alpha' \beta \delta' \gamma = \beta' \gamma' = \overline{\beta} \delta' \gamma \alpha' = \overline{\gamma} \alpha' \beta' \); thus \( \beta' \gamma' \) is nilpotent, if and only if so is \( \overline{\beta} \gamma \).

For \( \langle \overline{\beta}, \overline{\gamma} \rangle \in \text{Hom}(\overline{Z}, A(Y)) \times \text{Hom}(\overline{Y}, A(Z)) \), let \( \beta \times \gamma := (\beta \circ \text{alb}^0_Y) \times (\gamma \circ \text{alb}^0_Z) : Z \times Y \to A(Y) \times A(Z) \), and \( \overline{\beta} \times \overline{\gamma} := (\text{alb}^0_Y \circ \overline{\beta}) \times (\text{alb}^0_Z \circ \overline{\gamma}) : \overline{Z} \times \overline{Y} \to A(\overline{Y}) \times A(\overline{Z}) = \overline{\overline{Y}} \times \overline{Z} \).

Summing up, we obtain:

6.1.5 Theorem. Let \( Y, Z \) be reduced connected compact complex spaces with \( Y \notin \mathcal{J} \) and let \( \mathcal{A}(Y \times Z) := \{ (\alpha, \beta, \gamma, \delta) \in \text{Aut}(Y) \times \text{Hol}(\overline{Z}, A(Y)) \times \text{Hol}(\overline{Y}, A(Z)) \times \text{Aut}(Z) : \overline{\beta} \gamma \text{ nilpotent} \} \).
Then the map \( \mathcal{H}(Y \times Z) \rightarrow \text{Aut}_+(Y \times Z) \) given by \((\alpha, \beta, \gamma, \delta) \mapsto (\alpha \times \delta) \circ (\beta, \gamma)\) is \textit{well-defined} and bijective.

6.2. Automorphisms of projective varieties

6.2.1 Lemma. Let \( U \) be a connected complex space, and let \( T \) be a connected compact complex \textit{subgroup} of \( A(U) \). Assume there exists a \textit{line} bundle \( L \) on \( U \) that is ample on some orbit \( Tu_0 \) of \( T \).

Then there exists \((U \rightarrow T) \in \mathcal{F}_i\), where \( 1 := \dim T \).

\textit{Proof.} Denote by \( \widehat{L} \) the line bundle \((E_0^T L) \otimes ((\cdot u_0 \circ p_T)^* L)^{-1}\) on \( T \times U \) (where \( E = E_0 \) denotes the evaluation map). Evidently, \( \widehat{L} \big|_{T \times \{u_0\}} \) is topologically \textit{trivial} for all \( u \in U \); thus \( u \mapsto j_u^* \widehat{L} \) defines a holomorphic map \( \tau : U \rightarrow \text{Pic}_0(T) \). Let \( T_0 \) denote the connected component of \( T \) that contains \( u_0 \). As \( \widehat{L} \) is trivial along every \( T \times \{u\}, u \in T_0 \), there exists a line bundle \( L_1 \) on \( T_0 \) with \( \widehat{L} \big|_{T \times \{u_0\}} = p_{T_0}^* L_1 \); thus \( E^* L \mid T_0 \times T_0 = p_{T_0}^* L_1 \otimes (\cdot u_0 \circ p_T)^* L \). We conclude that \( L_1 \) is ample, since so is \( E^* L \big|_{\{u_0\} \times T_0} = p_{T_0}^* L_1 \big|_{\{u_0\} \times T_0} \).

This shows that \( \tau \big|_{\{u_0\}} \) is finite and hence surjective. In particular, there exists some finite holomorphic homomorphism \( \beta : \text{Pic}_0(T) \rightarrow T \) such that \( \alpha = \beta \circ \tau \) satisfies the condition of Lemma 2.4.2.

6.2.1.a Corollary. Let \( X \) be a projective variety. Then there exists \((X \rightarrow A(X)) \in \mathcal{F}_a\), where \( a := \dim A(X) \).

6.2.1.b Corollary. Let \( X \) be a projective variety with standard decomposition \( X \cong X_d \times X'_l \times X'_l \times \ldots \times X'_l \times X'_l = X^h_l \times \ldots \times X^n_l \times X'_l \) (compare 5.3.4.a).

