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ON PRODUCT DECOMPOSITIONS OF COMPLEX SPACES

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INTRODUCTION

Every connected complex space U admits a maximal decomposition $U \cong U_1 \times \ldots \times U_n$ with indecomposable $U_\nu \neq \mathbf{C}^0$, and it is natural to ask whether (or under which conditions) these factors are unique. When considering this question, one is of course tempted to copy the number-theoretic procedure, i.e., given two decompositions, to try at first to find a common factor and then to drop it. However, just this simplification tums out to be the real problem.

For general complex spaces, little can be said as to when $X \times Y \cong X \times Z$ implies $Y \cong Z$. Even in the compact case, no counterexamples were known unti1 1977. Then T. Shioda [12] and, some four years later, G. Parigi [10] presented various examples of compact complex manifolds $Y \ncong Z$ such that $X \times Y \cong X \times Z$ for some torus X. Shioda's manifolds Y and Z are tori as well, and Parigi's examples are tota1 spaces of fibre bundles with finite structure group and torus basis; we shall denote this class of complex spaces by $\mathscr{T}$.

Roughly during the same period, diverse criteria for cancellability in the category of reduced connected compact complex spaces have been proven ([1], [4], [13]). It tumed out that in this situation, Shioda and Parigi had already more or less exhausted the scope of counterexamples:

As was shown in [5], $X \times Y \cong X \times Z$ entails $Y \cong Z$, if $\{X, Y, Z\} \not\subset \mathscr{T}$. Conversely, for every $X \in \mathscr{T}$, there exist non-isomorphic Y, Z such that $X \times Y \cong X \times Z$ (see [11]). The proof of the above cancellation result simultaneously yielded the uniqueness of the maximal decomposition for compact varieties that are not contained in $\mathscr{T}$.

We shall now generalize the situation of [5] in several respects. Firstly, non-reduced complex spaces will be admitted, and the compactness condition will be weakened; for instance, in the cancellation problem, we only require one of the factors X, Y, Z to be compact. As one of the main results, we obtain that then the cancellation theorem of [5] carries over word for word (Theorem 5.2.1). Again, the proof brings about a (partial) answer to the decomposition problem: In any maximal decomposition of a connected complex space, the compact factors $\notin \mathscr{T}$ and the product of the other ones are unique (Theorem *5.3.2* and Theorem 5.3.4).

In the last chapter, we are concemed with a different type of generalization, which is inspired by the fact that in all counterexamples to the cancellation problem, the varieties Y and Z are still isogeneous, i.e. they can both be covered finitely by some common S. Thus one is led to suspect that Y and Z are isogeneous, if so are $X \times Y$ and $X \times Z$. This is indeed the case, if at least one of the factors X, Y or Z is compact (Theorem 7.2.3). As a by-product of the proof, we obtain again a congenial decomposition result (Theorem 7.3.1).

It is easily seen that both the cancellation and the decomposition problem boil down to the following question: If $X \times Y \cong U \times V$ is an isomorphism between connected complex spaces, what is the relation between the individua1 factors?

To cover also the non-reduced case, it is necessary to consider at first the corresponding local problem, where X, Y, U and V are replaced by germs of complex spaces with $\dim X = 0$. It is shown that $X \times Y$ and $U \times V$ admit a simultaneous subdecomposition, i.e. that there exist isomorphisms $X \cong X_U \times X_V, Y \cong Y_U \times Y_V, U \cong X_U \times Y_U, V \cong X_V \times Y_V$ (compare Theorem 1.4.1). The same assertion holds, if X, Y, U, V are again complex spaces with $\dim X = 0$ (see Chapter 4). This latter result starts the induction on dim X in the proof of Theorem 5.1.5, which states that $X \times Y$ and $U \times V$ with X compact admit a simultaneous subdecomposition, if e.g. $\{X, Y, U, V\} \not\subset \mathcal{T}$. The induction step is brought about by a construction presented in Chapter 3, which assigns an isomorphism $\overline{X} \times \overline{Y} \cong \overline{U} \times \overline{V}$ to the given one, such that $X \times Y$ and $U \times V$ admit a simultaneous subdecomposition, if so do $\overline{X} \times \overline{Y}$ and $\overline{U} \times \overline{V}$; it turns out that $\dim \overline{X} < \dim X$ if $\{X, Y, U, V\} \not\subset \mathcal{T}$. The background niaterial for this latter conclusion as well as for the construction of the isomorphism $\overline{X} \times \overline{Y} \cong \overline{U} \times \overline{V}$ is compiled in Chapter 2, the contents of which can be summed up as follows:

a) For every connected space S, there exists a largest compact connected complex Lie group $A(S)$ acting holomorphically and effectively on S.

b) If there exists a holomorphic $S \to A(S)$ that maps the orbit of some positive-dimensional closed complex subgroup T of $A(S)$ onto T, then $S \in \mathcal{T}$.

Even for a reduced compact $X \notin \mathcal{T}$, the unique indecomposable factors given by Theorem 5.3.4 are in general not unique as subspaces of X , i.e. an automorphism of X need not be a product of isomorphisms between the indecomposable factors. The relation between Aut (X) and the automorphism groups of the factors is investigated in Chapter 6. It turns out that the situation simplifies considerably, if X is a projective variety.

Finally, when dealing with the isogeny situation, we start again with connected complex spaces $X \times Y$, $U \times V$ which are now assumed to be isogeneous. Pursuing a similar line of reasoning as in Chapter 5, we show:

a) There exists a torus of maximal dimension which is a common isogeny factor of X, Y, U and V.

b) If this torus is zerodimensional, then there exist isogenies $X' \sim X$, $Y' \sim Y$, $U' \sim U$, $V' \sim V$ such that $X' \times Y' \cong U' \times V'$.

To this latter isomorphism, we can then apply the results of Chapter 5.

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0. PRELIMINARIES

0.1. Categories with (co-)products

Let $\mathscr{A}$ be a category with a (co-)product $\odot$. For $A, B \in \mathscr{A}$ the canonica1 morphisms $A \odot B \to A, A \odot B \to B$ $(A \to A \odot B, B \to A \odot B)$ will be denoted by p_1, p_2 (j_1, j_2) or, if unambiguous, by p_A, p_B (j_A, j_B), or simply by p (j). By $J_{A,B}$ *or*, if *the* meaning is clear from the context, by J we denote the natural isomorphism $A \odot B \to B \odot A$, and we let $J_A := J_{,,}$. Moreover, for every permutation σ of $\{1, . . . , n\}$, we let $J_\sigma = J_{A,\sigma} : \bigodot_n A \to \bigodot_n A$ be given by $p_{\sigma_1}, \dots, p_{\sigma_n}$ $(j_{\sigma_1}, \dots, j_{\sigma_n})$.

If $Z \in \mathscr{A}$ is a final (initial) element, then $p_A : A \odot Z \to A$ $(j_A : A \to A \odot Z)$ is an isomorphism for all $A \in \mathscr{A}$. If Z is a zero object, we denote by abuse of notation the morphism $A \overset{p_A^{-1}}{\to} A \odot Z \overset{\mathrm{id}_A \odot \mathrm{kan}}{\longrightarrow} A \odot B, B \to A \odot B$ $(A \odot B \to A \odot Z \to A, A \odot B \to B)$ by j', j_2 (p', p_2) or by j_A, j_B (p_A, p_B). There will be no confusion with the previous j, p, since we shall always consider only one category at a time with exactly one fixed product (coproduct) that is nota coproduct (product).

$A' \in \mathscr{A}$ is a *factor* of $A \in \mathscr{A}$, if $A \cong A' \odot A''$ for some $A'' \in \mathscr{A}$. If $\mathscr{A}$ hasafinal (initial) object Z, we shall say that $A \in \mathscr{A}$ is *indecomposable*, if every factor $\neq Z$ of A is isomorphic to A. **A** *decumposition* of $A \in \mathscr{A}$ is an isomorphism $A \to A_1 \odot . . . \odot A_n$ *in* $\mathscr{A}$.

A final (initial) object $Z \in \mathscr{A}$ is a *semi-zero object*, if Mor(Z, A) $\neq \emptyset$ (Mor(A, Z) $\neq 0$) for all $A \in \mathscr{A}$. If $\mathscr{A}$ has a semi-zero object Z, then a morphism $A \overset{\phi}{\to} B$ is called *constant*, if it admits a factorization $\phi = (A \to Z \to B)$.

0.2. Complex spaces and holomorphic mappings

0.2.1. Let $U = (|U|, \mathscr{O}_U)$, $V = (|V|, 8,)$ be complex spaces and let $f = (|f|,_* f) : U \to V$ be holomorphic. If U is reduced, we do not distinguish between U and $|U|$ and between f and $|f|$. **We** let $d_0(U) := \min_{u \in U}$ dim, U.

A (closed or open) complex subspace U' of U will be indicated by the symbol $U' \hookrightarrow U$ (which also denotes the inclusion map); if U' is reduced, connected and compact, we sometimes write $U' \underset{(rcc)}{\hookrightarrow} U$. If U is a complex Lie group and $U' \hookrightarrow U$ is a subgroup, we employ the symbol $U' \sqsubseteq U$.

For $V' \hookrightarrow V$ we denote by $f^{-1}(V')$ the largest subspace S of U such that there exists a holomorphic factorization $f|_S = (S \to V' \hookrightarrow V)$. If $U' \hookrightarrow U$ such that $f|_{U'}$ is proper, then there exists a smallest complex subspace S of V such that $f|U'$ admits a factorization through $S \hookrightarrow V$, and it will be denoted by $f(U')$.

f is a *quotient map*, if it satisfies the following condition: For every open $V' \subset V$ and every holomorphic g : $f^{-1}(V') \to W$ that factors set-theoretically through $f|_{f^{-1}(V')}$, there exists a unique holomorphic factorization of g through $f|_{f^{-1}(V')}$.

If f is proper with Stein factorization $(U \xrightarrow{\tau_f} S_f \xrightarrow{\bar{f}} V)$, then τ_f is a quotient map.

f is a *covering,* if U is connected and if f is finite and locally biholomorphic. Coverings are quotient maps.

Let $\phi : |U| \to S$ be a map of sets. We shall say that *the* ***analytic quotient*** $\phi : U \to S$ ***exists,*** if S can be endowed with a complex structure such that $\phi = |g|$ for some holomorphic quotient map $g : U \to S$.

Suppose that f is finite and factors through ϕ. If ϕ defines an analytic equivalence relation on U (i.e., if $\{(u, u') \in U \times U : \phi(u) = \phi(u')\}$ is analytic), then the analytic quotient $\phi : U \to S$ exists (see [8], Proposition 49. A 13).

0.2.2. The cartesian product of complex spaces is a product in the category of all complex spaces, and $\mathbf{C}^0$ is a semi-zero object. For $v \in V$ and every complex space W, we denote the constant holomorphic map $W \to \mathbf{C}^0 \cong \{v\} \hookrightarrow V$ by $[v]$. For $(u, v) \in U \times V$ we let $j_v := (\mathrm{id}_U, [v]) : U \to U \times V, j_u := ([u], \mathrm{id}_V) : V \to U \times V$, if the meaning is clear from the context. If $g : U \times V \to W$ is holomorphic, then the partial maps $g \circ j_u$, $g \circ j_v$ will be denoted by $g(u, .)$, $g(., v)$, respectively. Let $g : U \times V \to A \times B$ be holomorphic, $(u, v) \in U \times V$. Then we let $lg := p_A \circ g$ and $rg := p_B \circ g$; moreover, when no ambiguity arises, we let $\overleftarrow{v} := lg \circ j_v$, $\overrightarrow{v} := rg \circ j_v$, $\overleftarrow{u} := lg \circ j_u$ and $\overrightarrow{u} := rg \circ j_u$.

Lemma. *Let $g : U \times W \to V$ be a holomorphic map between connected complex spaces, and let $A \subset U \times W$ with $|p_U|(A) = |U|$ and $|p_W|(A) = |W|$.*

If f is constant on some open neighbourhood of A, then all partial maps $f(., w)$, $f(u, .)$ are constant.

Proof. For symmetry reasons, it suffices to consider the partial maps $f(., w)$; therefore we may assume that W is reduced and irreducible. Given $(u, w) \in U \times W$, we have to show that $f(., w)$ is constant on every infinitesimal neighbourhood of u; thus we may assume $U_{red} = \{u\}$. By assumption, there exists a non-empty open subset W' of W such that $f|_{U \times W'}$ is constant. Hence f is constant, since W is reduced and irreducible. $\square$

0.3. Products of groups

Although the groups considered in what follows need not be abelian, we denote the group composition by a '+' -sign. Then every group homomorphism $G \times H \to G' \times H'$ is given by a matrix $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ with $\alpha \in \mathrm{Hom}(G, G')$ etc. Note that $\alpha + \beta = \beta + \alpha$ (when no ambiguity can arise, we do not distinguish between α and $\alpha \circ p_G$, G and $j_G(G)$ etc.). The composition of two such homomorphisms is given by the product $\begin{pmatrix}\alpha' & \beta' \\ \gamma' & \delta'\end{pmatrix} \cdot \begin{pmatrix}\alpha & \beta \\ \gamma & \delta\end{pmatrix}$ (where $\alpha'\alpha + \beta'\gamma = \beta'\gamma + \alpha'\alpha$ etc.).

0.3.1 Lemma. ***Let $\phi : G \times H \to G' \times H'$ be an isomorphism of groups given by $\begin{pmatrix}\alpha & \beta \\ \gamma & \delta\end{pmatrix}$, and let ϕ^{-1} be given by $\begin{pmatrix}\alpha' & \beta' \\ \gamma' & \delta'\end{pmatrix}$.***

If α or $\alpha'\alpha$ is injective, then so is δ' or $\delta\delta'$, respectively; the same assertion holds, if «injective» is replaced by «surjective».

Proof. Let α be injective, and let $h' \in \mathrm{Ker}\,\delta\delta'$. Then $h' = \gamma\beta'(h') + \delta\delta'(h') = \gamma\beta'(h')$. If $\delta'(h') = 0$, then $0 = \alpha\beta'(h') + \beta\delta'(h') = \alpha\beta'(h')$, whence $\beta'(h') = 0$ and therefore $h' = \gamma\beta'(h') = 0$. If $\alpha'\alpha$ is injective, then the equation $0 = \alpha'\alpha\beta'(h') + \alpha'\beta\delta'(h') = \alpha'\alpha\beta'(h') - \beta'\delta\delta'(h') = \alpha'\alpha\beta'(h')$ again yields $\beta'(h') = 0$, whence $h' = 0$.

Let α' be surjective and let $h' \in H'$. Then $\beta'(h') = \alpha'(g')$ for some $g' \in G'$, and therefore $h' = \delta\delta'(h') + \gamma\beta'(h') = \delta\delta'(h') + \gamma\alpha'(g') = \delta\delta'(h') - \delta\gamma'(g') = \delta(\delta'(h') - \gamma'(g')) \in \mathrm{Im}\,\delta$. If $\alpha'\alpha$ is surjective, then $\beta'(h') = \alpha'\alpha(g)$ for some $g \in G$, and hence $h' = \delta\delta'(h') + \gamma\beta'(h') = \delta\delta'(h') + \gamma\alpha'\alpha(g) = \delta\delta'(h') - \delta\gamma'\alpha(g) = \delta\delta'(h') + \delta\delta'\gamma(g) \in \mathrm{Im}\,\delta\delta'$. ◊

0.3.2 Lemma. ***Let ϕ be as in 0.3.1.***

For all $m, n \in \mathbf{N}$, the map $G \times G \to G$, given by $(g_1, g_2) \mapsto (\alpha'\alpha)^m(g_1) + (\beta'\gamma)^n(g_2)$, is a surjective homomorphism of groups.

If $(\beta'\gamma)^n = 0$ for some n, then $(\alpha'\alpha)^m$ is an isomorphism for all m.

Proof. From $\alpha'\alpha + \beta'\gamma = \mathrm{id}_G$ we infer $\alpha'\alpha\beta'\gamma = \beta'\gamma\alpha'\alpha$ and hence $\mathrm{id}_G = (\alpha'\alpha + \beta'\gamma)^{mn} = ((\alpha'\alpha)^m + (\beta'\gamma) \circ \chi)^n = (\beta'\gamma)^n \circ \chi^n + (\alpha'\alpha)^m \circ \psi$ with suitable homomorphisms χ, ψ that commute with $\beta'\gamma$ and $\alpha'\alpha$. This proves everything. ◊

0.3.2.a Corollary. ***Let $g : V \times W \to V \times W$ be an endomorphism of K-vector spaces, and assume that $g|_{\mathrm{Im}\, g} = \lambda \cdot \mathrm{id}_{\mathrm{Im}\, g}$ for some $0 \neq \lambda \in K$.***

If $(W \xrightarrow{g \circ j_W} V \times W \xrightarrow{p_W} W)^n = 0$ for some n, then the composition $g \circ j_V \circ p_V | \mathrm{Im}\, g \to \mathrm{Im}\, g$ is an isomorphism.

Proof. Let $V' := \mathrm{Im}\, g$, $W' := \mathrm{Ker}\, g$ and define $\phi : V' \times W' \to V \times W$ by $\phi(v', w') := v' + w'$. Then ϕ is an isomorphism the inverse of which is given by $(v, w) \mapsto \left(\frac{1}{\lambda} \cdot g(v, w), (v, w) - \frac{1}{\lambda} \cdot g(v, w)\right)$. If ϕ is represented by the matrix $\begin{pmatrix}\alpha & \beta \\ \gamma & \delta\end{pmatrix}$ with inverse

$\begin{pmatrix}\alpha' & \beta' \\ \gamma & \delta'\end{pmatrix}$, then $\gamma\beta'(w) = \gamma\left(\frac{1}{\lambda}\cdot g(0,w)\right) = \frac{1}{\lambda}\cdot p_W(g(0,w))$; thus $(\gamma\beta')^n = 0$ for some n. By 0.3.2, $\alpha'\alpha$ is an isomorphism, since $(\beta'\gamma)^{n+1} = \beta'(\gamma\beta')^n\gamma = 0$. This proves the assertion, since $\alpha'\alpha(v') = \alpha'(p_V(v')) = p_V\left(\frac{1}{\lambda}\cdot g(p_V(v'),0)\right) = \frac{1}{\lambda}\cdot g(p_V(v'),0)$.

◊

0.33 Lemma. *Let $G_1 \times H_1 \supset G_2 \times H_2 \supset \ldots \supset G_n \times H_n \supset \ldots$ be a sequence of subgroups such that*

$$(*) \qquad R_{n+2} = R_n \cap (G_{n+1} \times H_{n+1}) \; for \; R \in \{G,H\}, n \in \mathbf{N}$$

Denore by P_n the homomorphism

$$G_{n+4} \xrightarrow{p_{G_{n+3}}} G_{n+3} \xrightarrow{p_{H_{n+2}}} H_{n+2} \xrightarrow{p_{H_{n+1}}} H_{n+1} \xrightarrow{p_{G_n}} G_n$$

If $P_n \circ P_{n+4} \circ \ldots \circ P_{n+4k} = 0$ forsome $n, k \in N$, then $G_m \times H_m = G_{m+1} \times H_{m+1}$ for all $m \gg 0$.

Proof. By assumption, the diagram

$$\begin{array}{ccccccccc}
G_{n+4} & \xrightarrow{p_{G_{n+3}}} & G_{n+3} & \xrightarrow{p_{H_{n+2}}} & H_{n+2} & \longrightarrow & H_{n+1} & \longrightarrow & G_n \\
\cap & & \cap & & \cap & & \cap & & \cap \\
G_{n+2} & \xrightarrow{p_{G_{n+1}}} & G_{n+1} & \xrightarrow{p_{H_n}} & H_n & \longrightarrow & H_{n-1} & \longrightarrow & G_{n-2}
\end{array}$$

is commutative for all $n \geq 2$. Thus, if $P_n \circ \ldots \circ P_{n+4k} = 0$ for some n, k, then $P_m \circ \ldots \circ P_{m+4k} = 0$ for all $m \geq n$ with $m - n$ even.