Then \( \text{Aut}(X) \cong \left( \prod_{\lambda=1}^l \text{Aut}(X'_\lambda) \right) \times \text{Aut}(X') \times \text{Hol} \left( \text{Alb}^0(X_c), A(X') \right) \) (where the isomorphism is given by \ref{6.1.5}) and \( \text{Aut}(X'_\lambda) \cong \bigcup_{o \in S(n_o)} J_o \circ \left( \text{Aut}(X'_\lambda) \right) n_o \) (compare \ref{6.1.6} (ii)).

6.2.1.c Example. Let \( T_1 \) be a two-dimensional torus of algebraic dimension 1, and let \( \tau : T_1 \rightarrow T \) denote its equivariant algebraic reduction. Let \( C \rightarrow T \) be a surjective holomorphic map from a compact Riemann surface of genus \( \geq 2 \) onto \( T_1 \) and let \( X := T_1 \times_{\tau|T} C \). Then \( X \) is a two-dimensional compact Kähler manifold, \( A(X) \cong \text{Ker} \tau \) is one-dimensional, and \( X \notin \mathcal{F}_i \).
7. ISOGENY DECOMPOSITIONS

In Shioda's as well as in Parigi's examples for $X \times Y \cong X \times Z$, the varieties $Y, Z$ always admit coverings $S \to Y, S \to Z$ (with the same $S$) and thus are still closely related to each other. We shall now see that this fact is not accidental.

7.1. Isogenous products

7.1.1 Definition. Let $S_1, S_2$ be connected complex spaces.

(i) $S_1$ and $S_2$ are isogenous, if there exist coverings (i.e. locally biholomorphic finite mappings with connected domain) $S \to S_1, S \to S_2$.

Notation: $S_1 \sim S_2$. A diagram $S \to S_1 \times S_2$ of coverings is called an isogeny between $S_1$ and $S_2$.

(ii) $S_1$ is an isogeny factor of $S_2$, if $S_2 \sim S_1 \times S'_1$ with suitable $S'_1 \sim S_2$ is strongly indecomposable, if it admits no isogeny factor $\not\exists C^0 \mid S'_1$.

7.1.1 Remarks. (i) $\sim$ is an equivalence relation.

(ii) If $(\pi : U \to T) \in \mathcal{P}_h$ then $T$ is an isogeny factor of $U$.

7.1.2 Lemma. Let $\phi : S \to X \times Y$ be a covering.

Then there exist coverings $\alpha : X' \to X, \beta : Y' \to Y$ with the following properties.

(i) $\alpha \times \beta$ factors through $\phi$.

(ii) If $7 : X'' \to X, \delta : Y'' \to Y$ are coverings such that $7 \times \delta$ factors through $\phi$, then 7 factors through $\alpha$ and $\delta$ factors through $\beta$.

(iii) If $\phi$ is biholomorphic, then $\alpha = \text{id}$, and $\beta = \text{id}$.

Proof. Let $\tilde{\alpha} : \tilde{X} \to X, \tilde{\beta} : \tilde{Y} \to Y$ be the universal coverings with deck transformation groups $G \cong \pi_1(X), H \cong \pi_1(Y)$. Then $G' := G \cap \pi_1(S), H' := H \cap \pi_1(S)$ have finite index in $G, H$, respectively, and $G' \times H'$ is a subgroup of $\pi_1(S)$. Thus there exist factorizations $\tilde{\alpha} = (\tilde{X} \to \tilde{X}/G', \tilde{\alpha}) \times (X) \to (\tilde{Y} \to \tilde{Y}/H', \tilde{\beta}) \to Y$, and the assertion follows with $X' = \tilde{X}/G', Y' = \tilde{Y}/H'$.