Furthermore, $G_{l+3} \cap G_{l+2} = G_{l+1} \cap (G_{l+2} \times H_{l+2}) \cap G_{l+2} = G_{l+1} \cap G_{l+2}$, $H_{l+3} \cap H_{l+2} = H_{l+1} \cap H_{l+2}$, $G_{2l+1} \cap H_{2l} = G_{2l-1} \cap (G_{2l} \times H_{2l}) \cap H_{2l} = G_{2l-1} \cap H_{2l}$ and $G_{2l+2} \cap H_{2l+1} = G_{2l} \cap H_{2l+1}$ for all $l \geq 1$. Therefore, we obtain a commutative diagram

$$\begin{array}{ccccccc}
G_1 \times H_1 & \supset & G_2 \times H_2 & \supset & G_3 \times H_3 & \supset & \ldots \\
\downarrow \text{kan} \times \text{kan} & & \downarrow \text{id} \times \text{kan} & & \downarrow \text{kan} \times \text{kan} & & \\
G'_1 \times H'_1 & \supset & G_2 \times H_2^+ & \supset & G'_3 \times H'_3 & \supset & \ldots
\end{array}$$

where $G'_{2l+1} = G_{2l+1}/(G_1 \cap H_2)$, $H'_{2l+1} = H_{2l+1}/(H_1 \cap H_2)$ and $H^+_{2l} = H_{2l}/(H_1 \cap H_2 + G_1 \cap H_2)$ (note that $G_1 \cap G_2$ and $G_1 \cap H_2$ commute).

Clearly, the lower line of this diagram again satisties the condition (*).

Let now $n, k \in \mathbf{N}$ with $P_n \circ \ldots \circ P_{n+4k} = 0$; obviously, we may assume that n is even.

For $l \in \mathbb{N}$ with $2l \geq n$, consider the commutative diagram

$$\begin{array}{ccccccccc}
G_{2l+4} & \xrightarrow{p} & G_{2l+3} & \to & H_{2l+2} & \to & H_{2l+1} & \to & G_{2l} \\
\| & & \downarrow \mathrm{kan} & & \downarrow \mathrm{kan} & & \downarrow \mathrm{kan} & & \| \\
G_{2l+4} & \xrightarrow{p} & G'_{2l+3} & \to & H^+_{2l+2} & \to & H'_{2l+1} & \to & G_{2l}
\end{array}$$

The arrow $H'_{2l+1} \to G_{2l}$ is injective, since $H_1 \cap H_2 = H_{2l+1} \cap H_{2l} = \mathrm{Ker}(H_{2l+1} \xrightarrow{p_{G_{2l}}} G_{2l})$. Furthermore, the arrow $H^+_{2l+2} \to H'_{2l+1}$ is injective, since $p_{H_{2l+1}}(H_1 \cap H_2 + G_1 \cap H_2) = p_{H_{2l+1}}(H_{2l+1} \cap H_{2l+2} + G_{2l+1} \cap H_{2l+2}) = H_{2l+1} \cap H_{2l+2} = \mathrm{Ker}(H_{2l+1} \xrightarrow{p_{G_{2l}}} G_{2l})$.

Now apply the same construction to the sequence $G'_1 \times H'_1 \supset G_2 \times H^+_2 \supset G'_3 \times H'_3 \supset \dots$ with G and H interchanged; this yields a commutative diagram

$$\begin{array}{ccccc}
G'_1 \times H'_1 & \supset & G_2 \times H^+_2 & \supset & G'_3 \times H'_3 & \supset \dots \\
\downarrow \mathrm{kan}\times\mathrm{km} & & \downarrow \mathrm{kan}\times\mathrm{id} & & \downarrow \mathrm{kan}\times\mathrm{km} & \\
G''_1 \times H''_1 & \supset & G^+_2 \times H^+_2 & \supset & G''_3 \times H''_3 & \supset \dots
\end{array}$$

with the lower line again satisfying (*).

Hence, for n and l as above, we obtain

$$\begin{array}{ccccccccc}
G_{2l+4} & \xrightarrow{p} & G_{2l+3} & \to & H_{2l+2} & \to & H_{2l+1} & \to & G_{2l} \\
\| & & \downarrow \mathrm{kan} & & \downarrow \mathrm{kan} & & \downarrow \mathrm{kan} & & \| \\
G_{2l+4} & \xrightarrow{p} & G'_{2l+3} & \to & H^+_{2l+2} & \to & H'_{2l+1} & \to & G_{2l} \\
\downarrow & & \downarrow & & \| & & \downarrow & & \downarrow \\
G^+_{2l+4} & \to & G''_{2l+3} & \to & H^+_{2l+2} & \to & H''_{2l+1} & \to & G^+_{2l}
\end{array}$$

where the compositions $G^+_{2l+4} \to G''_{2l+3} \to H^+_{2l+2}$ and $H^+_{2l+2} \to H'_{2l+1} \to G_{2l}$ are injective.

We conclude that $P^+_{2l} \circ \dots \circ P^+_{2l+4(k-1)} = 0$, if $P^+_{2l} : G^+_{2l+4} \to G^+_{2l}$ denotes the homomorphism given by the bottom line of the above diagram. Moreover, it is obvious from the construction, that $G_{2l+2} \times H_{2l+2} = G_{2l+1} \times H_{2l+1} = G_{2l} \times H_{2l}$ iff $G^+_{2l+2} \times H^+_{2l+2} = G''_{2l+1} \times H''_{2l+1} = G^+_{2l} \times H^+_{2l}$.

Thus, if we proceed by induction on the minimal k with $P_n \circ \dots \circ P_{n+4k} = 0$, it remains to consider the case $k = 0$, i.e. the case $G_n = 0$. Then $G_{2l} = 0$, $G_{2l+1} \times H_{2l+1} \subset H_{2l}$, $H_{2l+2} = H_{2l} \cap (G_{2l+1} \times H_{2l+1}) = G_{2l+1} \times H_{2l+1}$ and $R_{2l+3} = R_{2l+1} \cap H_{2l+2} = R_{2l+1} \cap (G_{2l+1} \times H_{2l+1}) = R_{2l+1}$ for $R \in \{G, H\}$ and all l with $2l \geq n$. ◊

1. LOCAL ALGEBRAS WITH ARTINIAN FACTORS

Let K be a field of characteristic zero with a complete valuation and denote by $\mathscr{L}_K$ the category of local analytic K-algebras. The analytic tensor product is a coproduct in $\mathscr{L}_K$ and K is a zero-object in $\mathscr{L}_K$.

For $A \in \mathscr{L}_K$ with maximal ideal $\mathbf{m}_A$ let $\mathbf{n}_A \subset A$ be the nilradical of A. The canonical projection $A \twoheadrightarrow A/\mathbf{n}_A =: A_{red}$ is denoted by red, or simply by red. The reduction of a homomotphism $f : A \twoheadrightarrow B$ in $\mathscr{L}_K$ is indicated by $f_{red} : A_{red} \twoheadrightarrow B_{red}$, its Jacobian $m_A/m_A^2 \twoheadrightarrow m_B/m_B^2$ by $Tf : T_A \twoheadrightarrow T_B$.

For any local subalgebra $A' \subset A$ we let $A/\!\!/A' := A/A \cdot m_{A'}$.

1.1. A surjectivity criterion

Let $(f : A \to B) \in \mathscr{L}_K$. It is well known that f is surjective, if and only if so is its Jacobian Tf.

1.1.1 Lemma. *Let $(g : A \otimes B \twoheadrightarrow A' \otimes B') \in \mathscr{L}_K$ such that $p_{A'}gj_A$ and $p_{B'}gj_B$ are surjecrive.*

If $p_{B'}gj_A$ or $p_{A'}gj_B$ is constant, then g is surjective.

Proof. Tg is given by a mauix that has the form $\begin{pmatrix} G_{11} & G_{12} \\ 0 & G_{22} \end{pmatrix}$ or $\begin{pmatrix} G_{11} & 0 \\ G_{21} & G_{22} \end{pmatrix}$ with surjective $G_{11} : T_A \twoheadrightarrow T_{A'}, G_{22} : T_B \to T_{B'}$. ◊

The following lemma provides the essential argument in the proof of the local cancellation theorem. Its assertion does no longer hold, if $\mathrm{char}(K) > 0$.

1.1.2 Lemma. *Let* $(f : A \twoheadrightarrow B \otimes C) \in \mathscr{L}_K$.

If $p_B f|_{\mathbf{n}_A}$ is injecfive with $p_B f(\mathbf{n}_A \setminus \mathbf{m}_A^2) \subset \mathbf{m}_B \setminus \mathbf{m}_B^2$, fhen $\mathbf{n}_A \setminus \mathbf{m}_A^2 \subset \mathrm{Ker}\, p_C f$.

Proof. Let $a \in (\mathbf{m}_A \setminus \mathbf{m}_A^2) \cap \mathbf{n}_A$, and let $m \in N$ be minimal with $a^{m+1} = 0$. If $p_C f(a) \neq 0$, there exists $l \in N$ with $p_C f(a) \in \mathbf{m}_C^l \setminus \mathbf{m}_C^{l+1}$.

Let $\overline{B} = N/\mathbf{m}_B^{m+1}, \overline{C} = C/\mathbf{m}_C^{l+1}$, and let $\overline{f} = \mathrm{kan} \circ f : A \to \overline{B} \otimes \overline{C}$. Then $\overline{f}(a) = b\otimes 1 + z + 1 \otimes c$ with $z \in \mathbf{m}_{\overline{B}} \otimes \mathbf{m}_{\overline{C}}, b^m \neq 0 = b^{m+1-\mu} \cdot z^{\mu}$ for $0 \leq \mu \leq m+1, c \neq 0 = c^2 = cz$, and hence $0 = \overline{f}(a^{m+1}) = ((b \otimes 1 + z) + 1 \otimes c)^{m+1} = (m+1) \cdot (b \otimes 1 + z)^m \cdot (1 \otimes c) = (m+1) \cdot b^m \otimes c$, a contradiction. ◊

1.1.2.a Corollary. *(compare [9]). Let f be as above with A artinian.*

If $p_B f$ and $T_{p_B} f$ are injective, then $p_C f$ is constant.

1.1.2-b Corollary. *(compare [9]). Let f be as in 1.1.2 with A_{red} regular.*

If $p_B f|_{\mathfrak{n}_A}$ is injective with $p_B f(\mathfrak{n}_A \setminus \mathfrak{m}_A^2) \subset \mathfrak{m}_B \setminus \mathfrak{m}_B^2$, then $p_C f$ factors through red A.

Proof. It suffices to show that every minimal set of generators $\{n_1, \ldots, n_s\}$ of $\mathfrak{n}_A$ is contained in $\mathfrak{m}_A \setminus \mathfrak{m}_A^2$. Let $\mathfrak{n}'$ be generated by $\{n_1, \ldots, n_s\} \cap (\mathfrak{m}_A \setminus \mathfrak{m}_A^2)$ and let $A' := A/\mathfrak{n}'$. Then dim A' = dim A = dim A_{red} = dim $T_{A_{red}}$ = dim $T_{A'}$, whence A' is reduced, i.e. $\mathfrak{n}' = \mathfrak{n}_A$. ◊

1.1.2.c Corollary. *Let $(g : A \otimes B \to A' \otimes B') \in \mathscr{L}_K$ with A artinian.*

If $p_{A'} g j_A$ is an isomorphism, and if $p_{B'} g j_B$ is surjective, then g is surjective.

Proof. Evident by 1.1.2.a and 1.1.1. ◊

1.2. Isomorphisms between coproducts in $\mathscr{L}_K$.

Let $f : A \otimes B \to C \otimes D$ be an isomorphism in $\mathscr{L}_K$, and let $f_A := p_A f^{-1} j_C p_C f j_A$, $f'_A := p_A f^{-1} j_D p_D f j_A$, $f_D^{-1} := (f^{-1})_D = p_D f j_B p_B f^{-1} j_D$, $f_C^{-1} := (f^{-1})'_C := p_C f j_B p_B f^{-1} j_C$.

1.2.1 Lemma. *If $T p_C f j_A$ or $T f_A$ is injective, then so is $T p_B f^{-1} j_D$ or $T f_D^{-1}$, respectively. The same assertion holds, if «injective» is replaced by «surjective».*

Proof. Compare 0.3.1. ◊

1.2.1.a Corollary. *$p_C f j_A$ or f_A is an isomorphism, if and only if $p_B f^{-1} j_D$ or f_D^{-1} is, respectively.*

Proof. Let $p_C f j_A$ or f_A be an isomorphism. Note at first that it suffices to show that $p_B f^{-1} j_D$ resp. f_D^{-1} is surjective: If $p_C f j_A$ is bijective and $p_B f^{-1} j_D$ is surjective, we obtain a sequence of surjective homomorphisms

$$A \otimes B \xrightarrow{f} C \otimes D \xrightarrow{(p_C f j_A)^{-1} \otimes \mathrm{id}_D} A \otimes D \xrightarrow{\mathrm{id}_A \otimes p_B f^{-1} j_D} A \otimes B$$

and we conclude that $p_B f^{-1} j_D$ is also injective.

The assertion is now evident by 1.2.1. ◊

1.2.1.b Corollary. *Let A or C be artinian.*

(i) If $p_C f j_A$ is an isomorphism, then so are $p_A f^{-1} j_C$, $p_D f j_B$ and $p_B f^{-1} j_D$; moreover, $p_D f j_A$ and $p_B f^{-1} j_C$ are constant.

(ii) If f_A is surjective, then f_A and f_D^{-1} are isomorphisms, and $p_D f j_A$ is constant.

Proof. In (i) as well as in (ii), A is artinian, if C is. Thus we may assume that A is artinian.

(i) If $p_C f j_A$ is an isomotphism, then so is $p_B f^{-1} j_D$ by 1.2.1.a, and C is artinian. By 1.1.2, $p_D f j_A$ is constant, whence $p_D f j_B$ is surjective and therefore bijective, since $B \cong D$ via $p_B f^{-1} j_D$. Thus $p_A f^{-1} j_C$ is an isomorphism, whence $p_B f^{-1} j_C$ is constant.

(ii) If f_A is an isomorphism, then so is f_D^{-1} by 1.2.1.a. If Tf_A and f_A are injective, then so are $Tp_C f j_A$ and $p_C f j_A$, whence $p_D f j_A$ is constant by 1.1.2.a. Θ

1.2.2 Lemma. *$p_B f^{-1} j_C$ defines an isomorphism $C/\!/p_C f(A) \to B/\!/p_B f^{-1}(D)$, whose inverse is given by $p_C f j_B$.*

Proof. $p_B f^{-1} p_C f(\mathbf{m}_A) \subset p_B f^{-1}(f(\mathbf{m}_A) + C \otimes \mathbf{m}_D) = p_B f^{-1}(C \otimes \mathbf{m}_D) = B \cdot p_B f^{-1}(\mathbf{m}_D)$, whence $p_B f^{-1}(C \cdot p_C f(\mathbf{m}_A)) \subset B \cdot p_B f^{-1}(\mathbf{m}_D)$.

Furthermore, $p_C f p_B f^{-1}(c) \in p_C f(f^{-1}(c) + \mathbf{m}_A \otimes B) = c + C \cdot p_C f(\mathbf{m}_A)$ for all $c \in \mathbf{m}_C$, and the assertion follows for symmetry reasons. Θ

1.2.3 Lemma. *For all $m, n \in N$ the multiplication map* mult: Im $f_A^m \otimes$ Im $f_A'^n \to A$ *is surjective.*

Proof. By 0.3.2, the Jacobian of mult is surjective. Θ

1.3. The structure of local algebras with artinian factors

Let $f : A \otimes B \to C \otimes D$ be an isomorphism in $\mathscr{L}_K$ and assume that A is artinian. Then $A_C :=$ Im f_A^m, $A_D :=$ Im $f_A'^m$ *are* welldefined for $m \gg 0$, and mult: $A_C \otimes A_D \to A$ is surjective by 1.2.3. Clearly, $p_C f|_{A_C}$, $p_D f|_{A_D}$ and their Jacobians are injective; thus $p_D f|_{A_C}$, $p_C f|_{A_D}$ are constant by 1.1.2.a. Therefore, $A_C = f_A(A)$, $A_D = f_A'(A)$, and $p_C f(A) = p_C f(A_C) =: C_A$, $p_D f(A) = p_D f(A_D) =: D_A$ are isomorphic to A_C, A_D via $p_C f$, $p_D f$, respectively. Conversely, $p_A f^{-1}$ induces isomorphisms $C_A \to A_C$, $D_A \to A_D$, whence,again by 1.1.2.a, $p_B f^{-1}|C_A$ and $p_B f^{-1}|D_A$ areconstant. Let $B_C := p_B f^{-1}(C)$, $B_D := p_B f^{-1}(D)$, $C_B := p_C f(B_C)$, $D_B := p_D f(B_D)$; then, by 1.2.3, the multiplications $B_C \otimes B_D \to B$, $C_A \otimes C_B \to C$, $D_A \otimes D_B \to D$ are surjective.

1.3.1 Lemma. *mult: $A_C \otimes A_D \to A$ is an isomorphism.*

Proof. Denote by χ the composition

$$A_C \otimes A_D \xrightarrow{\text{mult}} A \xrightarrow{fj_A} C \otimes D \xrightarrow{p_A f^{-1} j_C \otimes p_A f^{-1} j_D} A \otimes A \xrightarrow{p_C f j_A \otimes p_D f j_A} C_A \otimes D_A.$$

Then $p_{C_A} \chi j_{A_C} = p_C f j_A p_A f^{-1} j_C p_C f j_A|_{A_C} = p_C f j_A f_A|_{A_C}$ and $p_{D_A} \chi j_{A_D} = p_D f j_A p_A f^{-1} j_D p_D f j_A|_{A_D} = p_D f j_A f_A'|_{A_D}$ are isomorphisms. By 1.1.2.c, χ is surjective and thus bijective, whence mult: $A_C \otimes A_D \to A$ is injective and hence an isomorphism. Θ

1.3.2 Lemma. $(p_B f^{-1} p_D f | B_D \to B_D) = \mathrm{id}_{B_D}$, $(p_D f p_B f^{-1} | D_B \to D_B) = \mathrm{id}_{D_B}$, *and* $p_B f^{-1} p_D f|_{B_C}$ *is constant, and the corresponding statements hold after interchanging C and D. In particular,* B_C *and* C_B, *as well as* B_D *and* D_B *are isomorphic via* $p_C f$, $p_D f$, *with the respective inverse given by* $p_B f^{-1}$.

Proof. Let $s \in \mathbf{m}_{C\otimes D}$. Then $p_B f^{-1} p_D f p_B f^{-1}(s) = p_B f^{-1} p_D f(f^{-1}(s) + (p_B f^{-1}(s) - f^{-1}(s))) = p_B f^{-1} p_D(s)$, since $p_D f(p_B f^{-1}(s) - f^{-1}(s)) \in p_D f(\mathbf{m}_A \otimes B) = D \cdot \mathbf{m}_{D_A}$ and $p_B f^{-1}|_{D_A}$ is constant. Thus $p_B f^{-1} p_D f p_B f^{-1}|_C = p_B f^{-1} p_D|_C$ is constant, and $p_B f^{-1} p_D f p_B f^{-1}|_D = p_B f^{-1}|_D$, and we conclude that $p_B f^{-1} p_D f|_{B_C}$ is constant and $p_B f^{-1} p_D f|_{B_D} = id_{B_D}$. Then also $p_D f^{-1} p_B f|_{D_B} = id_{D_B}$ since $p_D f|B_D \to D_B$ is surjective by definition. ◊

1.3.2.a Corollary. *The multiplications* $B_C \otimes B_D \to B$, $C_A \otimes C_B \to C$, $D_A \otimes D_B \to D$ *are isomorphisms.*

Proof. Let

$$\chi := (C_A \otimes C_B \xrightarrow{\text{mult}} C \xrightarrow{f^{-1} j_C} A \otimes B \xrightarrow{f_A \otimes p_B f^{-1} j_C p_C f j_B} A_C \otimes B_C) .$$

Then $p_{A_C}\chi j_{C_A} = f_A \circ p_A f^{-1} j_C|_{C_A}$ and $p_{B_C}\chi j_{C_B} = p_B f^{-1} p_C f p_B f^{-1}|_{C_B} = p_B f^{-1}|_{C_B}$ are isomorphisms, and hence so are χ and mult (see 1.1.2.c).

Symmetrically, mult: $D_A \otimes D_B \to D$ is an isomoxphism.