Let now $X \times Y \xleftarrow{\psi} U \times V$ be an isogeny (between connected complex spaces), and construct the triangle

$$
\begin{array}{ccc}
X' \times Y' & \xrightarrow{\phi} & S \\
\downarrow \alpha \times \beta & & \downarrow \phi \\
X \times Y & & 
\end{array}
$$

as above. Let $(f_1 : X_1 \times Y_1 \to U_1 \times V_1) := (\psi \circ \phi' : X' \times Y' \to U \times V)$, and apply the
same construction to \( f_1 \), thus obtaining

\[
\begin{align*}
U_3 \times V_3 & \xrightarrow{f_1} X_2 \times Y_2 \\
\downarrow \alpha_1 \times \beta_1 & \Rightarrow f_1 \\
U_1 \times V_1 & \end{align*}
\]

Iterating this procedure, we arrive at

\[
\begin{align*}
\ldots & \rightarrow X_{2n+2} \times Y_{2n+2} & \xrightarrow{\alpha_2 \times \beta_{2n}} & X_{2n} \times Y_{2n} & \ldots \\
\downarrow \beta_{2n} & \Rightarrow f_{2n} \\
U_{2n+3} \times V_{2n+3} & \xrightarrow{\alpha_2 \times \beta_{2n-1}} & U_{2n+1} \times V_{2n+1} & \rightarrow & \ldots
\end{align*}
\]

By construction, if some \( f_n \) is biholomorphic, then so are all \( f_m \) for \( m \geq n \), and \( f_n \) and \( f_{n+1} \) are then inverse to each other.

Let \( ((x_{2n}, y_{2n})) \in \prod_{n \geq 1} (X_{2n} \times Y_{2n}) \) with \( \alpha_{2n} (x_{2n+2}) = x_{2n}, \beta(y_{2n+2}) = y_{2n} \) and let

\[
(u_{2n+1}, v_{2n+1}) = f_{2n+1} (x_{2n+2}, y_{2n+2}) \]

Consider the sequence

\[
(*) \ldots \rightarrow X_{2n+4} \xrightarrow{v_{2n+4}} U_{2n+3} \xrightarrow{v_{2n+3}} Y_{2n+2} \xrightarrow{v_{2n+2}} V_{2n+1} \xrightarrow{v_{2n+1}} X_{2n} \rightarrow \ldots
\]

and denote by \( R_n \overset{(e)}{\rightarrow} R'_n \) the map given by a subsequence of length 1, where \( R, R' \in \{ X, Y, U, V \} \) appropriately.

7.13 Definition. The isogeny \( X \times Y \overset{\eta}{\rightarrow} S \rightarrow U \times V \) degenerates (with respect to the family \( ((x_{2n}, y_{2n})) \)) if the reduction of \( R_n \overset{(e)}{\rightarrow} R'_n \) is constant for \( 1 \geq 0 \) and all \( n \).

7.1.3.a Remark. If \( (X_{2n+4k} \overset{(e)}{\rightarrow} X_{2n+2}) \) is constant, then so is \( (R_{n+l} \overset{(e)}{\rightarrow} R'_{n+l}) \) for all \( n \) and all \( l \geq 4k + 6 \).

Proof. Clearly, \( \alpha_{2n+2} \circ v_{2n+4} = v_{2n+2} \circ \alpha_{2n+3} \) and corresponding relations hold for \( x, u, v \).

Thus \( (X_{4k+2} \overset{(e)}{\rightarrow} X_{2n+2}) \circ \alpha_{4k+2} \circ \ldots \circ \alpha_{2n+4k+2} = \alpha_2 \circ \ldots \circ \alpha_{2n} \circ (X_{2n+4k+2} \overset{(e)}{\rightarrow} X_{2n+2}) \),

whence \( X_{4k+2} \overset{(e)}{\rightarrow} X_2 \) is constant, if and only if so is \( X_{2n+4k+2} \overset{(e)}{\rightarrow} X_{2n+2} \). Furthermore, every subsequence of (*) of length \( \geq 4k + 6 \) contains some \( X_{2n+4k+2} \overset{(e)}{\rightarrow} X_{2n+2} \).

From now on assume that \( X \) is compact.
7.1.4 Lemma. (compare 5.1.1).

(i) If \( |1| \geq 2 \), then \( R_{n+1} \) is isomorphic (set-theoretically) through \( \text{Hol} \left( \frac{R_{n+1}}{R_n} \right) \) to \( R_n \) where \( \gamma \in \{ \alpha, \beta \} \) according as \( R' \in \{ U, X \} \) or \( R' \in \{ V, Y \} \) with corresponding \( \tau \) factors.