Finally, mult $\otimes$ mult: $(A_C \otimes A_D) \otimes (B_C \otimes B_D) \to A \otimes B$ is surjective, mult $\otimes$ mult: $(C_A \otimes C_B) \otimes (D_A \otimes D_B) \to C \otimes D$ is an isomorphism, and $A_C \otimes A_D \otimes B_C \otimes B_D \cong C_A \otimes C_B \otimes D_A \otimes D_B$. Thus mult: $B_C \otimes B_D \to B$ is an isomorphism as well. ◊

In total, we have shown:

1.3.3 Theorem. *Let* $f : A \otimes B \to C \otimes D$ *be an isomorphism in* $\mathscr{L}_K$ *with A artinian. Then rhere exists a commutative diagram of isomorphisms in* $\mathscr{L}_K$

$$\begin{array}{ccc} (A_C \otimes A_D) \otimes (B_C \otimes B_D) & \xrightarrow{\bar{f}} & (C_A \otimes C_B) \otimes (D_A \otimes D_B) \\ \downarrow \text{mult}\otimes\text{mult} & & \downarrow \text{mult}\otimes\text{mult} \\ A \otimes B & \xrightarrow{f} & C \otimes D \end{array}$$

where $R_A = p_R f(A) = p_R f(A_R)$, $B_R = p_B f^{-1}(R) = p_B f^{-1}(R_B)$, $R_B = p_R f(B_R)$, $A_R = p_A f^{-1}(R_A)$ *for* $R \in \{C, D\}$. *In particular,* $R_S \cong S_R$ *for* $R \in \{C, D\}$, $S \in \{A, B\}$.

1.3.3.a Corollary. *(Cancellation theorem, see [6]). Let* $R, R', S \in \mathscr{L}_K$ *such that* R, R' *or* S *is artinian.*

If $R \otimes S \cong R' \otimes S$, *then* $R \cong R'$.

Proof. We may assume that S is indecomposable. Then either $R_S \cong S_R = K = R'_S \cong S_{R'}$ or $R_S \cong S_R = S = S_{R'} \cong R'_S$. In the first case, $R \cong R_{R'} \cong R'_R \cong R'$, and in the second one $R \cong R_{R'} \otimes R_S \cong R'_R \otimes R'_S \cong R'$. ◊

1.3.3.b Corollary. *(Decomposition theorem, see [6]). Every $S \in \mathscr{L}_K$ admits a unique decomposition (up to reordering) $S \cong S_1 \otimes \ldots \otimes S_n \otimes S'$ with indecomposable artinian $S_1, \ldots, S_n \in \mathscr{L}_K \setminus (K)$ and with $S' \in \mathscr{L}_K$ having no artinian factor $\neq K$.*

Proof. Of course, we need only verify the uniqueness part. Using induction on n, it suffices to show: If $S \cong \tilde{S}_1 \otimes \ldots \otimes \tilde{S}_m \otimes S''$ is another decomposition of the same type, then there exists $1 \leq \nu \leq n$ with $S_\nu \cong \tilde{S}_1$ and $\bigotimes_{\nu \neq \nu} S_\nu \otimes S' \cong \tilde{S}_2 \otimes \ldots \otimes \tilde{S}_m \otimes S''$.

The case $n = 0$ being trivial, we may assume that the assertion is proven for some $n-1 \geq 0$. Let $B := \bigotimes_{\nu \geq 2} S_\nu \otimes S'$, $D := \bigotimes_{\mu \geq 2} \tilde{S}_\mu \otimes S''$ and let $f : S_1 \otimes B \to \tilde{S}_1 \otimes D$ be some isomorphism. If $S_1 = (S_1)_{\tilde{S}_1}$, then let $\nu := 1$. Otherwise, $S_2 \otimes \ldots \otimes S_n \otimes S' = B \cong \tilde{S}_1 \otimes B_D \cong \tilde{S}_1 \otimes D_B$ and $\tilde{S}_2 \otimes \ldots \otimes \tilde{S}_m \otimes S'' = D = S_1 \otimes D_B \cong S_1 \otimes B_D$. From the induction hypothesis, we infer that $\tilde{S}_1 \cong S_{\nu'}$, $\bigotimes_{\nu \neq \nu \geq 2} S_\nu \otimes S' \cong B_D$ for some $2 \leq \nu \leq n$, and hence $\bigotimes_{\nu \neq \nu} S_\nu \otimes S' = S_1 \otimes B_D \cong D = \bigotimes_{\mu \geq 2} \tilde{S}_\mu \otimes S''$. ◊

In view of the applications we have in mind, it is advisable to reformulate 1.3.3 in terms of quotient algebras.

1.3.4 Theorem. *Let $f : A \otimes B \to C \otimes D$ be an isomorphism in $\mathscr{L}_K$ with A artinian, and let R_S, S_R be as in 1.3.3, where $R \in \{A, B\}$, $S \in \{C, D\}$. Let $\{R, R'\} = \{A, B\}$, $\{S, S'\} = \{C, D\}$. Then*

(i) $\mathrm{kan} \circ f j_R : R \to C/\!/C_{R'} \otimes D/\!/D_{R'}$ and $\mathrm{kan} \circ f^{-1} j_S : S \to A/\!/A_{S'} \otimes B/\!/B_{S'}$ are isomorphisms.

(ii) The composition $R \xrightarrow{p_S f j_R} S \xrightarrow{\mathrm{kan}} S/\!/S_{R'}$ factors through $R \to R/\!/R_{S'}$ with an isomorphism $R/\!/R_{S'} \to S/\!/S_{R'}$, and $S \xrightarrow{p_B f^{-1} j_S} B \xrightarrow{\mathrm{kan}} B/\!/B_{S'}$ factors through $S \to S/\!/S_A$ with an isomorphism $S/\!/S_A \to B/\!/B_{S'}$. The homomorphism $f_A : A \to A$ resp. $f'_A : A \to A$ factors through $\mathrm{kan} : A \to A/\!/A_D$ resp. $\mathrm{kan} : A \to A/\!/A_C$ and the resulting composition $A/\!/A_D \to A \to A/\!/A_D$ resp. $A/\!/A_C \to A \to A/\!/A_C$ is an isomorphism.

Proof. $\mathrm{kan}|R_S \to R/\!/R_{S'}$, $\mathrm{kan}|S_R \to S/\!/S_{R'}$ are isomorphisms, since so are mult: $R_S \otimes R_{S'} \to R$, mult $: S_R \otimes S_{R'} \to S$.

(i) Let

$$\phi_R := (R_C \otimes R_D \xrightarrow{\mathrm{mult}} R \xrightarrow{f j_R} C \otimes D \xrightarrow{\mathrm{kan}} C/\!/C_{R'} \otimes D/\!/D_{R'})$$

and

$$\psi_S := (S_A \otimes S_B \xrightarrow{\mathrm{mult}} S \xrightarrow{f^{-1} j_S} A \otimes B \xrightarrow{\mathrm{kan}} A/\!/A_{S'} \otimes B/\!/B_{S'}).$$

Then $p_{S/\!/S_{R'}}\phi_R j_{R_S} : R_S \to S/\!/S_{R'}$ and $p_{R/\!/R'_S}\phi_S j_{S_R} : S_R \to R/\!/R_{S'}$ are isomorphisms, whence so are ϕ_A, ψ_C, ψ_D by 1.1.2.c.

From the commutative diagram

$$\begin{array}{ccccc} B_D & \hookrightarrow & B & \xrightarrow{p_C f} & C \\ \downarrow \text{kan} & & \downarrow \text{kan} & & \downarrow \text{kan} \\ K & \hookrightarrow & B/\!/B_D & \xrightarrow{\cong} & C/\!/C_A \end{array}$$

(compare 1.2.2 and the definition of C_A, B_D), we infer that $p_{C/\!/C_A}\phi_B\, j_{B_D}$ is constant; symmetrically, so is $p_{D/\!/D_A}\phi_B\, j_{B_C}$. In particular, the Jacobian of ϕ_B is surjective, and hence so is ϕ_B.

(ii) The case $R = B$ follows from 1.2.2 and the definition of B_C, B_D. The case $R = A$ follows from $S_A = p_S f(A)$ and $p_S f(A_{S'}) = K$ for $\{S, S'\} = \{C, D\}$. The morphism f_A is constant on A_D and hence factors through $A \to A/\!/A_D$; furthermore, $f_A(A) = A_C$ and $A_C \to A/\!/A_D$ is an isomorphism. The corresponding statement for f'_A follows symmetrically. ◊

1.4. Germs of complex spaces with zero-dimensional factors

1.4.1 Theorem. *Let $\phi : X \times Y \to U \times V$ be an isomorphism between germs of complex spaces, and assume that X is zero-dimensional. Let $\{S, S'\} = \{U, V\}$, $\{R, R'\} = \{X, Y\}$, and denote by*

$$\begin{array}{ccc} X_S & & X \to S' \to X \\ Y_S & \textit{the fibre of} & Y \to S' \\ S_X & & S \to Y \to S \\ S_Y & & S \to X \end{array}$$

(where each arrow denotes the corresponding partial map given by ϕ or ϕ^{-1}).

Then

(i) $p_S\phi|X_S \times Y_S \to S$ and $p_R\phi^{-1}|U_R \times V_R \to R$ are isomorphisms.

(ii) The partial map $S \to R$ defines an isomorphism $S_R \to R_S$, the partial map $Y \to S$ defines an isomorphism $Y_S \to S_Y$, and the composition of partial maps $X \to S \to X$ factors through the inclusion $X_S \hookrightarrow X$, inducing an isomorphism $X_S \to X_S$.

The proof is evident by 1.3.4.

1.4.1.a Corollary. *Let X, Y, Z be germs of complex spaces such that at least one of them is zero-dimensional.*

If $X \times Z \cong Y \times Z$, then $X \cong Y$.

1.4.2-b Corollary. *Every germ U of a complex space admits a unique decomposition (up to reordering) $U \cong U_1 \times \ldots \times U_n \times U'$ with zero-dimensional indecomposable $U_\nu \neq \mathbb{C}^0$ and with U' having no zero-dimensional factor $\neq \mathbb{C}^0$.*

2. FAMILIES OF HOLOMORPHIC MAPPINGS

When considering the complex analytic cancellation problem, one is faced immediately with various families of holomorphic mappings – eight at first sight. but actually a lot more. In this chapter, we prepare the way for dealing with them.

2.1. The simultaneous Stein factorization

Let $\phi : W \times U \to V$ be a holomorphic map between connected complex spaces. Then W and U can be interpreted as parameter spaces of holomorphic maps from U or W into V with evaluation map ϕ. In general, we consider U to be the common domain of the maps parametrized by W. Sometimes, however, it is advisable to interchange the roles of the two factors, and it will be done without further comment.

Mostly, we shall not distinguish between $w \in W$ and the partial map $\phi(w, .)$ (or between u and $\phi(., u)$).

2.1.1 Lemma and Definition. *Assume that $\Phi := (p_W, \phi) : W \times U \to W \times V$ is proper. Then the partial maps $w \in W$ admit a* simultaneous Stein factorization, *i.e. the Stein factorization $\Phi = (W \times U \xrightarrow{\tau_\Phi} S_\Phi \xrightarrow{\overline{\Phi}} W \times V)$ satisfies $\tau_\Phi = \mathrm{id}_W \times \tau_w$, $\overline{\Phi}(w, .) = (p_W, \overline{w})$ for all $w \in W$, where $w = (U \xrightarrow{\tau_w} S_w \xrightarrow{\overline{w}} V)$ is the Stein factorization of $w \in W$.*

Proof. By ([5], 4.3), the assertion is true for reduced U, W. Thus there exists for every $w \in W$ a commutative diagram

$$\begin{array}{ccc} W \times U & \xrightarrow{\tau_\Phi} & S_\Phi \\ \downarrow \mathrm{id} \times \tau_w & \nearrow h_w & \\ W \times S_w & & \end{array}$$

with some homeomorphism h_w. The mapping h_w is biholomorphic, since both $\mathrm{id} \times \tau_w$ and τ_Φ are quotient maps. ◊

For the remainder of this work, we let therefore $\pi_\phi := \tau_w : U \to U_\phi := S_w$ for $w \in W$ arbitrary, and $\phi_{sd} := p_V \circ \overline{\Phi} : W \times U_\phi \to V$.

2.2. Effectively parametrized families

Let $\phi : W \times U \to V$ be a holomorphic map between connected complex spaces, and let $\mathrm{Hol}(U, V) := \{\alpha : U \to V : \alpha$ holomorphic$\}$, $\mathrm{Hol}(U) := \mathrm{Hol}(U, U)$, $\mathrm{Aut}(U) := \{\alpha \in$

Hol (U) : α biholomorphic) . The evaluation map $\mathrm{Hol}(U, V) \times U \to V$ will be denoted by $E_{U,V}$ or by E_U, if $U = V$ (or by E, if the meaning is clear from the context). For $u \in U$, $U' \subset U$, $H \subset \mathrm{Hol}(U, V)$, we denote by $\cdot u$ the composition $E_{U,V} \circ j_u : \mathrm{Hol}(U, V) \to \mathrm{Hol}(U, V) \times U \to V$, and we let $HU' := E_{U,V}(H \times U')$, and $Hu := H\{u\}$. We shall say that W is *(almost) effectively parametrized*, if the natural map $\rho_\phi : W \ni w \mapsto \phi(w, .) \in \mathrm{Hol}(U, V)$ is injective (or, respectively, has discrete fibres).

Let $\phi_1 : W_1 \times V \to V_1$ be another holomorphic map between connected complex spaces. When no ambiguity can arise, we denote by $W_1 \circ W$ the image of $W_1 \times W$ under $\rho_{\phi_1} \circ (\mathrm{id}_{W_1} \times \phi)$; in particular, $\alpha \circ W := \{\alpha\} \circ W = \rho_{\alpha\circ\phi}(W)$ for $\alpha \in \mathrm{Hol}(V, V_1)$.

If U is compact, then $\mathrm{Hol}(U, V)$ admits a unique complex structure such that $E_{U,V}$ and all possible ρ_ϕ are holomorphic; if U is moreover reduced, then the complex space $Hol(U, V)$ carries the compact-open topology (see [2]). For compact U, we henceforth tacitly assume $\mathrm{Hol}(U, V)$ to be endowed with this complex structure. Note that then, according to 0.2, HU' and $\rho_\phi(W')$ carry the analytic image structure, whenever $H \hookrightarrow \mathrm{Hol}(U, V)$, $U' \hookrightarrow U$, $W' \hookrightarrow W$, with proper $E_{U,V}|_{H\times U'}$ or $\rho_\phi|W'$.

It is well known that $\mathrm{Aut}(U)$ is open in $\mathrm{Hol}(U)$ for compact U; if, in addition, U is reduced, then $\mathrm{Aut}(U)$ is also closed in $\mathrm{Hol}(U)$.

2.2.1 Lemma. *Suppose that $(p_U, \phi) : W \times U \to U \times V$ is proper.*

(i) Let W be almost effectively parametrized. Then $\dim W \leq \dim V$. *If moreover every irreducible component of W contains a surjective $w : U \to V$, then* $\dim W \leq d_u(V) = \min_{v\in V} \dim_v V$.

(ii) If some $u_0 \in U$ is finite (resp. surjective), then every $u \in U$ is finite (resp. surjective).

Proof. We may assume that U, V, W are reduced; furthermore, a trivial argument shows that W can be assumed irreducible. Applying 2.1.1 to the family $(\phi(., u))_{u\in U}$ yields (ii) and the first part of(i). Let V' be an irreducible component of V. If $w \in W$ is surjective, there exists an irreducible component U' of U with $w(U') \subset V'$. Then $\phi(U' \times W) \subset V'$, whence $\dim W \leq \dim V'$ by the first part of(i).

2.2.2 Lemma and Notation. *If W is compact, then the analytic quotient $W \to \rho_\phi(W)$ exists. $\rho_\phi(W)$ together with this complex structure will be denoted by $\rho_\phi[W]$. The evaluation map $E_{U,V} : \rho_\phi[W] \to V$ is holomorphic.*

Proof. Let $\phi = (W \times U \xrightarrow{\pi_\phi\times\mathrm{id}} W_\phi \times U \xrightarrow{\phi_\mathrm{st}} V)$ be the simultaneous Stein factorization of the partial maps $u : W \to V$. Obviously, ρ_ϕ factors through π_ϕ, and ρ_ϕ defines an analytic equivalence relation on W. If $\rho_\phi(W)$ is endowed with the quotient topology, then the natural map $W_\phi \to \rho_\phi(W)$ and the orbit maps $\cdot u : \rho_\phi(W) \to V$ are finite; hence, by ([8], 49.A

13), the analytic quotient $W_\phi \to \rho_\phi(W)$ exists. Denote the corresponding complex space by $\rho_\phi[W]$; then $\rho_\phi : W \to \rho_\phi[W]$ and $\rho_\phi \times \mathrm{id}_U$ are quotient maps, since $\mathcal{O}_{W_\phi} = (\pi_\phi)_*\mathcal{O}_W$. In particular, the evaluation map is holomorphic. ◊

Note that, if U is compact, $\rho_\phi[W]$ need not coincide with the complex subspace $\rho_\phi(W) \hookrightarrow \mathrm{Hol}(U, V)$ -in general, the latter structure is a substructure of that on $\rho_\phi[W]$. Nevertheless, for non-compact U, when no such rivalry can occur, we shall introduce the notion of a reduced connected complex subspace of Hol (U, V):

2.23 Definition. ***Assume that U is non-compact. If W is compact, reduced and weakly normal, and if W is effectively parametrized, then W is called a reduced connected compact wmplex subspace of $\mathrm{Hol}(U, V)$, expressed by the symbol $W \underset{(rcc)}{\hookrightarrow} \mathrm{Hol}(U, V)$.***

2.2.3.a Remarks.

(i) If W is reduced and compact, then the weak normalization of $\rho_\phi[W]$ is a reduced compact complex subspace of Hol (U, V).

(ii) Let $W_1 \underset{(rcc)}{\hookrightarrow} \mathrm{Hol}(U, V)$, $W_2 \underset{(rcc)}{\hookrightarrow} \mathrm{Hol}(V, V_1)$. Then $W_2 \circ W_1$ carries a unique structure of a reduced connected compact complex subspace of $\mathrm{Hol}(U, V_1)$, with which we shall always assume it to be endowed. Note that, in contrast to the case U compact, the inclusion $W_2 \circ w_1 \to W_2 \circ W_1$ need not be an embedding; it is, though, if it is bijective.

2.3. Action of compact complex Lie groups

Let U *be* a connected complex space.

23.1 Lemma and Notation. ***There exists $A(U) \underset{(rcc)}{\hookrightarrow} \mathrm{Hol}(U)$ with $\mathrm{id}_U \in A(U)$ such that the following condition holds: If $\phi : W \times U \to U$ is holomorphic with reduced compact connected W, such that $\mathrm{id}_U \in \rho_\phi(W)$, then $\rho_\phi(W) \subset A(U)$ and $\rho_\phi : W \to A(U)$ is holomorphic.***

In particular, $A(U)$ admits no proper complex substructure, with respect to which the evaluation map E_U remains holomorphic.

$A(U)$ is a compact complex Lie group and $A(U)$ is a norma1 subgroup of $\mathrm{Aut}(U)$; if U is compact, then $A(U)$ is centra1 in the identity component $\mathrm{Aut}_0(U)$ of $\mathrm{Aut}(U)$.

Proof. By 2.2.2 and 2.2.1(i), there exists an irreducible $A(U) \underset{(rcc)}{\hookrightarrow} \mathrm{Hol}(U)$ of maximal dimension with $\mathrm{id}_U \in A(U)$. Then $\mathrm{id}_U \in A(U) \circ A(U) \underset{(rcc)}{\hookrightarrow} \mathrm{Hol}(U)$, whence the inclusion $\alpha \circ A(U) \to A(U) \circ A(U)$ is bijective and therefore biholomorphic for all $\alpha \in A(U)$. Thus $A(U) \subset \mathrm{Aut}(U)$ and $A(U)$ is a compact complex Lie group.

Let $W \underset{(rcc)}{\hookrightarrow} \mathrm{Hol}(U)$ with $\mathrm{id}_U \in W$ and let W' be an irreducible component of W that meets $A(U)$. Then the composition $A(U) \xrightarrow{\cong} w_0 \circ A(U) \to W' \circ A(U)$ is bijective and hence biholomorphic for all $w_0 \in W' \cap A(U)$. Thus $|W'| \subset A(U)$ and hence $|W| \subset A(U)$. On the other hand, the composition $W \xrightarrow{\cong} W \circ \mathrm{id}_U \to W \circ A(U) = A(U)$ is injective and holomorphic, whence $(W, \mathscr{O}'_W) \hookrightarrow A(U)$ for a suitable complex substructure $\mathscr{O}'_W$ of $\mathscr{O}_W$. By 2.1.1, the orbit maps $A(U) \xrightarrow{u} A(U)u \hookrightarrow U$ are finite and hence locally biholomorphic. Thus no reduced subspace of $A(U)$ can admit a proper complex substructure with respect to which the evaluation map remains holomorphic. We conclude that, if $\phi : W \times U \to U$ is as postulated, then $\rho_\phi(W) \subset A(U)$ and the inclusion $\rho_\phi[W] \to A(U)$ is holomorphic, and hence so is $\rho_\phi : W \to A(U)$.

$A(U)$ is normal in $\mathrm{Aut}(U)$, since $\alpha \circ A(U) \circ \alpha^{-1} \underset{(rcc)}{\hookrightarrow} \mathrm{Aut}(U)$ for every $\alpha \in \mathrm{Aut}(U)$.

If U is compact, then $A(U)$ is a compact connected complex subgroup of the connected complex Lie group $\mathrm{Aut}_0(U)$ and hence is central. ◊

In the last chapter, we shall make use of the following generalization of the above result:

2.3.2 Lemma. *Let* $\ldots \to U_{n+1} \xrightarrow{\alpha_n} U_n \to \ldots \xrightarrow{\alpha_0} U_0$ ***be a sequence of coverings, and let*** $W_n \underset{(rcc)}{\hookrightarrow} \mathrm{Hol}(U_{n+1}, U_n)$ ***with*** $\alpha_n \in W_n, n \in \mathbb{N}$.

Then $|W_n| \subset A(U_n) \circ \alpha_n$ ***for*** $n \gg 0$, ***and the inclusion is holomorphic.***

In particular, ***any*** $W \underset{(rcc)}{\hookrightarrow} \mathrm{Hol}(U)$ *containing* ***a*** *covering* α *lies* ***in*** $A(U) \circ \alpha$.

Proof. We may assume that all W_n are irreducible. It suffices to show that $W_n \circ \ldots \circ W_{n+k} \subset A(U_n) \circ \alpha_n \circ \ldots \circ \alpha_{n+k}$ for some $k \geq 1$, since the α_ν are surjective and locally biholomoxphic. On the other hand, by 2.2.1(i), $W_n \circ \ldots \circ W_{n+k} \circ \ldots \circ W_{n+k+l} = W_n \circ \ldots \circ W_{n+k} \circ \alpha_{n+k+1} \circ \ldots \circ \alpha_{n+k+l}$ for all n, l 1, and for k sufficiently large (depending on n). Thus, after suitably condensing the given sequence, we may assume that $W_n \circ W_{n+1} = W_n \circ \alpha_{n+1}$ for all n, whence, in particular, $\dim W_{n+1} \leq \dim W_n$. Cutting off a sufficiently long initial sequence, we can assume that $\dim W_n = \dim W_{n+1}$ for all n. The inclusion $W_n \circ \alpha_{n+1} \to W_n \circ W_{n+1}$ is bijective and hence biholomorphic, and, utilizing its inverse, we obtain a holomorphic

$$\phi := (W_n \times W_{n+1} \to W_n \circ W_{n+1} \xrightarrow{\cong} W_n \circ \alpha_{n+1} \xrightarrow{\cong} W_n)$$

with $\phi(\cdot, \alpha_{n+1}) = \mathrm{id}_{W_n}$. Thus $\rho_\phi(W_{n+1}) \subset A(W_n)$ and ρ_ϕ is finite, since so is $W_{n+1} \to \alpha_n \circ W_{n+1}$. From $\dim W_n \geq \dim A(W_n) \geq \dim W_{n+1} = \dim W_n$, we infer that $W_n \cong A(W_n)$ is a torus. Denote by $g_n : \mathbb{C}^k \to W_n$ the universal covering and assume that $g_n(0) = \alpha_n$. Let $E_{n+1} := E_{U_{n+1}, U_n} \circ (g_{n+1} \times \mathrm{id}_{U_{n+1}}) : \mathbb{C}^k \times U_{n+1} \to U_n$, and denote by $h_n : \mathbb{C}^k \to \mathbb{C}^k$

the linear lifting of $\phi(\alpha_n, .) : W_{n+1} \to W_n$. Then there exists a unique $E'_{n+1} : \mathbf{C}^k \times U_{n+1} \to U_{n+1}$ with $\alpha_n \circ E'_{n+1} = E_{n+1}$ and $E'_{n+1}(0, .) = \mathrm{id}_{U_{n+1}}$.

The simple-arrow part of the diagram

$$\begin{array}{ccccc} \mathbf{C}^k \times U_{n+2} & \xrightarrow{E_{n+2}} & U_{n+1} & & \\ & & & \searrow \alpha_n & \\ \downarrow h_n \times \alpha_{n+1} & & \Downarrow \mathrm{id} & & U_n \\ & & & \nearrow \alpha_n & \\ \mathbf{C}^k \times U_{n+1} & \xrightarrow{E'_{n+1}} & U_{n+1} & & \end{array}$$

is commutative, and from $E_{n+2}(0,.) = \alpha_{n+1} = E'_{n+1} \circ (h_n \times \alpha_{n+1})(0,.)$, we infer that the entire diagram is commutative, since α_n is a covering. Thus $\rho_{E'_n}(\mathbf{C}^k) \circ \alpha_n = W_n$ (as subsets of $\mathrm{Hol}(U_{n+1}, U_n)$), and we can endow $V_n := \rho_{E'_n}(\mathbf{C}^k)$ with the complex structure given by the bijection $V_n \to V_n \circ \alpha_n = W_n$.

The diagram

$$\begin{array}{ccc} V_{n+1} \times U_{n+1} & \xrightarrow{E_{U_{n+1}}} & U_{n+1} \\ \downarrow & & \downarrow \\ W_n \times U_{n+1} & \xrightarrow{E_{U_{n+1}, U_n}} & U_n \end{array}$$

is commutative with locally biholomorphic vertical arrows; thus $E_{U_{n+1}}$ is holomorphic, whence $V_n \underset{(rcc)}{\hookrightarrow} \mathrm{Hol}(U_n)$. As $\mathrm{id}_{U_n} \in V_n$ the assertion follows. ◊

2.3.2.a Remark. Assume that U and W *are* compact and that some $w_0 \in W$ is a covering $U \to V$.

If $\rho_\phi(W) \subset \mathrm{Hol}(V) \circ w_0$, then the corresponding map $\overline{\rho} : W \to \mathrm{Hol}(V)$ is holomorphic with image in $\mathrm{Aut}(V)$.

Proof. Evidently, $\rho_\phi = (W \xrightarrow{\overline{\rho}} \mathrm{Hol}(V) \to \mathrm{Hol}(V) \circ w_0 \hookrightarrow \mathrm{Hol}(U, V))$, and $\mathrm{Hol}(V) \to \mathrm{Hol}(V) \circ w_0$ is biholomorphic, since w_0 is surjective and locally biholomorphic. Furthermore, $\overline{\rho}(W) \subset A(V)$, and $\mathrm{Aut}(V)$ is open in $\mathrm{Hol}(V)$. ◊

2.3.3 Definition. Let $g : U \to V$ be a holomorphic map between connected complex spaces, and let $T \sqsubseteq A(U)$.

g is $T-$*equivariant*, if there exists a map $g_* : T \to A(V)$ with $g_*(0) = 0$ and $g_*(\alpha) \circ g = g \circ \alpha$ for all $\alpha \in T$.

g is $T - T'$ -equivariant, if g is T -equivariant with $g_*(T) \subset T' \sqsubseteq A(V)$.

2.3.3.a Remarks. Let g be a T -equivariant.

(i) g_* is uniquely determined and is a homomorphism of complex Lie groups.

(ii) If $f_{md} g = 0$, then g_* is finite.

Proof. Let $u_0 \in U$ and consider the commutative diagram

$$\begin{array}{ccc} T & \xrightarrow{g_*} & A(V) \\ \downarrow \cdot u_0 & & \downarrow \cdot g(u_0) \\ Tu_0 & \xrightarrow{g} & A(V)g(u_0) \end{array}$$

The mapping $\cdot g(u_0)$ is locally biholomorphic and $g_*(0) = 0$; thus g_* is a homomorphism of complex Lie groups. In particular, g_* is uniquely determined by the equation $\cdot g(u_0) \circ g_* = g \circ \cdot u_0$.

If g is finite in u_0, then g_* is finite in 0 and hence everywhere. ◊

2.3.4 Lemma. ***Let $g: U \to V$ be a holomorphic map between connected complex spaces.***

(i) Let $T \sqsubseteq A(U)$, $T'' \sqsubseteq A(V)$ such that $g(Tu_0) \subset T''g(u_0)$ for some $u_0 \in U$. If $\bigcap_{u\in U} T''_{g(u)} = 0$ (e.g. if g is surjective), then g is $T - T''$-equivariant.

(ii) If g is proper with $g_*\mathcal{O}_U = \mathcal{O}_V$, then g is A(U)-equivariant.

(iii) Let $T'' \sqsubseteq A(V)$. If g is a covering, then there exists a unique $T \sqsubseteq A(U)$ such that g is $T - T''$-equivariant. In particular, dim ***$A(U) \geq$*** dim ***$A(V)$.***

(iv) Let $h: V \to V''$ be a covering, and let $T \sqsubseteq A(U)$. If g is surjective and $h \circ g$ is T-equivariant, then g is T-equivariant.

Proof. (i) We may assume $g(Tu_0) = T''g(u_0)$. Then $T'' \circ g \circ T \underset{(rcc)}{\hookrightarrow}$ Hol (U, V) with $(g \circ T)u_0 = (T'' \circ g \circ T)u_0 = (T'' \circ g)u_0$; thus $T'' \circ g \circ T = T'' \circ g = g \circ T$ by 2.2.1(i), and we conclude that $g(Tu) = T''g(u)$ for all $u \in U$. The maps $g_u := (g|Tu \to T''g(u))$ are $A(Tu)$-equivariant with $(g_u)_*: A(Tu) \to A(T''g(u)) = T''/T''_{g(u)}$. As every $T''_{g(u)}$ is finite and $\bigcap_{u\in U} T''_{g(u)} = 0$, there exists a holomorphic homomorphism $g_*T \to T''$ such that every composition $T \overset{\text{kan}}{\to} A(Tu) \overset{(g_u)_*}{\to} T''/T''_{g(u)}$ factors through g_*. By construction, $g_*(\alpha) \circ g = g \circ \alpha$ for all $\alpha \in T$.

(ii) Let $\phi := g \circ E_U = (T \times U \overset{\text{id}\times\pi_\phi}{\to} T \times U_\phi \overset{\phi_{st}}{\to} V)$ be the simultaneous Stein factorization. Then every $\phi_{st}(t, .)$ is biholomorphic, since $\mathcal{O}_V = g_*\mathcal{O}_U = g_*t_*\mathcal{O}_U = \phi(t, .)_*\mathcal{O}_U = \phi_{st}(t, .)_*(\pi_\phi)_*\mathcal{O}_U = \phi_{st}(t, .)_*\mathcal{O}_{U_\phi}$. By 2.3.1, the assertion follows with $g_* := \rho_{\phi_{st}}$.

Assertion (iii) follows from (i) by applying Lemma 2.3.2 to the sequence of coverings

$$\ldots \to U \overset{\text{id}}{\to} U \to \ldots \to U \overset{g}{\to} V$$

with $W_n := T'' \circ g$.

Assertion (iv) is evident by (i) and (iii). ◊

2.3.4.a Corollary. $A(U \times V) = A(U) \times A(V)$.

Proof. The inclusion $A(U) \times A(V) \subset A(U \times V)$ is obvious. To show the converse, let $\phi := p_U \circ E = (A(U \times V) \times (U \times V) \xrightarrow{\pi_\phi \times \mathrm{id}} A_0 \times (U \times V) \xrightarrow{\phi_{st}} U)$ be the simultaneous Stein factorization, and let $\psi := \phi_{st} \circ (\mathrm{id}_{A_0} \times j_{v_0}) : A_0 \times U \to A_0 \times (U \times V) \to U$ for some fixed $v_0 \in V$. Then $\mathrm{id}_U \in \rho_\psi(A_0)$, whence $\rho_\psi(A_0) \subset A(U)$ by 2.3.1, and therefore $p_U(A(U \times V)(u, v_0)) = (\rho_\psi(A_0))u \subset A(U)u$ for all $u \in U$. By Lemma 2.3.4(i), p_U is $A(U \times V)$-equivariant, and, symmetrically, so is p_V. Evidently, $((p_U)_*, (p_V)_*) : A(U \times V) \to A(U) \times A(V)$ is injective, and the assertion follows. ◊

2.3.4.b Corollary. *Let T, T' be tori and let $\phi : T' \times U \to T$ be a holomorphic. If some $\phi(t_0, .) : U \to T$ is constant, then ϕ factors through $p_{T'}$.*

Proof. We may assume $t_0 = 0$, $\phi(t_0, .) = [0]$. T' acts effectively on $T' \times U$ via addition in the first factor. By Lemma 2.3.4(i), ϕ is T'-equivariant; thus $[0] = \phi(0, .) = (\phi \circ (-t))(t, .) = \phi_*(-t) \circ \phi(t, .)$, i.e. $\phi(t, .) = \phi_*(t)$. ◊

Let $T \subset A(U)$. By ([7], Satz IV.10.1), there exists a holomorphic structure on $|U|/T$ such that the quotient map g becomes holomorphic. Replacing this structure by $q_*\mathcal{O}_U$, we conclude that the analytic quotient $U \to |U|/T$ exists; it will be denoted by $q_T : U \to U/T$. We shall employ the following notation: $(Q_U : U \to U_\infty) := (q_{A(U)} : U \to U/A(U))$.

2.3.4.c Corollary.

(i) $Q_{U \times V} = Q_U \times Q_V$.

(ii) The mapping $U \to U_\infty$ is functorial with respect to proper holomorphic mappings that satisfy $g_*\mathcal{O}_U = \mathcal{O}_{U'}$.

(iii) There exists a covering $U' \to U$ such that every covering $g : U_1 \to U'$ is $A(U_1)$-equivariant. In particular, there exists a covering $g_\infty : (U_1)_\infty \to (U')_\infty$ with $g_\infty \circ Q_{U_1} = Q_{U'} \circ g$.

Proof. (i) follows from 2.3.4.a, (ii) from 2.3.4(ii); to prove (iii), note that, by 2.2.1(i), every covering $U' \to U$ satisfies $\dim A(U') \le d_0(U') = d_0(U)$. Thus, if $U' \to U$ is a covering with $\dim A(U')$ maximal, then every covering $U_1 \to U'$ is $A(U_1)$-equivariant by 2.3.4(iii) and 2.3.4(i). ◊

2.4. Torsion bundles over tori

Let U be a connected complex space.

2.4.1. Definition. Let $\pi : U \to T$ be holomorphic, T a k-dimensional torus. We shall say that π is a *torsion bundle* over *T with fibre* U_0, if π is a U_0-bundle with finite structure group such that the total space of the associated principal bundle is connected.

Notation. $(\pi : U \to T) \in \mathcal{F}_k$ with fibre U_0. Sometimes we also say $U \in \mathcal{F}_k$, if there exists $(\pi : U \to T) \in \mathcal{F}_k$ with some fibre. With this convention we let $\mathcal{F} := \bigcup_{k>1} \mathcal{F}_k$.

2.4.1.a Remarks, examples, and notations

(i) Every connected complex space lies in $\mathcal{F}_0$. If $(\pi : U \to T) \in \mathcal{F}_k$ with fibre U_0, and $(\tau : V \to T') \in \mathcal{F}_l$ with fibre V_0, then $\pi \times \tau \in \mathcal{F}_{k+l}$ with fibre $U_0 \times V_0$. In particular, $U \times V \in \mathcal{F}$ if $U \in \mathcal{F}$ or $V \in \mathcal{F}$. We shall see later on that the converse holds, too.

(ii) Let $T := \mathbf{C}/\mathbf{Z} + i\mathbf{Z}$ and let $\pi_j : T \to T/\left\langle\frac{1}{5}\right\rangle$ be the $\mathbf{Z}_5$-principal bundle given by the $\mathbf{Z}_5$-action $\mathbf{Z}_5 \times T \ni (n, t) \mapsto t + \frac{nj}{5} \in T$, where $1 \leq j \leq 4$. for every complex space U_0 with non-trivial $\mathbf{Z}_5$-action, the U_0-bundle $\pi_j\langle U_0\rangle$ associated to π_j is a torsion bundle over $T/\left\langle\frac{1}{5}\right\rangle$ with fibre U_0. The bundles $\pi_j\langle U_0\rangle$ and $\pi_{5-j}\langle U_0\rangle$ are isomorphic via $t \mapsto -t$, whereas $\pi_j\langle U_0\rangle$ and $\pi_k\langle U_0\rangle$ are not isomorphic for $k \neq 5, 5-j$. The associated fibre spaces, however, and, a fortiori, their total spaces, may be isomorphic. For instance, if $U_0 = V^5$ for some V, where $\mathbf{Z}_5$ acts by cyclic permutation of the coordinates, then the fibre spaces associated to the $\pi_j\langle U_0\rangle$ are all isomorphic. On the other hand, if $U_0 = \mathbf{P}_1$ with $\mathbf{Z}_5$-action $(n,(x_0 : x_1)) \mapsto (x_0 : \varepsilon^n x_1)$ where $\varepsilon = \exp\left(\frac{2\pi i}{5}\right)$, then not even the tota1 spaces of $\pi_1\langle\mathbf{P}_1\rangle$ and $\pi_2\langle\mathbf{P}_1\rangle$ are isomorphic (see [5], 6.2).

(iii) Let $(\pi : U \to T) \in \mathcal{F}_k$ with fibre U_0, and let $\pi' : \mathbf{T'} \to \mathbf{T}$ be the associated principal bundle. Then T' is a k-dimensiona1 torus and π' is a covering. We may assume that π' is a homomorphism and identify the structure group Γ of π with Ker π'. Assume that the Γ-action on T' is given by $(\gamma, t) \mapsto t + \tilde{\chi}(\gamma)$ with some $\tilde{\chi} \in \mathrm{Aut}(\Gamma)$, and define $\chi : \Gamma \to \mathrm{Aut}(U_0)$ by $\chi(\gamma) := \tilde{\chi}^{-1}(-\gamma)$ (where we consider Γ as a subgroup of $\mathrm{Aut}(U_0)$). Then thenatural map $T' \times U_0 \to U$ given by $(t,u) \sim (t + \tilde{\chi}(\gamma), (-\gamma)(u))$, coincides with the quotient map $q : T' \times U_0 \to (T' \times U_0)/\mathrm{graph}(\chi)$. Consider the cartesian square

$$\begin{array}{ccc} T' \times U_0 & \xrightarrow{q} & U \\ \downarrow p_{T'} & & \downarrow \pi \\ T' & \xrightarrow{\pi'} & T \end{array}$$

and let T' act on $T' \times U_0$ via addition in the first factor. Applying Lemma 2.3.4(iv) to $T' \times \{u_0\} \xrightarrow{q} q(T' \times \{u_0\}) \xrightarrow{\pi} T$, we infer from 2.3.4(i) that q is T'-equivariant. Moreover, q_* is injective, since Ker $q_* \subset \Gamma$ and $q_*(\gamma, u) \neq q_*(\gamma', u)$ for all $\gamma, \gamma' \in \Gamma, \gamma \neq \gamma'$.

Therefore, we shall from now on consider $T' \subset A(U)$ in this sense.

(iv) Let S be a k-dimensional torus, and let $\chi : \Gamma \to \mathrm{Aut}(U_0)$ be a monomorphism from a finite subgroup Γ of S into the automorphism group of some complex space U_0. Then, evidently, the map $(S \times U_0)/\mathrm{graph}(\chi) \to S/\Gamma$, given by $(s, u) \mapsto s + \Gamma$, is a torsion bundle over S/Γ with fibre U_0.