(ii) If \( n \neq 0 \) and \( R_{n+1} \) contains \( (X_{n+1} \to Y_m) \), then \( R_{n+1} \) factors through \( \text{Aut}( \frac{X_{n+1}}{X_n} ) \).

Proof. The proof of (i) does not require \( X \) to be compact; thus we may assume \( R_n = X_n \) for symmetry reasons.

Let \( \phi := \rho n \circ (\gamma_{n+1} : \rho X_{n+1} \to \gamma_{n+1}) : X_{n+1} \times Y_{n+1} \to X_n \); then \( \phi(X_{n+1}, \cdot) = \gamma_{n+1} \circ \gamma_{n+1} \) and \( \phi(\cdot, Y_{n+1}) = \rho n \circ \rho n + 1 (\gamma_{n+1}, \gamma_{n+1}) = \alpha_n \).

Thus we obtain a commutative diagram

\[
\begin{array}{ccc}
R_{n+1} & \xrightarrow{\rho n} & R_n \\
\downarrow & \downarrow \phi & \downarrow \gamma \\
Y_{n+1} & \xrightarrow{\rho n + 1} & X_n \\
\alpha_n & \xrightarrow{\text{Hol}(X_{n+1} \to X_n)} & \alpha_n \\
\end{array}
\]

which proves (i).

Consider now \( \psi := (X_{n+1} \times X_{n+1} \xrightarrow{\text{id} \times \phi} X_{n+1} \times Y_{n+1}) \), let \( \overline{W}_n := \rho \psi \), and denote by \( \overline{W}_n \) the weak normalization of \( \rho \psi \). Applying 2.3.2 to the sequence \( \ldots \to X_{n+2} \to X_{n+1} \to X_n \to \ldots \), we conclude that \( \overline{W}_n \) factors through \( \text{Aut}(X_n) \) and from 2.3.2a we infer that the natural map \( \overline{W}_n \to \text{Hol}(X_n) \) is holomorphic with image contained in \( \text{Aut}(X_n) \). This yields a commutative diagram

\[
\begin{array}{ccc}
X_{n+4} & \xrightarrow{\text{id} \times \phi} & X_{n+1} \\
\downarrow & \nearrow \uparrow \rho n & \uparrow \rho n + 1 \\
\text{Aut}(X_{n+4}) & \xrightarrow{\text{red}} & \text{Aut}(X_{n+1}) \\
\end{array}
\]

and we conclude that \( X_{n+4} \) factors through \( \text{Aut}(X_{n+1}) \) since the orbit map \( \alpha_{n+1} : \text{Aut}(X_{n+1}) \to X_{n+1} \) factors through \( X_{n+4} \).
From the commutative diagram
\[
\begin{array}{cccccc}
R_{n+1} & \overset{(*)}{\rightarrow} & X_{m+1} & \overset{(*)}{\rightarrow} & R_{n+2} & \overset{(*)}{\rightarrow} & R^n \\
\downarrow & & \downarrow X_m & & \downarrow \gamma_{m+2} & & \\
A(X_m) & & \text{Hol}(R'_{n+2}, R^n) & & \\
\end{array}
\]
we infer that there exists \( V_n \overset{\gamma}{\rightarrow} \text{Hol}(R'_{n+2}, R^n) \) with \( \gamma_n \in V_n \) such that \( X_{m+8} \overset{(*)}{\rightarrow} R^n \) factors holomorphically through \( \gamma_{m+2} : V_n \rightarrow R^n \). Now assertion (ii) follows by applying 2.3.2 to the sequence \( \cdots \rightarrow R'_{n+2} \overset{\gamma_n}{\rightarrow} R'_{n+1} \rightarrow \cdots \) and to the family \( (V_n) \).

**7.1.5 Proposition.** Let \( l := \lim_{k \to \infty} \dim \text{Im}(X_{2+4k} \overset{(*)}{\rightarrow} X_k) \).

Then there exists an \( l \)-dimensional torus \( T \) which is an isogeny factor of \( X \mid Y, U \) and \( V \).