Conversely, by (iii), every $\pi \in \mathcal{T}_k$ arises in this way.

(v) Let $(\pi : U \to T) \in \mathcal{T}_k$ with fibre U_0, and let V be a connected component of $\pi^{-1}(0) \cong U_0$. Denote by $\Delta \subset T' \subset A(U)$ the isotropy group of V; then $\Delta \supset \Gamma := \mathrm{Ker}\,\pi'$ (π' as in (iii)), and Δ stabilizes every connected component of $\pi^{-1}(0)$. In particular, Δ contains every isotropy group Γ_u for $u \in \pi^{-1}(0)$. Thus $\pi_* = \pi' = (T' \overset{\mathrm{kan}}{\to} T'/\Gamma \overset{\lambda}{\to} T)$ with some homomorphism λ, and the restrictions $\pi : T'u \to T$, $u \in \pi^{-1}(0)$, all factor through λ. As U is the disjoint union of the $T'u$, $u \in \pi^{-1}(0)$, we obtain a map (of sets) $\pi_c : U \to T'/\Delta$ with $\pi = \lambda \circ \pi_c$; π_c is holomorphic, since λ is locally biholomorphic.

The commutative diagram

$$\begin{array}{ccccccc} T' \times U_0 & \hookleftarrow & T' \times V & \overset{q}{\to} & U & & \\ & \searrow p_{T'} & \downarrow p_{T'} & & \downarrow \pi_c & \searrow \pi & \\ & & T' & \overset{\mathrm{kan}}{\to} & T'/\Delta & \overset{\lambda}{\to} & T \end{array}$$

immediately yields that $\pi_c \in \mathcal{T}_k$ with fibre V and structure group Δ. If $U = (T' \times U_0)/\mathrm{graph}(\chi)$ (according to (iv)), then $U = (T' \times V)/\mathrm{graph}(\psi)$, where $\psi := (\chi|\Delta \to \mathrm{Aut}(V))$.

Note that for compact reduced U the equation $\pi = \lambda \circ \pi_c$ is just the Stein factorization of π.

The following characterization of $\mathcal{T}_k$ is one of the essential ingredients of the investigations performed in Chapter 5:

2.4.2 Lemma. *Let $\pi : U \to T$ be a holomorphic map into a k-dimensional torus T, and assume that there exists a k dimensional $T' \subset A(U)$ with $\pi(T'u_0) = T$ for some $u_0 \in U$.*

Then $\pi \in \mathcal{T}_k$ with fibre $\pi^{-1}(\pi(u_0)) =: U_0$.

Proof. By Lemma 2.3.4(i), the map π is T'-equivariant with $\pi_*(T') = T$; in particular, π is locally trivial, and the diagram

$$\begin{array}{ccccc} T' \times U_0 & & \overset{E_U}{\to} & & U \\ \downarrow p_{T'} & & & & \downarrow \pi \\ T' & & \overset{\pi_*}{\to} & & T \\ & \searrow \cdot u_0 & & \nearrow \pi & \\ & & T'u_0 & & \end{array}$$

commutes. Again by 2.3.4(i), E_U is T'-equivariant (with respect to the addition in the first factor), whence E_U is a covering. Thus every $t \in T'$ defines a map $\chi_t : \Gamma = \operatorname{Ker}\pi_* \to \operatorname{Aut}(U_0)$ such that $E_U^{-1}(E_U(t,u)) = \chi_t(\Gamma)u$ for all $u \in U$. Now $t(E_U(\gamma,\chi_0(\gamma)(u))) = t(E_U(0,u)) = E_U(t,0) = E_U(t+\gamma,\chi_t(\gamma)(u)) = t(E_U(\gamma,\chi_t(\gamma)(u)))$ for all $t \in T', u \in U$, whence $\chi_t = \chi_0$ for all $t \in T'$; in particular, $\chi := \chi_0$ is a homomorphism, since $\chi(\gamma + \gamma') = \chi_\gamma(\gamma') \circ \chi_0(\gamma) = \chi(\gamma') \circ \chi(\gamma)$ for all γ, γ'. Evidently, $E_U : T' \times U_0 \to U$ factors through $T' \times U_0 \xrightarrow{\mathrm{kan}\times\mathrm{id}} (T/\operatorname{Ker}\chi) \times U_0$, and we conclude that $\operatorname{Ker}\chi = 0$. Therefore π can be represented as in 2.4.1.a(iv).

2.4.2.a Corollary. *Let $\alpha : U \to A(U)$ be holomorphic. Fix some $u_0 \in U$ and define $\alpha_n : U \to U$ by $\alpha_n := (U \xrightarrow{\alpha} A(U) \xrightarrow{\cdot u_0} U)^n$ for all $n \in \mathbb{N}$.*
There exists $k \in \mathbb{N}$ such that $(\alpha \circ \alpha_n : U \to \alpha(\alpha_{n-1}(A(U)u_0))) \in \mathscr{T}_k$ for all $n \gg 0$.

Proof. We may assume $\alpha(u_0) = 0$. Then $T_n := \alpha(\alpha_n(A(U)u_0)) \sqsubseteq A(U)$ and $T_{n+1} \sqsubseteq T_n$, whence $T_n = T_{n+1}$ for $n \gg 0$. Letting $k := \dim T_n$ for $n \gg 0$, the assertion follows from Lemma 2.4.2, since $\alpha_n : U \to U$ factors through $\cdot u_0 : T_{n-1} \to U$. □

2.4.2.b Corollary. *Let $U \times V \in \mathscr{T}_k$. If $V \notin \mathscr{T}$, then $U \in \mathscr{T}_k$.*

Proof. Let $(\pi : U \times V \to T) \in \mathscr{T}_k$. Composing π with some covering $T \to T'$, we may assume $T \sqsubset A(U \times V)$ (compare 2.4.1.a(iii)), and that $\pi_* : T \to T$ is homothetic. Fix some $(u_0, v_0) \in U \times V$ and considering $g := (A(U \times V) = A(U) \times A(V) \xrightarrow{\cdot(u_0,v_0)} U \times V \xrightarrow{\pi} T)$; evidently, $g(T) = T$. For $S \in \{U, V\}$ let $d_S := \lim_{n\to\infty} \dim Im(T \xrightarrow{p_{A(S)}} A(S) \xrightarrow{\jmath} A(U) \times A(V) \xrightarrow{g} T)^n$; then $S \in \mathscr{T}_{d_S}$ by 2.4.2.a, whence $d_V = 0$. Thus the lifting $\tilde{g} : \widetilde{A(U)} \times \widetilde{A(V)} \to \tilde{T}$ to the universal coverings with $\tilde{g}(0,0) = 0$ satisfies the condition of 0.3.2.a, and we conclude that $T \xrightarrow{p_{A(U)}} A(U) \xrightarrow{\jmath} A(U \times V) \xrightarrow{g} T$ is surjective, whence $d_U = k$. □

2.4.3 Definition. Let $(\pi_j : U_j \to T_j) \in \mathscr{T}_k$ with fibre V_j, $j = 1, 2$. A holomorphic map $f : U_1 \to U_2$ is a $\mathscr{T}$-morphism, if it is $T_1 - T_2$-equivariant and fibre-preserving with respect to π_1, π_2.

For brevity of expression, we employ the notation: $f : \pi_1 \to \pi_2$.

2.4.3.a Remarks.

(i) If $f : \pi_1 \to \pi_2$, there exists a commutative diagram of holomorphic mappings

$$\begin{array}{ccccc} T_1' \times V_1 & \xrightarrow{q_1} & U_1 & \xrightarrow{\pi_1} & T_1 \\ \downarrow f_* \times f_0 & & \downarrow f & & \downarrow \bar{f} \\ T_2' \times V_2 & \xrightarrow{q_2} & U_2 & \xrightarrow{\pi_2} & T_2 \end{array}$$

(ii) $f : U_1 \to U_2$ is fibre-preserving, if f maps at least one fibre of π_1 into one of π_2.

(iii) A surjective holomorphic $f : U_1 \to U_2$ is a $\mathcal{T}$-morphism, if and only if it maps some fibre of π_1 into one of π_2, and some orbit of T_1' into one of T_2'.

Proof. (i) The existence of the righthand rectangle is obvious. Let $f_0 := f|\pi_1^{-1}(0) \to \pi_2^{-1}(0)$. Then the lefthand rectangle commutes, since $q_2(f_*(t), f_0(u)) = q_2(f_*(t)(0, f_0(u))) = f_*(t)(q_2(0, f_0(u))) = f_*(t)(f(q_1(0,u))) = f(t(q_1(0,u))) = f(q_1(t, u))$.

(ii) and (iii) follow from 2.3.4.b, 2.4.1.a(v), and from 2.3.4(i). ◊

3. PRELIMINARIES ON ISOMORPHISMS BETWEEN PRODUCTS

Let $f = (lf, rf) : X \times Y \to U \times V$ be a biholomorphic map between connected complex spaces. This is the starting position for both the cancellation and the decomposition problem. We shall now develop some techniques for reducing the situation to a simpler one.

3.1. Relations between the partial maps

3.1.1 Lemma. *Let $(x, y) \in X \times Y$, $(u, v) := f(x, y)$.*

(i) $\vec{y} = rf(\cdot, y)$ induces a biholomorphic map from $F := \overleftarrow{y}^{-1}(u)$ onto $\vec{u}^{-1}(y)$, whose inverse is given by $\overleftarrow{u} = lf^{-1}(u, \cdot)$.

(ii) If $\overleftarrow{y}$ is biholomorphic in x, then $\vec{u}$ is biholomorphic in v.

(iii) If every $\overleftarrow{y'}$, where $y' \in Y$, is biholomorpic, then so is every $\vec{u'}$, where $u' \in U$.

Proof. (i) From $id_{F \times \{v\}} = f^{-1} \circ f|_{F \times \{v\}} = f^{-1} \circ ([u], rf)|_{F \times \{v\}}$ we infer $\overleftarrow{u}\,\vec{y}|_F = id$, and $\vec{u}\,\vec{y}\,|_F = [y]\,|_F$, and the assertion follows with a symmetry argument.

Assertion (ii) is evident by 1.2.1.a.

(iii) The partial maps $\overleftarrow{y'}$ resp. $\vec{u'}$ are all biholomorphic, if and only if $(lf, p_Y) : X \times Y \to U \times Y$ resp. $(p_U, rf^{-1}) : U \times V \to U \times Y$ is biholomorphic. Thus the assertion follows from $(p_U, rf^{-1}) \circ f = (lf, p_Y)$. ◊

3.1.1.a Corollary. *If all $\overleftarrow{y}$ are biholomorphic, then $Y \cong V$.*

3.1.1.b Remark. Let

$$\begin{array}{ccc} X \times Y & \xrightarrow{f} & U \times V \\ \downarrow p_1 \times q_1 & & \downarrow p_2 \times q_2 \\ X' \times Y' & \xrightarrow{f'} & U' \times V' \end{array}$$

be commutative and let $y' := q_1(y)$.

Then $\overleftarrow{y'} \circ p_1 = p_{U'} \circ f' \circ (p_1 \times q_1)(\,.\,, y) = p_{U'} \circ (p_2 \times q_2) \circ f(\,.\,, y) = p_2 \circ \overleftarrow{y}$. In particular, if p_1 is surjective, then $\overleftarrow{y'}$ is constant, if $\overleftarrow{y}$ is.

3.2. Degenerating isomorphisms

3.2.1 Definition. Let $(x, y) \in X \times Y$, $(u, v) = f(x, y)$. *f degenerates with respect to (x, y)*, if the reduction of the map $(\overleftarrow{u}\,\overrightarrow{x}\,\overrightarrow{v}\,\overleftarrow{y})^n$ is constant for $n \gg 0$. We say that *f degenerates*, if f degenerates with respect to some (x, y).

3.2.1.a Examples.

(i) If f is a product of isomorphisms $X \to U$, $Y \to V$, then e.g. every $\overrightarrow{v}$, $v \in V$, is constant, whence f degenerates with respect to every (x, y).

(ii) Let $X = Y = U = V$ be a one-dimensional torus, and let f be given by $f(x, y) = (2x + y, x + y)$. Then $(\overleftarrow{u}\,\overrightarrow{x}\,\overrightarrow{v}\,\overleftarrow{y})(x') = 2x' + 3x + 4y$ for all $x, x' \in X, y \in Y, (u, v) = f(x, y)$ Thus f does not degenerate.

3.2.1.b Remarks.

(i) If f degenerates with respect to (x, y), then $J \circ f \circ J$ degenerates with respect to (y, x), and $J \circ f^{-1}$ degenerates with respect to $(u, v) = f(x, y)$. It is not clear, whether e.g. f^{-1} degenerates.

(ii) Let X be compact, $(x_0, y_0, u_0, v_0) \in X \times Y \times U \times V$. If $|(\overleftarrow{u_0}\,\overrightarrow{x_0}\,\overrightarrow{v_0}\,\overleftarrow{y_0})^n|$ is constant for some fixed $n \in \mathbb{N}$, then so is every $|(\overleftarrow{u}\,\overrightarrow{x}\,\overrightarrow{v}\,\overleftarrow{y})^n|$ (compare Lemma 2.1.1). In particular, if f degenerates with respect to some (x_0, y_0), then f degenerates with respect to every (x, y), and the minimal n from the definition does not depend on (x, y).

(iii) Let

$$\begin{array}{ccc} X \times Y & \xrightarrow{f} & U \times V \\ \downarrow p_1 \times q_1 & & \downarrow p_2 \times q_2 \\ X' \times Y' & \xrightarrow{f'} & U' \times V' \end{array}$$

be commutative with surjective p_1 and biholomorphic f', and assume that f degenerates with respect to some (x, y). Then, by 3.1.1.b, f' degenerates with respect to $(p_1 \times q_1)(x, y)$.

3.3. Simultaneous subdecompositions

3.3.1 Lemma and Notation. *Let* $Y_1 := Y \times V$, $y_1 := (y, v) \in Y_1$, *und let* $S_1 f : X \times Y_1 \to X \times Y_1$ *be given by* $S_1 f := (f^{-1} \times \mathrm{id}_V) \circ J_V \circ (f \times \mathrm{id}_V)$. *For* $n \geq 1$ *let* $Y_{n+1} := Y_n \times Y_n$, $y_{n+1} := (y_n, y_n)$ *und* $S_{n+1} f := S_1(S_n f)$. *Then*

(i) $S_{n+1} f \circ S_{n+1} f = \mathrm{id}_{X \times Y_{n+1}}$, *and*

(ii) $p_X \circ S_{n+1} f(\cdot, y_{n+1}) = (\overleftarrow{v}\, \overleftarrow{y})^{2^n}$

for all $n \geq 0$.

Proof. It suffices to consider the case $n = 0$. Then (i) is evident from the definition of $S_1 f$, and (ii) follows from $S_1 f(\cdot,(y,v)) = (f^{-1} \times \mathrm{id}) \circ J \circ (f(\cdot,y),[v]) = (f^{-1} \times \mathrm{id}) \circ (lf(\cdot,y),[v],rf(\cdot,y)) = (\overleftarrow{v}\, \overleftarrow{y}, \overrightarrow{v}\, \overleftarrow{y}, \overrightarrow{y})$. ◊

3.3.2 Definition. Let $(x, y) \in X \times Y$, $(u, v) = f(x, y)$. We shall say that (x, y) *decomposes* f, if the following conditions are fulfilled:

(i) For $\{(A,B),(C,D)\} = \{(X,Y),(U,V)\}$ with $a,b,c,d \in \{x,y,u,v\}$ accordingly, the systems of complex subspaces

$$\{((\overleftarrow{d}\,\overleftarrow{b})^n)^{-1}(a) : n \in \mathbb{N}\}, \qquad \{((\overleftarrow{c}\,\overrightarrow{b})^n)^{-1}(a) : n \in \mathbb{N}\},$$

$$\{((\overrightarrow{d}\,\overleftarrow{a})^n)^{-1}(b) : n \in \mathbb{N}\}, \qquad \{((\overrightarrow{c}\,\overrightarrow{a})^n)^{-1}(b) : n \in \mathbb{N}\}$$

have maximal elements A_D, A_C, B_D, B_C respectively.

(ii) For $\{\{A, B\}, \{C, D\}\} = \{\{X,Y\},\{U, V\}\}$ with $A \in \{X, U\}$, the maps $A_C \times B_C \to C$ given by $p_C \circ f$ (if $A = X$) or by $p_C \circ f^{-1}$ (if $A = U$) are biholomorphic.

(iii) The isomorphism $\tilde{f} : U_X \times V_X \times U_Y \times V_Y \to X_U \times Y_U \times X_V \times Y_V$ induced by f via (ii) satisfies:

Each of the partial maps $R_S \to S_R$, $S_R \to R_S$ given by $\tilde{f}, \tilde{f}^{-1}$ and x, y, u, v (where $R \in \{U, V\}$, $S \in \{X,Y\}$) is biholomorphic (i.e. the composition $U_X \to U_X \times \{(u, u, v)\} \xrightarrow{\tilde{f}} X_U \times Y_U \times X_V \times Y_V \xrightarrow{p} X_U$, etc.).

f induces a simultaneous subdecomposition, if some (x, y) decomposes f.

3.3.2.a Remarks.

(i) The condition 3.3.2(iii) is well-defined, since by construction $s \in S_R \cap S_{R'}$ for all possible combinations (i.e. $\overleftarrow{v}\, \overleftarrow{y}(x) = lf^{-1}(lf(x,y), v) = lf^{-1}(f(x,y)) = x$ etc.).

(ii) f as in 3.2.1.a(i) induces a simultaneous subdecomposition, f as in 3.2.1.a(ii) does not.

(iii) If (x, y) decomposes f, then (x, y) also decomposes $J \circ f$, and $f(x, y)$ decomposes f^{-1}.

(iv) Assume that f induces a simuhaneous subdecomposition and that $X \cong U$ is indecomposable. Then $Y \cong V$.

In fact, if $X \ncong \mathbf{C}^0$, then either $X = X_U$, and hence $U = U_X$, or $X = X_V$ and $U = U_Y$. In the first case, we conclude $V = V_Y \cong Y_V = Y$; in the second one, $Y \cong U_Y \times V_Y = U \times V_Y \cong X \times Y_V = X_V \times Y_V \cong V$.

3.3.2.b Example. With the notations of 2.4.1.a, let $X = U = T$, and Y be the total space of $\pi_1 \langle \mathbf{P}_1 \rangle$, V that of $\pi_2 \langle \mathbf{P}_1 \rangle$. Then $X \cong U$ is indecomposable, $X \times Y$ is isomorphic to $U \times V$ via the map induced by $T \times T \times \mathbf{P}_1 \ni (s,t,x) \mapsto (3s+5t, s+2t, x) \in T \times T \times \mathbf{P}_1$, but Y is not isomorphic to V.

3.3.3 Lemma. *Let*

$$\begin{array}{ccc} X \times Y & \xrightarrow{f} & U \times V \\ \downarrow p_1 \times p_2 & & \downarrow q_1 \times q_2 \\ X' \times Y' & \xrightarrow{f'} & U' \times V' \end{array}$$

be a commutative diagram of holomorphic maps between connected complex spaces with f' biholomorphic. Assume that $p_1 = \mathrm{id}_X$ or $p_2 = \mathrm{id}_Y$, and that $q_1 = \mathrm{id}_U$ or $q_2 = \mathrm{id}_V$.

If $(x', y') = (p_1(z), p_2(y))$ decomposes f', then (x, y) decomposes f.

Proof. By 3.3.2.a(iii), we need only consider the case $p_2 = \mathrm{id}_Y$, $q_2 = \mathrm{id}_V$. Then V_X and Y_U exist and are equal to $V'_{X'}$ resp. $Y'_{U'}$. From the commutative diagram

$$\begin{array}{ccccccc} X & \xrightarrow{\vec{y}} & V & \xrightarrow{\overleftarrow{u}} & X & \xrightarrow{\vec{y}} & V \\ \downarrow p_1 & & \| & & \downarrow p_1 & & \| \\ X' & \xrightarrow{\vec{y}'} & V' & \xrightarrow{\overleftarrow{u}'} & X' & \xrightarrow{\vec{y}'} & V' \end{array}$$

(compare 3.1.1.b), we infer that V_Y exists and is equal to $V'_{Y'}$, and that X_U exists and is equa1 to $p_1^{-1}(X'_{U'})$.

Symmetrically: Y_V exists and is equa1 to $Y'_{V'}$, and U_X exists and is equal to $q_1^{-1}(U'_{X'})$.

From the commutative diagram

$$\begin{array}{ccc} \overleftarrow{y}^{-1}(u) & \xrightarrow{\vec{y}\,\cong} & \vec{u}^{-1}(y) \\ \downarrow p_1 & & \downarrow \mathrm{id}_V \\ \overleftarrow{y}'^{-1}(u') & \xrightarrow{\vec{y}'\,\cong} & \vec{u}'^{-1}(y') \end{array}$$

(compare 3.1.1.(i)), we infer that $p_1 | \overleftarrow{y}^{-1}(u) \to \overleftarrow{y'}^{-1}(u')$ is welldefinedand biholomorphic.

Let now $S_n f$, $S_n f'$ be as in 3.3.1. By construction, the diagram

$$\begin{array}{ccc} X \times Y_n & \xrightarrow{S_n f} & X \times Y_n \\ \downarrow p_1 \times id & & \downarrow q_1 \times id \\ X' \times Y'_n & \xrightarrow{S_n f'} & X' \times Y'_n \end{array}$$

is well-defined and commutative.

Applying the above remark to $S_n f$, we conclude that

$p_1 | \overleftarrow{y}_n^{-1}(x) \to \overleftarrow{y'}_n^{-1}(x')$ is well-defined and biholomorphic (where $\overleftarrow{y}_n = lS_n f(\,.\,, y_n) : X \to X$). Thus, by 3.3.1.(ii), X_V exists and $p_1 | X_V \to X'_{V'}$ is well-defined and biholomorphic. Symmetrically: U_Y exists and $q_1 | U_Y \to U'_{Y'}$, is well-defined and biholomorphic.

From the commutative diagram

$$\begin{array}{ccc} X_V \times Y_V & \xrightarrow{rf} & V \\ \cong \downarrow p_1 \times id & & \| \\ X'_{V'} \times Y'_{V'} & \underset{\cong}{\xrightarrow{rf'}} & V' \end{array}$$

we infer that $rf | X_V \times Y_V \to V$ is biholomorphic. Symmetrically: $rf^{-1} | U_Y \times V_Y \to Y$ is biholomorphic.

The commutative diagram

$$\begin{array}{ccc} X_U \times Y_U & \xrightarrow{lf} & U \\ \downarrow p_1 \times id & & \downarrow q_1 \\ X'_{U'} \times Y'_{U'} & \underset{\cong}{\xrightarrow{lf'}} & U' \end{array}$$

yields that $lf | X_U \times Y_U \to U$ is biholomorphic, since $X_U \times Y_U = (p_1 \times id)^{-1}(X'_{U'} \times Y'_{U'})$, $U = q_1^{-1}(U')$. Symmetrically, $lf^{-1} | U_X \times V_X \to X$ is biholomorphic.

To verify condition 3.3.2.(iii), let $R \in \{U, V\}$, $S \in \{X, Y\}$, and denote by j : $R_S \to U_X \times V_X \times U_Y \times V_Y$ the natural embedding given by (u, v) (i.e. $U_X \to U_X \times \{(v, u, v)\}$ etc.), with corresponding j' : $R'_{S'} \to U'_{X'} \times V'_{X'} \times U'_{Y'} \times V'_{Y'}$. Consider the commutative diagram

$$\begin{array}{ccccc}
R_S & & \longrightarrow & & R'_{S'} \\
\downarrow i & & & & \downarrow j' \\
U_X \times V_X \times U_Y \times V_Y & & \xrightarrow{P} & & U'_{X'} \times V'_{X'} \times U'_{Y'} \times V'_{Y'} \\
& \searrow (f^{-1} \times f^{-1}) & & \nearrow & \\
& X \times Y & \xrightarrow{p_1 \times \mathrm{id}} & X' \times Y' & \\
\downarrow \tilde f & \downarrow f & & \downarrow f' & \downarrow \tilde f' \\
& U \times V & \xrightarrow{q_1 \times \mathrm{id}} & U' \times V' & \\
& \nearrow (f \times f) & & \nwarrow & \\
X_U \times Y_U \times X_V \times Y_V & & \xrightarrow{Q} & & X'_{U'} \times Y'_{U'} \times X'_{V'} \times Y'_{V'} \\
\downarrow p & & & & \downarrow p \\
S_R & & \longrightarrow & & S'_{R'}
\end{array}$$

with $P := q_1 \times \mathrm{id} \times q_1 \times \mathrm{id}$, $Q := p_1 \times \mathrm{id} \times p_1 \times \mathrm{id}$.

If $R_S \neq U_X$, then P, Q define isomorphisms $j(R_S) \to j'(R'_{S'})$, resp. $j(S_R) \to j'(S'_{R'})$, whence $p \circ \tilde f \circ j : R_S \to S_R$ is biholomorphic.

Let now $R_S = U_X$. The diagram

$$\begin{array}{ccc}
X_U \times Y_U \times X_V \times Y_V & \xrightarrow{p} & X_U \\
\downarrow Q & & \downarrow p_1 \\
X'_{U'} \times Y'_{U'} \times X'_{V'} \times Y'_{V'} & \xrightarrow{p} & X'_{U'}
\end{array}$$

is clearly cartesian, and from $\tilde f(j(U_X)) = Q^{-1}(\tilde f'(j'(U'_{X'})))$ we infer that $p|\tilde f(j(U_X)) \to X_U$ is biholomorphic, since $p|\tilde f'(j'(U'_{X'})) \to X'_{U'}$ is. □

3.4. Dimension-decreasing constructions

3.4.1. Consider at first the double-arrow part of the diagram

$$\begin{array}{ccccc}
X \times Y & \overset{f}{\Rightarrow} & U \times V & \overset{f^{-1}}{\Rightarrow} & X \times Y \\
& \searrow p' \times \mathrm{id} & & \searrow \mathrm{id} \times q' & & \searrow p' \times \mathrm{id} \\
\Downarrow (f, p_Y) & X' \times Y & \Downarrow (p_U, f^{-1}) & U \times V' & \Downarrow (f, p_Y) & X' \times Y \\
& \swarrow & & \swarrow & & \swarrow \\
U \times Y & \overset{\mathrm{id}}{\Rightarrow} & U \times Y & \overset{\mathrm{id}}{\Rightarrow} & U \times Y
\end{array}$$

which is clearly commutative. In particular, (lf, p_Y) is proper, if and only if so is (p_U, rf^{-1}). Assume now that (lf, p_Y) and (p_U, rf^{-1}) are proper, and let their Stein factorizations be given by the simple arrows (compare 2.1.1). As p', q' are quotient maps, the above diagram can be commutatively enlarged by uniquely determined holomorphic arrows $f' : X' \times Y \to U' \times V'$, $(f^{-1})' : U' \times V' \to X' \times Y$ which are obviously inverse to each other.

Interchanging U and V, if allowed (i.e. if the corresponding arrows are proper), we obtain

$$
\begin{array}{ccccccccc}
X \times Y & & \xrightarrow{f} & U \times V & & \xrightarrow{f^{-1}} & X \times Y & & \\
 & \searrow_{p''\times\mathrm{id}} & & & \searrow_{q''\times\mathrm{id}} & & & \searrow_{p''\times\mathrm{id}} & \\
\Downarrow (p_Y, rf) & & X'' \times Y & \Downarrow (rf^{-1}, p_V) & & U'' \times V & \Downarrow (p_Y, rf) & & X'' \times Y \\
 & \swarrow & & & \swarrow & & & \swarrow & \\
Y \times V & & \xrightarrow{\mathrm{id}} & Y \times V & & \xrightarrow{\mathrm{id}} & Y \times V & &
\end{array}
$$

and again we can insert unique holomorphic maps $f'' : X'' \times Y \to U'' \times V$, $(f^{-1})'' = (f'')^{-1} : U'' \times V \to X'' \times Y$.

3.2.1.b (iii) and 3.3 immediately yield:

3.4.1.a Remark. Let $(x, y) \in X \times Y$ and let $(x', y') = (p'(x), y)$, $(x'', y'') = (p''(x), y)$.

(i) If f degenerates with respect to (x, y), then f' degenerates with respect to (x', y'), and f'' degenerates with respect to (x'', y'').

(ii) If (x', y') decomposes f', or if (x'', y'') decomposes f'', then (x, y) decomposes f.

Assume now that X is compact, i.e. that both constructions can be performed. We shall see that they commute (in the obvious sense). Applying the $''$-construction to f' yields just as above

$$
\begin{array}{ccccccccc}
X' \times Y & & \xrightarrow{f'} & U' \times V' & & \xrightarrow{f'^{-1}} & X' \times Y & & \\
 & \searrow_{(p')''\times\mathrm{id}} & & & \searrow_{p''\times\mathrm{id}} & & & \searrow_{(p')''\times\mathrm{id}} & \\
\Downarrow (p_Y, rf') & & (X')'' \times Y & \Downarrow (rf'^{-1}, p_{V'}) & & U'' \times V' & \Downarrow (p_Y, rf') & & (X')'' \times Y \\
 & \swarrow & & & \swarrow & & & \swarrow & \\
Y \times V' & & \xrightarrow{\mathrm{id}} & Y \times V' & & \xrightarrow{\mathrm{id}} & Y \times V' & &
\end{array}
$$

since the Stein factorization of $(rf'^{-1}, \mathrm{id}_{V'})$ is evidently given by the corresponding simple arrows. Symmetrically, we obtain:

$$
\begin{array}{ccccccccc}
X'' \times Y & & \overset{f''}{\Rightarrow} & & U'' \times V & & \overset{f^{-1}}{\Rightarrow} & & X'' \times Y \\
& \searrow_{(p'')' \times \mathrm{id}} & & & & \searrow_{\mathrm{id} \times q'} & & & & \searrow_{(p'')' \times \mathrm{id}} \\
\Downarrow (lf'', p_Y) & & (X'')' \times Y & \Downarrow (p_{U''}, rf''^{-1}) & & U'' \times V' & \Downarrow (lf'', p_Y) & & (X'')' \times Y \\
& \swarrow & & & \swarrow & & & \swarrow & \\
U'' \times Y & & \overset{\mathrm{id}}{\Rightarrow} & & U'' \times Y & & \overset{\mathrm{id}}{\Rightarrow} & & U'' \times Y
\end{array}
$$

and we conclude that $(f')'' = (f'')'$.

Let now $|f| := (f')'' : |X| \times |Y| \to |U| \times |V|$ (although $Y = |Y|$ it is convenient to mark each entry with the same symbol), and let $|P := (|p| \times \mathrm{id}_{,}) := ((p')'' \times \mathrm{id}_{,}) : X \times Y \to |X| \times |Y|$, $|Q := (q'' \times q') : U \times V \to |U| \times |V|$.

3.4.1.b Remark. Let $(x, y) \in X \times Y$, and let $(|x, |y) = |P(x, y)$.

(i) If f degenerates with respect to (x, y), then $|f|$ degenerates with respect to $(|x, |y)$.

(ii) If $(|x, |y)$ decomposes $|f|$ then (x, y) decomposes f.

(iii) The Stein factorization of every $\vec{v}\,\overleftarrow{y} : X \to U \to Y$ and every $\overleftarrow{u}\,\vec{y} : X \to V \to Y$ (with arbitrary $(u, v) \in U \times V$) has the form $X \overset{|p}{\to} |X \to Y$.

Proof. (i) and (ii) follow again from 3.2.1.b(iii) and 3.3.

(iii) By construction, all partial maps $U'' \to Y$, $V' \to Y$, $(X'')' \to U''$, $(X')'' \to V'$ are finite, and hence so are the compositions $|X \to |U \to |Y = Y$, $|X| \to |V| \to |Y| = Y$. On the other hand, $|p|$ is a quotient map with connected fibres. ◊

3.4.2. We shall now present a similar construction that will take care of the non-compact factors.

Let $(x, y) \in X \times Y, (u, v) = f(x, y)$, denote by X_0, Y_0, U_0, V_0 the orbits $A(X)x$, $A(Y)y, A(U)u, A(V)v$, and let $f_0 := f|X_0 \times Y_0 \to U_0 \times V_0$ (compare 2.3.4.a). Applying the '-construction (3.4.1) to f_0 o J, we obtain a commutative diagram

$$
\begin{array}{ccc}
X_0 \times Y_0 & \overset{f_0}{\to} & U_0 \times V_0 \\
\downarrow \mathrm{id} \times 'p_0 & & \downarrow \mathrm{id} \times 'q_0 \\
X_0 \times Y_0 & \overset{'f_0}{\to} & U_0 \times V_0 \\
& \cong &
\end{array}
$$

As Y_0, V_0 are orbits of $A(Y)$, $A(V)$, respectively, there exist connected compact complex subgroups $A' \sqsubseteq A(Y)$, $B' \sqsubseteq A(V)$ such that $'p_0 = (q_{A'} : Y_0 \to Y_0/A')$ and $'q_0 = (q_{B'} :$

$V_0 \twoheadrightarrow V_0/B'$ (compare 2.3.4.c). Applying 2.1.1 to the composition $(X \times Y) \times A(Y) \overset{\mathrm{id}_X \times E}{\to} X \times Y \overset{{}^l f}{\to} U$ and to $(U \times V) \times A(V) \overset{\mathrm{id}_U \times E}{\to} U \times V \overset{{}^l f^{-1}}{\to}$ X, we see that A', B' do not depend on the choice of (x, y). Moreover, by 2.3.4.(i), f is $A' - B'$ -equivariant. Thus, denoting by 'p,' q the quotient maps $Y \twoheadrightarrow Y/A'$, $V \twoheadrightarrow V/B'$, respectively, we arrive at a commutative diagram

$$\begin{array}{ccccc} X \times Y & \overset{f}{\to} & U \times V & \overset{f^{-1}}{\to} & X \times Y \\ \downarrow \mathrm{id} \times {}'p & & \downarrow \mathrm{id} \times {}'q & & \downarrow \mathrm{id} \times {}'p \\ X \times Y/A' & \overset{'f}{\to} & U \times V/B' & \overset{'(f^{-1})}{\to} & X \times Y/A' \end{array}$$

where $'f$ and $'(f^{-1})$ are holomorphic and inverse to each other.

Again, we may interchange U and V to obtain

$$\begin{array}{ccc} X \times Y & \overset{f}{\to} & U \times V \\ \downarrow \mathrm{id} \times {}''p & & \downarrow {}''q \times \mathrm{id} \\ X \times Y/A'' & \overset{''f}{\to} & U/B'' \times V \end{array}$$

Finally, we can construct "('f) and '("f), which again coincide, and will be denoted by $f| : X| \times Y| \to U| \times V|$. The quotient maps $X \times Y \to X| \times Y|$, $U \times V \to U| \times V|$ will be indicated by $P| = (\mathrm{id} \times p|)$, $Q| = (''q \times {}'q)$, respectively.

3.4.2.a Remark. Let $(x, y) \in X \times Y$, and let $(x|, y|) = P|(x, y)$.

(i) If f degenerates with respect to (x, y), then $f|$ degenerates with respect to $(x|, y|)$.

(ii) If $(x|, y|)$ decomposes $f|$, then (x, y) decomposes f.

(iii) Every ${}^u\!f_x : Y \to X$ and every ${}^v\!f_x : Y \to X$ factors through $p| : Y \to Y|$ such that the corresponding map $Y| \to X$ is finite on the images $p|(A(Y)y)$ of the orbits of $A(Y)$.

3.4.3. Let now X be compact and let $\overline{f} := |(f|) : \overline{X} \times \overline{Y} \to \overline{U} \times \overline{V}$ (which does not coincide with $(|f)|!$); moreover, let $\overline{P} := |(P|) : X \times Y \to \overline{X} \times \overline{Y}$ and $\overline{Q} := |(Q|) : U \times V \to \overline{U} \times \overline{V}$.

Summing up, we arrive at the commutative diagram

$$
\begin{array}{ccccc}
X \times Y & & \overset{\overline{P}}{\to} & & \overline{X} \times \overline{Y} \\
& \searrow^{\mathrm{id} \times p} & & \nearrow^{p \times \mathrm{id}} & \\
& & X \times Y & & \\
\downarrow f & & \downarrow f & & \downarrow \overline{f} \\
& & U \times V & & \\
& \nearrow & & \searrow & \\
U \times V & & \overline{Q} & & \overline{U} \times \overline{V}
\end{array}
$$

3.4.3.a Remark. Let $(x, y) \in X \times Y$, and let $(\overline{x}, \overline{y}) = \overline{P}(x, y)$.

(i) If f degenerates with respect to (x, y) , then $\overline{f}$ degenerates with respect to $(\overline{x}, \overline{y})$.

(ii) If $(\overline{x}, \overline{y})$ decomposes $\overline{f}$, then (x, y) decomposes f .

4. COMPLEX SPACES WITH ZERO-DIMENSIONAL FACTORS

This chapter provides the connecting link between the local and the global situation.

Let $f : X \times Y \to U \times V$ be a biholomorphic map between connected complex spaces, and assume that $X_{red} = \{x\}$. For $y \in Y$ let $\overline{y} := (x, y)$.

Theorem. *f induces a simultaneous subdecomposition.*

More explicitly, we have:

(i) Every $(x, y) \in X \times Y$ decomposes f .

(ii) Let $\overline{y} = (x, y) \in X \times Y, (u, v) = f(\overline{y})$. For $R \in \{X, Y\}, S \in \{U, V\}$ denote by $R_S(\overline{y}), S_R(\overline{y})$ the subfactors given by $\overline{y}$ according to 3.3.2. then

$$U_X(\overline{y}) = (\overleftarrow{x}\overrightarrow{v})^{-1}(u), \qquad X_U(\overline{y}) = (\overleftarrow{u}\,\overrightarrow{y})^{-1}(x),$$

and $\overleftarrow{v}$ induces an isomorphism $U_X(\overline{y}) \to X_U(\overline{y})$;

$$V_X(\overline{y}) = (\overrightarrow{x}\,\overrightarrow{u})^{-1}(v), \qquad X_V(\overline{y}) = (\overleftarrow{v}\,\overleftarrow{y})^{-1}(x),$$

and $\overleftarrow{u}$ induces an isomorphism $V_X(\overline{y}) \to X_V(\overline{y})$;

$$U_Y(\overline{y}) = \overleftarrow{v}^{-1}(x), \qquad Y_U(\overline{y}) = \overrightarrow{x}^{-1}(v),$$

and $\overrightarrow{v}$ ***induces an isomorphism*** $U_Y(\bar{y}) \to Y_U(\bar{y})$ ***whose inverse is given by*** $\overleftarrow{x}$;

$$V_Y(\bar{y}) = \overleftarrow{u}^{-1}(x), \qquad Y_V(\bar{y}) = \overleftarrow{x}^{-1}(u),$$

and $\overrightarrow{u}$ ***induces an isomorphism*** $V_Y(\bar{y}) \to Y_V(\bar{y})$ ***whose inverse is given by*** $\overrightarrow{x}$ ***.***

(iii) Let y' ∈ Y, S ∈ $\{U, V\}$***. Then*** $X_S(\bar{y}) = X_S(\bar{y}') =: X_S$ ***and*** $S_Y(\bar{y}) = S_Y(\bar{y}') =: S_Y$.

Proof. Let y ∈ Y be fixed, and let S ∈ $\{U, V\}$. By Theorem 1.4.1, $X_S(\bar{y})$ and $S_X(\bar{y})$ exist, and

(1) the relations postulated in (ii) are satisfied.

(2) lf^{-1} defines an isomorphism $U_X(\bar{y})$ x $V_X(\bar{y}) \to$ ***X,***

(3) the compositions $X_U(\bar{y}) \hookrightarrow X \overset{\overleftarrow{v}\,\overleftarrow{y}}{\to} X_U(\bar{y})$ and $X_V(\bar{y}) \hookrightarrow X \overset{\overleftarrow{u}\,\overrightarrow{y}}{\to} X_V(\bar{y})$ are well-defined and biholomorphic.