In particular, if \( X \mid Y, U \) and \( V \) do not admit a common torus isogeny factor (of positive dimension), then every isogeny between \( X \times Y \) and \( U \times V \) degenerates.

**Proof.** Evidently, \( l = \lim_{k \to \infty} \dim \text{Im}(S_{m+k} \overset{(*)}{\rightarrow} S'_m) \) for all \( m \in \mathbb{N} \) and all \( S, S' \in \{X, Y, U \mid V\} \) (compare 7.1.3.a).

By Lemma 7.1.4.(ii), there exists a commutative diagram
\[
\begin{array}{cccc}
S_{m+2k} & \overset{(*)}{\rightarrow} & S'_{m+k} & \overset{(*)}{\rightarrow} & S''_m \\
\downarrow & & \downarrow \gamma'_{m+k} & & \uparrow s''_m \\
A(S'_m) & & A(S''_m) & & \\
\end{array}
\]
for \( m \) sufficiently large and \( k \geq 16 \). Increasing \( k \), we may assume that \( \dim \text{Im}(S_{m+2k} \overset{(*)}{\rightarrow} S''_m) = 1 \); then \( \text{Im}(S_{m+k} \overset{(*)}{\rightarrow} S''_m) \) coincides with the image of the orbit \( A(S'_{m+k}) \text{Im}(S_{m+k}) \), and hence is the orbit of some \( T(S''_m) \square A(S''_m) \). Thus, for all \( m \gg 0 \) and all \( R \in \{X \mid Y, U \mid V\} \), there exists an \( l \)-dimensional \( T(R) \square A(R) \) such that every \( R'_{m+k} \overset{(*)}{\rightarrow} R_{m+k} \) factors through \( T(R) \rightarrow R_{m+k} \), if \( k \) is sufficiently large. Using Lemma 2.4.2 and 7.1.1.a(ii), we conclude that \( T(R_{m+k}) \) is an isogeny factor of \( R_{m+k} \); clearly, \( T(R_{m+k}) \) and \( T(R_{m+k}) \) are isogenous for all \( R, R' \in \{X \mid Y, U \mid V\} \).

**7.1.6 Lemma.** If the isogeny \( X \times Y \hookrightarrow S \rightarrow U \mid V \) degenerates, then \( f_n \) is a degenerating isomorphism for \( n \in \mathbb{N} \).

**Proof.** By 7.1.2(iii) and 7.1.3.b, it suffices to show that \( f_n \) is biholomorphic for \( n \in \mathbb{N} \). For this, in turn, we need only show that \( f_n \) induces an isomorphism between the corresponding fundamental groups.
Let $G_{2n} = \pi_1(X_{2n})$, $H_{2n} = \pi_1(Y_{2n})$, $G_{2n+1} = \pi_1(U_{2n+1})$, $H_{2n+1} = \pi_1(V_{2n+1})$. By construction of the sequence ( $f_n$ ), the sequence ( $G$, $H_n$ ) satisfies the condition of Lemma 0.3.3, if the isogeny $X \times Y \rightarrow S \rightarrow U \times V$ degenerates.

7.2. Cancellation

7.2.1 Lemma. Let $U_1$, $U_2$ be connected complex spaces, and let $T_1$, $T_2$ be tori such that $T_1 \sim U_1 \sim T_2 \times T_2$.

If there exists no positive-dimensional torus that is an isogeny factor of both $U_1$ and $U_2$, then $T_1 \sim T_2$ and $U_1 \sim U_2$.

Proof: It is easily seen (e.g. by using 7.1.2) that every isogeny factor of a torus is isogenous to a torus. Thus, by 7.1.5, every isogenous between $T_1 \times U_1$ and $T_2 \times U_2$ degenerates. By 7.1.6, we may assume that there exists a degenerating isomorphism $f : T_1 \times U_1 \rightarrow T_2 \times U_2$, which, by 5.1.5, induces a simultaneous subdecomposition. As neither $U_1$ nor $U_2$ admits a positive-dimensional torus factor, we conclude that (with the notations of 3.3.2) $T_1 = T_1 \times T_1 \cong T_2 T_2 = T_2$ and $U_1 = U_1 U_2 \cong U_2 U_1 = U_2$.