Let now Y' be the irreducible component of Y that contains y and assume from now on that y satisfies the following condition:

(*) For every y' in some neighbourhood of y , any embedding $X_U(\bar{y}') \hookrightarrow X_U(\bar{y})$ is an isomorphism.

Such points y exist, since dim X = 0.

Let $\phi := (U$ x $V \overset{lf^{-1}}{\to} X \overset{\overleftarrow{v}\,\overleftarrow{y}}{\to}$ X,(y)). Then $\phi(\,.,v)|U_X(\bar{y}) \to X_U(\bar{y})$ is biholomorphic by (1) and (3); therefore $\phi(\,.,v')\,|U_X(\bar{y}'') \to X_U(\bar{y})$ is an embedding and hence, by (1) and (*), an isomorphism for $(v'$, y") sufficiently close to $(v$, y) . Using 1.1.2.a, we conclude that $\phi($u", .) is constant on $V_X(\bar{y}')$ for (u", y') sufficiently close to $(u$, y) ; in particular, if y' is close to y and (u', v') = $f(x$, y'), then $X_V(\bar{y}') = \overleftarrow{u}'(V_X(\bar{y}'))$ c X,(y),, and as shown above, $X_U(\bar{y}') \cong X_U(\bar{y})$. On the other hand, by (1) and (2), X is isomorphic to every $X_U(\bar{y}'')$ x $X_V(\bar{y}''), y'' \in$ Y; thus $X_V(\bar{y}') = X_V(\bar{y})$ for y' close to y. This means (see (1)) that $\overleftarrow{u}'\,\overleftarrow{y}'$ is constant on $X_V(\bar{y})$ for y' close to y and hence for all y' ∈ Y'. Thus, if y is chosen according to (*), then $X_V(\bar{y})$ is contained in every $X_V(\bar{y}')$, for y' close to y or y' ∈ Y'; in particular, any embedding $X_V(\bar{y}') \hookrightarrow X_V(\bar{y})$ is an isomorphism for y' close to y . We can therefore interchange U and ***V*** in the above considerations and obtain that every $X_U(\bar{y}')$ contains $X_U(\bar{y})$ for y' ∈ Y'. Using again (2), we conclude that $X_U(\bar{y}') = X_U(\bar{y}), X_V(\bar{y}') = X_V(\bar{y})$ for all y' ∈ Y', and hence, as Y is connected:

(4) $X_U(\bar{y}') = X_U(\bar{y}) =: X_U$ and $X_V(\bar{y}') = X_V(\bar{y}) =: X_V$ fora11 y' ∈ Y.

Let $\psi := lf^{-1}$ o $(lf$ o (id, x $p_1), rf$ o (id, x $p_2))$: X x Y x Y → X; then $\psi(\,.,$ (y, y)) = ***id,*** for all y. Using 1.1.2.a, we see that $\psi(x, ., .)$ is constant on some neighbourhood of the diagonal in Y x Y, and from the lemma in 0.2.2 we infer that every partial map $\psi(x$, (y, .)), $\psi(x$, (., y)) is constant. Thus $f(x, .)$ defines an embedding

$Y \to \bigcap_{v\in V} \overleftarrow{v}^{-1}(x) \times \bigcap_{u\in U} \overleftarrow{u}^{-1}(x)$, since every $\overleftarrow{v}\,\overleftarrow{x}: Y \to X$, $\overleftarrow{u}\,\overrightarrow{x}: Y \to X$ is constant.

On the other hand, by 3.1.1, the maps $\overleftarrow{x}, \overrightarrow{x}$ induce isomorphisms $\overrightarrow{x}^{-1}(v) \to \overleftarrow{v}^{-1}(x)$, $\overleftarrow{x}^{-1}(u) \to \overleftarrow{u}^{-1}(x)$, respectively, for all $(u,v)\in U\times V$. We conclude that $\overleftarrow{v}^{-1}(x)$, $\overleftarrow{u}^{-1}(x)$ do not depend on (u,v), and that $f(x,.)$ defines an isomorphism $Y \to \overleftarrow{v}^{-1}(x) \times \overleftarrow{u}^{-1}(x)$ with inverse $rf^{-1}|\,\overleftarrow{v}^{-1}(x)\times \overleftarrow{u}^{-1}(x) \to Y$. This yields

$$\overrightarrow{x}^{-1}(v) = ((\overrightarrow{u}\,\overrightarrow{x})^n)^{-1}(y), \qquad \overleftarrow{x}^{-1}(u) = ((\overrightarrow{v}\,\overleftarrow{x})^n)^{-1}(y),$$

$$\overleftarrow{v}^{-1}(x) = ((\overleftarrow{y}\,\overleftarrow{v})^n)^{-1}(u), \qquad \overleftarrow{u}^{-1}(x) = ((\overrightarrow{y}\,\overleftarrow{u})^n)^{-1}(v),$$

for all $n \geq 1$. Thus

(5) $S_Y(\bar{y})$ and $Y_S(\bar{y})$ exist for $S \in \{U, V\}$ and satisfy the relations postulated in (ii). Furthermore,

(6) $S_Y(\bar{y}) =: S_Y$ does not depend on $\bar{y}$, and $f(x,.)$ defines an isomorphism $Y \to U_Y \times V_Y$ whose inverse is given by rf^{-1}.

(5) and (6) immediately yield:

(7) $Y_S(\bar{y}) = Y_S(\bar{y}')$ for all $y' \in Y_S(\bar{y})$.

By 1.4.1, the restriction $lf|X_U(\bar{y}) \times Y_U(\bar{y}) \to U$ is biholomorphic in (x, y), and hence, by (4) and (7), is biholomorphic in every (x, y') with $y' \in Y_U(\bar{y})$. On the other hand, the reduction $((lf)_{red}|(X_U(\bar{y}) \times Y_U(\bar{y}))_{red} \to U_{red}) = (lf(x,.)|(Y_U(\bar{y}))_{red} \to (U_Y)_{red})$ is biholomorphic by (5). Thus:

(8) $lf|X_U \times Y_U(\bar{y}) \to U$ is biholomorphic, and, symmetrically, so is $rf|X_V \times Y_V(\bar{y}) \to V$.

Collecting what we have shown up to now, we observe that

(ii) is proven by (1) and (5),

(iii) is proven by (4) and (6), and,

by (ii), (iii), (2), (6) and (8), every (x, y) satisfies the conditions 3.3.2.(i) and 3.3.2.(ii).

To complete the proof, it remains to verify the condition 3.3.2.(iii). Consider the commutative diagram of biholomorphic mappings

$$\begin{array}{ccc} (U_X(\bar{y}) \times V_X(\bar{y})) \times (U_Y \times V_Y) & \xrightarrow{\bar{f}} & (X_U \times Y_U(\bar{y})) \times (X_V \times Y_V(\bar{y})) \\ \downarrow lf^{-1}\times rf^{-1} & & \downarrow lf\times rf \\ X \times Y & \xrightarrow{f} & U \times V \end{array}$$

and denote every partial embedding $R_S(\bar{y}) \hookrightarrow U_X(\bar{y}) \times V_X(\bar{y}) \times U_Y \times V_Y$, $S_R(\bar{y}) \hookrightarrow X_U \times Y_U(\bar{y}) \times X_V \times Y_V(\bar{y})$ by j (i.e. $U_X(\bar{y}) \to U_X(\bar{y}) \times \{(v, u, v)\}$ etc.).

By (ii), there exists a commutative diagram

$$\begin{array}{ccccc} U_X(\bar y) \times \{v\} & \xrightarrow{{}^l\tilde f(\cdot,(u,v))} & X_U \times Y_U(\bar y)) & \hookleftarrow & X_U \times \{y\} \\ \downarrow {}^lf^{-1} & & \downarrow {}^lf & \swarrow {}^lf & \\ X_U & \xrightarrow{{}^lf(\cdot,y)} & U & & \end{array}$$

with biholomorphic vertical arrows. Thus ${}^l\tilde f \circ j$ maps $U_X(\bar y)$ biholomorphically onto $X_U \times \{y\}$.

All we have used to derive this diagram from the preceding one, was the fact that $\overleftarrow{v}$ induces an isomorphism $U_X(\bar y) \to X_U$. Hence, by (ii), the same type of diagram exists, mutatis mutandis, for $V_X(\bar y)$, U_Y, V_Y, $Y_U(\bar y)$, $Y_V(\bar y)$, and we conclude:

$$\begin{array}{llll} {}^rf \circ j & & V_X(\bar y) & & X_V \times \{y\} \\ {}^l\tilde f \circ j & & U_Y(\bar y) & & \{x\} \times Y_U(\bar y) \\ {}^r\tilde f \circ j & \textit{maps} & V_Y(\bar y) & \textit{biholomorphically onto} & \{x\} \times Y_V(\bar y) \\ {}^r\tilde f^{-1} \circ j & & Y_U(\bar y) & & U_Y \times \{v\} \\ {}^r\tilde f^{-1} \circ j & & Y_V(\bar y) & & \{u\} \times V_Y \end{array}$$

Finally, there exists a commutative diagram

$$\begin{array}{ccccc} X_U \times \{y\} & \xrightarrow{{}^l\tilde f^{-1}} & U_X(\bar y) \times V_X(\bar y)) & \hookleftarrow & U_X(\bar y) \times \{v\} \\ & \searrow^{\overleftarrow{v} \circ {}^lf} & & & \\ \downarrow {}^lf & X_U & \downarrow {}^lf^{-1} & & \downarrow {}^lf^{-1} \\ & & \searrow & & \\ U & \xrightarrow{v} & X & \hookleftarrow & X_U \end{array}$$

(compare (3) for the diagonal in the lefthand rectangle), and we conclude that ${}^l\tilde f^{-1} \circ j$ maps X_U biholomorphically onto $U_X(\bar y) \times \{v\}$. Symmetrically: ${}^r\tilde f^{-1} \circ j$ maps X_V biholomorphically onto $\{u\} \times V_X(\bar y)$.

Thus, an even stronger condition than 3.3.2.(iii) is fulfilled. ◊

5. COMPLEX SPACES WITH COMPACT FACTORS

Generalizing the situation of the preceding chapter, weconsider now biholomorphic mappings $f : X \times Y \to U \times V$ with compact X. As demonstrated by Example 3.2.1.a(ii) (see also 3.3.2.a(ii)), f need no longer induce a simultaneous subdecomposition; it will, however, if $\{X, Y, U, V\} \not\subset \mathscr{T}_k$ for all $k \geq 1$ – a condition that is of course fulfilled, if dim $X = 0$. This result is the basis for the subsequent investigations conceming cancellability and decomposability.

5.1. The structure induced by two decompositions

Let $f : X \times Y \to U \times V$ be a biholomorphic map between connected complex spaces, and assume that X is compact. Fix some $(x_0, y_0) \in X \times Y$, let $(u_0, v_0) := f(x,y)$, and consider the sequence of holomorphic maps

$$(*)\ \dots \to X \overset{\overleftarrow{y_0}}{\to} U \overset{\overrightarrow{v_0}}{\to} Y \overset{\overrightarrow{x_0}}{\to} V \overset{\overleftarrow{u_0}}{\to} X \overset{\overleftarrow{y_0}}{\to} U \to \dots$$

To simplify the notations, we denote by $S \overset{(*,l)}{\to} S'$ the map $S \to S'$ given by a subsequence of (*) that starts at S, consists of l arrows, and ends at S' (where $\{S, S'\} \subset \{X, Y, U, V\}$); furthermore, we let $(S \overset{(*,0)}{\to} S) := \mathrm{id}_S$, and we say that $S \overset{(*,l)}{\to} S'$ contains $S_1 \overset{(*,m)}{\to} S_1'$, if $(S \overset{(*,l)}{\to} S') = (S \overset{(*,k)}{\to} S_1 \overset{(*,m)}{\to} S_1' \overset{(*,n)}{\to} S')$ with suitable k, n.

5.1.1 Lemma. *Let $\{S, S'\} \subset \{X, Y, U, V\}$ with corresponding $s_0, s_0' \in \{x_0, y_0, u_0, v_0\}$.*

(i) If $l \geq 2$, then $S \overset{(*,l)}{\to} S'$ factors through $\cdot s_0' : \mathrm{Hol}(S') \to S'$ with $s_0 \mapsto \mathrm{id}_{S'}$.

(ii) If $S \overset{(*,l)}{\to} S'$ contains $X \overset{(*,10)}{\to} Y$, then $S \overset{(*,l)}{\to} S'$ factors holomorphically through $s_0' : A(S') \to S'$ with $s_0 \mapsto \mathrm{id}_{S'}$.

Proof. See 7.1.4. ◇

5.1.2 Proposition. *Let* $l_f := \lim_{n\to\infty} \dim \mathrm{Im}(X \overset{(*,4n)}{\to} X)$.

For every $S \in \{X, Y, U, V\}$ there exists $(\pi_S : S \to T_S) \in \mathscr{T}_{l_f}$ with some connected fibre F_S.

In particular, if $\{X, Y, U, V\} \not\subset \mathscr{T}_k$ for all $k \geq 0$, then f degenerates with respect to (x_0, u_0).

Proof. By 5.1.1, 2.4.2.a and 2.4.1.a(v), the map $S \overset{(*,16)}{\to} S$ gives rise to some $(\pi_S : S \to T_S) \in \mathscr{T}_{l(S)}$ with connected fibre F_S. As $S \overset{(*,4n+4)}{\to} S$ contains $X \overset{(*,4n)}{\to} X$, we conclude that $l(S) = l(X) = l_f$ for all $S \in \{X, Y, U, V\}$. ◇

5.1.2.a Remark. Let $S \in \{X, Y, U, V\}$ with corresponding $s_0 \in \{x_0, y_0, u_0, v_0\}$, and let $\pi_S : S \to T_S$ be as in 5.1.2 with corresponding $T_S' \sqsubset A(S)$ (compare 2.4.1.a(iii)). By construction, $S \overset{(*,4n)}{\to} S$ factors through the inclusion $T_S' s_0 \hookrightarrow S$ of the orbit $T_S' s_0$ for $n \gg 0$. Furthermore, $\overleftarrow{y_0}(T_X' x_0) = T_U' u_0$, $\overrightarrow{v_0}(T_U' u_0) = T_Y' y_0$, $\overrightarrow{x_0}(T_Y' y_0) = T_V' v_0$, $\overleftarrow{u_0}(T_V' v_0) = T_X' x_0$.

5.1.2.b Corollary. *If* $\dim \mathrm{Im}(X \overset{(*,4)}{\to} X) = \dim X$, *then* $(\pi_X)_{red} : X_{red} \to T_X$ ***is** biholomorphic.*

Proof. By 2.2.1, $\dim X \geq d_0(X) \geq \dim$ ***A(X),*** and by 5.1.1.(i) and 2.3.1, $\dim$ ***A(X)*** $\geq \dim X$. Thus $X_{red} = A(X)x_0$ and $\dim X = l_f$, and we conclude that $(\pi_X)_{red}$ is locally biholomorphic, and hence biholomorphic, since F_X is connected. □

Recall now the diagram

$$
\begin{array}{ccccc}
X \times Y & & \overset{\overline{P}}{\to} & & \overline{X} \times \overline{Y} \\
& \searrow_{\mathrm{id} \times p} & & \nearrow_{p \times \mathrm{id}} & \\
& & X \rtimes Y & & \\
\downarrow f & & \downarrow f & & \downarrow \overline{f} \\
& & U \rtimes V & & \\
& \nearrow & & \searrow & \\
U \times V & & \overline{Q} & & \overline{U} \times \overline{V}
\end{array}
$$

that was constructed in 3.4.3.

5.1.3 Lemma. *There exist finite holomorphic maps* $g, h : \overline{X} \to X$ *such that*

(i) the Stein factorization of $X \overset{(*,4)}{\to} X$ *is given by* $(X \overset{(*,4)}{\to} X) = g \circ p$ *and*

(ii) the Stein factorization of $\overleftarrow{v_0}\,\overleftarrow{x_0}\,\overrightarrow{u_0}\,\overrightarrow{y_0} : X \to X$ *is given by* $\overleftarrow{v_0}\,\overleftarrow{x_0}\,\overrightarrow{u_0}\,\overrightarrow{y_0} = h \circ p$

Proof. Consider the following commutative diagrams derived from the above one:

$$
\begin{array}{ccccc}
X & \overset{(*,2)}{\to} & Y & \overset{(*,2)}{\to} & X \\
\| & & \downarrow p & \nearrow & \| \\
X & \to & Y & \to & X \\
\downarrow p & \nearrow & \| & & \\
\overline{X} & \to & \overline{Y} & &
\end{array}
$$

$$\begin{array}{ccccc}
X & \overset{\vec{u}_0\,\vec{y}_0}{\longrightarrow} & Y & \overset{\overleftarrow{v_0}\,\overleftarrow{x_0}}{\longrightarrow} & X \\
\| & & \downarrow p & \nearrow & \| \\
X & \to & Y & \to & X \\
\downarrow p & \nearrow & \| & & \\
\overline{X} & \to & \overline{Y} & &
\end{array}$$

By 3.4.1.b(iii), the diagonal arrows $\overline{X} \to Y$ are finite for both diagrams; moreover, by 3.4.2.a(iii), the diagonal arrows $Y \to X$ are finite on every $(p)(A(Y)y)$. Now, by 5.1.1.(i) and 2.3.1, the image of X under (∗2) is contained (set-theoretically) in the orbit $A(Y)\,y_0$, and, for symmetry reasons, $(\overleftarrow{u_0}\ \overleftarrow{x_0})(X) \subset A(Y)\,y_0$ as well. Tbus, if we denote by g resp. h the composite of the diagonal arrows in the corresponding diagram, the assertion is proven, since $|p|$ is a quotient map with connected fibres. ◊

5.13.a Corollary. *If* dim $\overline{X}$ = dim X, *then* $(\pi_X)_{red}$ *is biholomorphic.*

Proof. Evident by 5.1.2.b. ◊

5.1.3.b Corollary. *If f degenerates, then so do J o f and f^{-1} (compare 3.2.1.b(i) and (ii)).*

Proof. Evident by 5.1.3 and 3.2.1.b(i). ◊

Let now $(\pi_R : R \to T_R) \in \mathcal{T}_l$ be as in 5.1.2 (where $R \in \{X, Y, U, V\}$), with corresponding $T'_R \sqsubset A(R)$ (compare 2.4.1 .a(iii)).

5.1.3.c Corollary. *f is $T'_X \times T'_Y - T'_U \times T'_V$ -equivariant.*

Proof. The group isomorphism $f_* : A(X) \times A(Y) \to A(U) \times A(V)$ is given by a matrix $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ with inverse $\begin{pmatrix} \alpha' & \beta' \\ \gamma' & \delta' \end{pmatrix}$. From 5.1.2.a, we infer $\alpha(T'_X) = T'_U$ and $\delta(T'_Y) = T'_V$; obviously, it remains only to show that $\beta(T'_Y) \subset T'_U$ and $\gamma(T'_X) \subset T'_V$. Applying 5.1.2 and 5.1.2.a to J o f_*, we obtain subgroups $T''_R \sqsubset A(R)$ (where $R \in \{X, Y, U, V\}$), with T''_X = Im $(\alpha'\beta\delta'\gamma)^n$ for $n \gg 0$ and $T''_V = \gamma(T''_X), T''_Y = \delta'(T''_V), T''_U = \beta(T''_Y), T''_X = \alpha'(T''_U)$. Now, $\gamma'\alpha + \delta'\gamma = 0$ and $\alpha'\beta + \beta'\delta = 0$, whence $\alpha'\beta\delta'\gamma = \beta'\delta\gamma'\alpha$. **We** conclude that $T''_X = T'_X$, whence, for symmeuy reasons, $T''_R = T'_R$ in all other cases, and the assertion follows. ◊

In general, however, f need not be a $\mathcal{T}$-morphism between $\pi_X \times \pi_Y$ and $\pi_U \times \pi_V$:

5.1.4.a Example. Let T be a one-dimensional torus, and let X = Y = U = $V = T \times P_2$, where $P_2 \hookrightarrow \mathbb{C}$ denotes the double point. Defining $f : X \times Y \to U \times V$ by $f((s,x),(t,y)) := ((2s+t+xy,x),(s+t+xy,y))$, one easily checks that $(\pi_R : R \to T_R) = (p_T : T \times P_2 \to T)$ for all factors R. Thus f is not fibre-preserving with respect to $\pi_X \times \pi_Y, \pi_U \times \pi_V$.