For any connected complex space $U$ denote by $t(U)$ the maximal $m \in \mathbb{N}$ such that there exists an $m$-dimensional torus that is an isogeny factor of $U$. Thus $U_1$ is isogenous to $T(U_1) \times U_1$, where $T(U)$ is a $t(U)$-dimensional torus and $U_1$ is a connected complex space with $t(U_1) = 0$.

7.2.1-a Corollary. Let $U_1 \sim T \times U'$ with some torus $T$. If $\dim T = t(U)$ or if $t(U') = 0$, then $T(U) \sim T$ and $U_1 \sim U'$.

(ii) $T(U) \times T(V) \sim T(U \times V)$ and $U_1 \times U_2 \sim (U \times V)_+$ for all connected complex spaces $U$ and $V$.

Proof: The assertion (i) is obvious by 7.2.1. To prove (ii), consider any isogeny between $U_1 \times V_1$ and $T \times Y$, where $T$ is a torus and $Y$ a suitable connected complex space. By 7.1.5, this isogeny degenerates, and using 7.1.6 and 5.1.5, we conclude that $U_1$ and $T$ or $V_1$ and $T$ possess a common isogeny factor. Thus $\dim T = 0$ and we can apply 7.2.1 to $T(U \times V) \times (U \times V)_+$, $(T(U) \times T(V)) \times (U_1 \times V_1)_+$.

7.2.2 Lemma. Let $T_1$, $T_2$, $T_3$ be tori with $T \times T_1 \sim T \times T_2$. Then $T_1 \sim T_2$.

Proof: We proceed by induction on $\dim T \times T_1$. In the induction step, we may assume that $\dim T_1 > 0$, and that $T_1$ and $T_2$ have no common torus isogeny factor. Then, by 7.1.5, any isogeny between $T \times T_1$ and $T \times T_2$ degenerates, whence, by 7.1.6, we may assume that there...
exists a degenerating isomorphism \( T \times T_1 \rightarrow T' \times T_2 \) with some torus \( T' \sim T \). From 5.1.5 we infer \( T \cong T_2 \times T_2 \) and \( T' \cong T_2 \times T_1 \), since \( T_1 \) and \( T_2 \) have no positive-dimensional common factor. Thus \( T_1 \sim T_2 \) by induction hypothesis.

7.23 Theorem. Let \( X, Y, Z \) be connected complex spaces, such that \( X, Y \) or \( Z \) is compact. If \( X \times Y \) and \( X \times Z \) are isogenous, then so are \( Y \) and \( Z \).

Proof. By 7.2.1.a, we have \( T(X) \times T(Y) \sim T(X \times Y) \sim T(X \times Z) \sim T(X) \times T(Z) \) and \( X_+ \times Y_+ \sim (X \times Y)_+ \sim (X \times Z)_+ \sim X_+ \times Z_+ \). Thus \( T(Y) \sim T(Z) \) by 7.2.2. By 7.1.5, every isogeny between \( X_+ \times Y_+ \) and \( X_+ \times Z_+ \) degenerates (note that \( X_+ \), \( Y_+ \) or \( Z_+ \) is compact). Using 7.1.6, we may assume \( X_+ \times Y_+ \cong X_+ \times Z_+ \), whence \( Y_+ \cong Z_+ \) by 5.2.1. Thus \( Y \sim T(Y) \times Y_+ \sim T(Z) \times Z_+ \sim Z \).

7.2.3.a Corollary. If \( X \times Y \cong X \times Z \) with \( X, Y \) or \( Z \) compact, then \( Y \) and \( Z \) are isogenous.

73. Decomposition

73.1 Theorem. Every connected complex space \( U \) admits a unique isogeny decomposition (up to reordering) \( U \sim X \times \ldots \times X_n \times T(U) \times U'(n \geq 0) \), such that

(i) \( T(U) \) is a (possibly zero-dimensional) torus and \( U' \) has no compact isogeny factor \( \neq C^0 \),

(ii) every \( X_+ \mid 1 \leq \nu \leq n \) is compact, strongly indecomposable, \( \neq C^0 \), and not isogenous to any torus.

Proof. Evident by 7.2.1.a, 7.1.6, and 5.3.4.
REFERENCES


On product decompositions of complex spaces

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