Fortunately, nothing of this kind can happen in the reduced case:

5.1.4 Lemma. *f is a $\mathcal{T}$-morphism $\pi_X \times \pi_Y \to \pi_U \times \pi_V$, if one of the following conditions is fulfilled:*

(i) $\overline{f} : \pi_{\overline{X}} \times \pi_{\overline{Y}} \to \pi_{\overline{U}} \times \pi_{\overline{V}}$.

(ii) π_X is biholomorphic.

(iii) X is reduced.

Proof. By 5.1.3.c, we need only show that f is fibre-preserving.

(i) Let $\lambda_S : S \to \overline{S}$ be the canonical projection (i.e. $\lambda_X = |p|$ etc.). As $S \overset{(*.4)}{\to} S$ factors through λ_S, the construction of π_S immediately yields $\pi_S = \pi_{\overline{S}} \circ \lambda_S$. Thus, if $(\pi_{\overline{U}} \times \pi_{\overline{V}}) \circ \overline{f} = \overline{f}_0 \circ (\pi_{\overline{X}} \times \pi_{\overline{Y}})$ with a suitable holomorphic $\overline{f}_0 : T_{\overline{X}} \times T_{\overline{Y}} \to T_{\overline{U}} \times T_{\overline{V}}$, then $(\pi_U \times \pi_V) \circ f = (\pi_{\overline{U}} \times \pi_{\overline{V}}) \circ (\lambda_U \times \lambda_V) \circ f = (\pi_{\overline{U}} \times \pi_{\overline{V}}) \circ \overline{f} \circ (\lambda_X \times \lambda_Y) = \overline{f}_0 \circ (\pi_{\overline{X}} \times \pi_{\overline{Y}}) \circ (\lambda_X \times \lambda_Y) = \overline{f}_0 \circ (\pi_X \times \pi_Y)$, i.e. f is fibre-preserving.

(ii) Let $S \in \{U, V\}$. Every composition $X \to S \to Y$ of partial maps is an immersion of the form $T_X \to T'_Y$ y with suitable y; therefore every $Y \to S \to X$ factors through $\pi_Y : Y \to T_Y$. We conclude that $\overleftarrow{x}$ resp. $\overrightarrow{x}$ maps every fibre of π_Y into one of π_U resp. π_V; in other words, $lf(\pi_X^{-1}\pi_X(x) \times \pi_Y^{-1}\pi_Y(y)) = lf(\{x\} \times \pi_Y^{-1}\pi_Y(y)) \subset \pi_U^{-1}\pi_U(lf(x,y))$, and $rf(\pi_X^{-1}\pi_X(x) \times \pi_Y^{-1}\pi_Y(y)) \subset \pi_V^{-1}\pi_V(rf(x,y))$.

Assertion (iii) follows from 5.1.3.a and from (i) and (ii) by induction on dim X − dim T_X, since $T_{\overline{X}} = T_X$. ◊

5.1.5 Theorem. *Let $f : X \times Y \to U \times V$ be a biholomorphic map between connected complex spaces with X compact.*

If f degenerates, rhen every (x, y) ∈ X x Y decomposes f.

In particular, f induces a simultaneous subdecomposition, if $\{X, Y, U, V\} \not\subset \mathcal{T}_k$ for all $k \geq 1$.

Proof. We proceed by induction on dim X, noting that the case dim X = 0 has been settled in Chapter 4.

Let dim X $\geq$ 1. Then dim $\overline{X}$ < dim X by Corollary 5.1.3.a, and $\overline{f}$ degenerates by 3.4.3.(i). Thus, by induction hypothesis, every $(\bar{x}, \bar{y}) \in \overline{X} \times \overline{Y}$ decomposes $\overline{f}$, and the assertion follows from 3.4.3.(ii). ◊

5.1.5.a Corollary. *If f is a $\mathcal{T}$-morphism $\pi_X \times \pi_Y \to \pi_U \times \pi_V$ (e.g. if X is reduced), then the corresponding isomorphism $F_X \times F_Y \to F_U \times F_V$ between the jibres induces a simultaneous subdecomposition.*

Proof. Clearly, $F_X \times F_Y \to F_U \times F_V$ degenerates. ◻

Note that in Example 5.1.4.a, there still exists some isomorphism $F_X \times F_Y \to F_U \times F_V$ that induces a simultaneous subdecomposition. It would be (mildly) interesting, whether at least this statement remains true in general.

5.2. Cancellation

5.2.1 Theorem. *Let $g : X \times Y \to X \times Z$ be a biholomorphic map between connected complex spaces, and assume that X, Y or Z is compact.*

If $\{X, Y, Z\} \not\subset \mathcal{T}_k$ for all $k \geq 1$, then $Y \cong Z$.

Proof. We may assume that X is indecomposable. Then the assertion follows from 5.1.3.b, 5.1.5 and 3.3.2.a(iv). ◻

5.2.1.a Examples. X cancels in the sense of 5.2.1, if dim X = 0, if X has vanishing first Betti number or non-vanishing Euler characteristic, if dim $A(X) = 0$ (in particular, if X admits at most countably many holomorphic automorphisms), if X is Stein, etc. Further examples (with X compact and reduced) can be found in ([5], 1.3).

Conversely, G. Parigi has shown that for any $X \in \mathcal{T}$ there exist non-isomorphic Y, Z with $X \times Y \cong X \times Z$ (see [11]; he states this fact for compact reduced X only, but his proof is easily seen to work for general X as well).

An interesting question arising in this context is the following: If $X \times Y \cong X \times Z$, what is the relation between Y and Z ?

In view of Example 5.1.4.a, it seems reasonable to restrict one's attention at first to the reduced case, where one can find at least some structural similarity. By 5.1.2, 5.1.4 and 5.1.5.a, we obtain then commutative diagrams

$$\begin{array}{ccc} X \times Y & \overset{\cong}{\to} & X \times Z \\ \downarrow{\scriptstyle \pi_X \times \pi_Y} & & \downarrow{\scriptstyle \tau_X \times \pi_Z} \\ T_X \times T_Y & \overset{\cong}{\to} & \tilde{T}_X \times T_Z \end{array}$$

$$\begin{array}{ccc} (F \times F_1) \times (F'' \times F_1') & \overset{\cong}{\to} & (F \times F_1'') \times (F'' \times F_1) \\ \downarrow{\scriptstyle \cong} & & \downarrow{\scriptstyle \cong} \\ \pi_X^{-1}\pi_X(x) \times \pi_Y^{-1}\pi_Y(y) & \overset{\cong}{\to} & \tau_X^{-1}\tau_X(x') \times \pi_Z^{-1}\pi_Z(z) \end{array}$$

However, there is no reason for $\pi_X^{-1}\pi_X(x)$ and $\tau_X^{-1}\tau_X(x')$ to be isomorphic. Thus we are faced with a much more difficult question than the decomposition problem, namely:

Given $(\pi_j : X \to T_j) \in \mathscr{T}_k$ with fibre $X_j, j = 1, 2$, such that there exists a $\mathscr{T}$-morphism $h : \pi_1 \to \pi_2$ with $h_*(T_1') = T_2'$, what is the relation between X, and X_2 ?

In Chapter 7, at least a necessary condition for Y, Z to satisfy $X \times Y \cong X \times Z$ with suitable X will be given.

A more restricted version of the cancellation problem is the question of whether $X \times X \cong X \times Y$ implies $X \cong Y$. No counterexample with compact X, Y seems to be known. Shioda proved that no counterexample with tori X, Y can exist ([12]). Parigi's varieties $Y \ncong Z$ with $X \times Y \cong X \times Z$ satisfy by construction $Y \ncong X \ncong Z$.

5.3. Decomposition with respect to $\mathscr{P}$-categories

Denote by $\mathscr{C}$ the category of all compact connected complex spaces.

53.1 Definition. A subcategory $\mathscr{K} \subset \mathscr{C}$ is a $\mathscr{P}$-category, if it has the following property: $X \times Y \in \mathscr{K}$ if and only if $X, Y \in \mathscr{K}$.

53.1.a Remarks and Examples.

(i) $\mathbb{C}^0$ lies in every non-empty $\mathscr{P}$-category. The intersection of $\mathscr{P}$-categories is a $\mathscr{P}$-category.

(ii) Each of the following is a $\mathscr{P}$-category: $\mathscr{C}$, $\mathscr{C}_{\text{red}}$:= {X $\in \mathscr{C}$: X = X,}, $\mathscr{C}_0$:= {X $\in \mathscr{C}$: dim X = 0}, $\mathscr{C} \setminus \mathscr{T}$ (see 2.4.2.b), {X $\in \mathscr{C}$: X projective), {X $\in \mathscr{C}$: X Moisezon}, {X $\in \mathscr{C}$: trdeg $\mathscr{M}(X)$ = 0}, {tori}.

53.2 Theorem. *Let U be a connected complex space, and let $\mathscr{K} \subset \mathscr{C} \setminus \mathscr{T}$ be a $\mathscr{P}$-category.*

There exists a unique decomposition $U \cong U_{\mathscr{K}} \times U'$ with $U_{\mathscr{K}} \in \mathscr{K}$ such that U' has no factor in $\mathscr{K} \setminus \{\mathbb{C}^0\}$.

If $f = (lf, rf) : U_{\mathscr{K}} \times U' \to U_{\mathscr{K}} \times U'$ is biholomorphic, then every partial map $lf^j(\cdot, u')$, $rf^j(u, \cdot)$ (where $j = \pm 1$) is biholomorphic, and every composition $(lf^j(u, \cdot) \circ rf^{-j}(\cdot, u'))^n$ is constantfor n sufficiently large.

Proof. Let $f : U_{\mathscr{K}} \times U' \to U^1_{\mathscr{K}} \times U'_1$ be biholomorphic, where $U_{\mathscr{K}}, U^1_{\mathscr{K}} \in \mathscr{K}$, such that U', U'_1 have no factor in $\mathscr{K} \setminus \{\mathbb{C}^0\}$. f degenerates with respect to every $(u, u') \in U_{\mathscr{K}} \times U'$, since $\mathscr{K} \subset \mathscr{C} \setminus \mathscr{T}$. Therefore, every (u, u') decomposes f, and hence gives rise to a commutative diagram

$$\begin{array}{ccc} (F_1^1 \times F_{-1}^1) \times (F_{-1}^{-1} \times F_1^{-1}) & \to & (F_1^1 \times F_1^{-1}) \times (F_{-1}^{-1} \times F_{-1}^1) \\ \downarrow & & \downarrow \\ U_\kappa \times U' & \to & U_\kappa^1 \times U_1' \end{array}$$

according to 3.3.2 (we choose this new notation, in order to avoid e.g. $U^1_{\mathscr{K}}$ appearing as an index; moreover, we do not distinguish between the subfactors that are biholomorphically correlated by 3.3.2.(iii)).

If $lf(.,u')$ were not biholomorphic, then $F^1_{-1} \in \mathscr{K} \setminus \{\mathbf{C}^0\}$ or $F^{-1}_1 \in \mathscr{K} \setminus \{\mathbf{C}^0\}$, whence U'_1 or U' would admit a factor in $\mathscr{K} \setminus \{\mathbf{C}^0\}$.

Thus all $lf(.,u')$ and, symmetrically, all $lf^{-1}(.,u'_1)$ are biholomorphic, whence, by Lemma 3.1.1.(iii), so are all $rf^{-1}(u_1,.)$, $rf(.,u)$.

The theorem is now completely proven, since, in particular, $U_{\mathscr{K}} \cong F^1_1 \cong U^1_{\mathscr{K}}$ and $U' \cong F^{-1}_{-1} \cong U'_1$ (compare 3.3.2.(i)). ◊

53.3 Lemma. *Let $f : X \times Y \to U \times V$ be an isomorphism in $\mathscr{C}$, and assume that $X \neq \mathbf{C}^0$ is indecomposable and not contained in $\mathscr{T}$.*

There exists a unique $S \in \{U, V\}$ with $S \cong X \times S_0$ such that the resulting isomorphism $\bar{f} : X \times Y \to X \times (S_0 \times S')$ (where $\{S, S'\} = \{U, V\}$) satisfies: Every partial map $l\bar{f}^j(.,b)$, $r\bar{f}^j(x,.)$ is biholomorphic, and every $(l\bar{f}^{-j}(x,.) \circ r\bar{f}^j(.,b))^n$ is constant for $n > 0$ (where $b \in Y$ or $b \in S_0 \times S'$, according as $j = 1$ or $j = -1$).

Proof. Fix some $(x_0, y_0) \in X \times Y$ and consider the diagram corresponding to the simultaneous subdecomposition given by (x_0, y_0) (note that f degenerates):

$$\begin{array}{ccc} (X_U \times X_V) \times (Y_U \times Y_V) & \to & (U_X \times U_Y) \times (V_X \times V_Y) \\ \downarrow & & \Downarrow \\ X \times Y & \to & U \times V \end{array}$$

We may assume that $X = X_U$, since X is indecomposable; denote by $\bar{f}$ the resulting isomorphism $X \times Y = X_U \times Y \to U_X \times (U_Y \times V_Y) \cong X \times (U_Y \times V_Y)$. Then $l\bar{f}(.,y_0)$ and $l\bar{f}^{-1}(.,r\bar{f}(x_0,y_0))$ are biholomorphic by 3.3.2.(iii). As Aut (X) is open in Hol (X), the holomorphic maps $Y \ni y \mapsto l\bar{f}(.,y)$, $U_Y \times V_Y \ni (u,v) \mapsto l\bar{f}^{-1}(.,(u,v))$ ***both*** have their image in Aut(X). Thus all $l\bar{f}(.,y), l\bar{f}^{-1}(.,(u,v))$ are biholomorphic, whence, by 3.1.1, so are all $r\bar{f}^{-1}(x,.)$, $r\bar{f}(x,.)$. Now X is indecomposable and (x_0, y_0) decomposes $\bar{f}$; therefore (compare 3.3.2.(i), (ii)) all $(l\bar{f}^j(x,.) \circ r\bar{f}^{-j}(.,b))^n$ become constant for n sufficiently large.

Assume now that in addition $V = V_Y \cong X \times V_0$ with all the postulated properties for the resulting isomorphism $\hat{f} : X \times Y \to X \times (U \times V_0) = X \times (X \times U_Y \times V_0)$. Fix some $(u,v) \in U_Y \times V_0$, and let $\phi := l\hat{f}^{-1}(.,(.,u,v)) : X \times X \to X$. By construction, both

$\phi(x, .)$ and $\phi(., x)$ are contained in $A(X)$ for all $x \in X$, which can only happen, if every orbit map $\cdot x : A(X) \to X_{red}$ is biholomorphic and if X is reduced in every smooth point of X_{red} (see 1.1.2.b). Thus $\mathbf{C}^q \neq X \cong A(X)$ in contradiction to $X \notin \mathcal{T}$. ◊

5.3.4 Theorem. *Let* $X \in \mathcal{C} \setminus \mathcal{T}$.

Then X admits a unique decomposition (up to reordering) $X \cong X_1' \times \ldots \times X_l'$ ***such that*** $X_\lambda' \cong Y_\lambda^{n_\lambda}$ ***with*** $n_\lambda \geq 1$ ***and*** $X_\lambda \neq \mathbf{C}^q$ ***indecomposable and pairwise non-isomorphic for*** $1 \leq \lambda \leq l$.

If $f \in \mathrm{Aut}(X)$***, then every partial map*** $X_\lambda' \to X_\lambda'$ ***given by*** f ***or*** f^{-1} ***is biholomorphic, and every composition of partial maps*** $\left(X_{\lambda'}' \to \prod_{\lambda \neq \lambda'} X_\lambda' \to X_{\lambda'}'\right)^n$ ***is constant for n*** $\gg 0$.

Moreover, there exist permutations σ_λ ***of*** $\{1, \ldots, n_\lambda\}$ ***such that***

$$\bar{f} := (J_{\sigma_l} \times \ldots \times J_{\sigma_1}) \circ f : X_{1,1} \times \ldots \times X_{1,n_1} \times \ldots \times X_{l,1} \times \ldots \times X_{l,n_l} \to X_{1,1} \times \ldots \times X_{l,n_l}$$

(where $X_{\lambda,\nu} = X_\lambda$***) satisfies: All partial maps*** $X_{\lambda,\nu} \to X_{\lambda,\nu}$ ***given by*** $\bar{f}$ ***or*** $\bar{f}^{-1}$ ***are biholomorphic, and all compositions*** $\left(X_{\lambda',\nu'} \to \prod_{(\lambda,\nu)\neq(\lambda',\nu')} X_{\lambda,\nu} \to X_{\lambda',\nu'}\right)^n$ ***are constant for*** $n \gg 0$.

Proof. Evident by Lemma 5.3.3. ◊

5.3.4.a Let now $\mathcal{K} := \mathcal{C} \setminus \mathcal{T}$, $U_c := U_{\mathcal{K}}$ and U' according to 5.3.2, with $U_c = X_1' \times \ldots \times X_l' = X_1^{n_1} \times \ldots \times X_l^{n_l}$ according to 5.3.4. Every isomorphism $U \cong X_1^{n_1} \times \ldots \times X_l^{n_l} \times U'$ will be called a *standard* ***decomposition*** of U.

5.4. Some Examples

Let p, q with $p \neq q$ be primes, and let ***A,*** B be connected complex spaces such that $\mathbf{Z}_p$ acts non-trivially on ***A*** and $\mathbf{Z}_q$ acts non-trivially on B. Fix some generators $\alpha \in \mathbf{Z}_p$, $\beta \in \mathbf{Z}_q$, and let $T := \mathbf{C}/\mathbf{Z} + i\mathbf{Z}$.

For $1 \leq r \leq p-1, 1 \leq s \leq q-1$ define $\alpha_r \in \mathrm{Aut}(T \times A)$ by $\alpha_r(t, a) := \left(t + \frac{1}{p}, \alpha^r(a)\right)$, $\beta_s \in \mathrm{Aut}(T \times B)$ by $\beta_s(t, b) := \left(s + \frac{1}{q}, \beta^s(b)\right)$, and let $\gamma \in Aut(T \times A \times B)$ be given by $\gamma(t, a, b) := \left(t + \frac{1}{pq}, \alpha(a), \beta(b)\right)$. Then the quotients $A_r := (T \times A)/\alpha_r$, $B_s := (T \times B)/\beta_s$, $AB := (T \times A \times B)/\gamma$ are total spaces of torsion bundles over $T/\left(\frac{1}{p}\right)$, $T/\left(\frac{1}{q}\right)$, $T/\left(\frac{1}{pq}\right)$, respectively. Evidently, $A_r \cong A_{p-r}$ via $t \mapsto -t$.

5.4.1 Lemma.

(i) $T \times A_r \cong T \times A_{r'}$ *for all* $r, r' \in \{1; \ldots, p-1\}$.

(ii) $A_r \times B_s \cong A_r \times B_{s'}$ *for all* $r \in \{1, \ldots, p-1\}$, *and* $s, s' \in \{1, \ldots, q-1\}$.

(iii) $A_r \times A_r \cong A_{r'} \times A_{r'}$ *if* $r^2 \equiv \pm r'^2$ (mod p).

(iv) Assume that $p = 2$, $q = 3$, *and let C, D be connected complex spaces with non-trivial* $\mathbf{Z}_2$*-resp.* $\mathbf{Z}_3$*-action. Then* $AB \times CD \cong AD \times CB$.

Proof. Let $\Phi : T \times T \to T \times T$ be given by the matrix

(i) $\begin{pmatrix} \lambda & p \\ \lambda' & p+\rho \end{pmatrix}$, where $r'\rho \equiv r$ (mod p), and $(\lambda - \lambda')p + \lambda\rho = 1$,

(ii) $\begin{pmatrix} \lambda p + 1 & \mu q \\ \lambda p & \mu q + \rho \end{pmatrix}$, where $s'p \equiv s$ (mod q), and $\lambda p\rho + \mu q = 1 - \rho$,

(iii) $\begin{pmatrix} \rho & \lambda p \\ p & \mu\rho \end{pmatrix}$, where $r' = \rho r$ and $\mu\rho^2 = \lambda p^2 \pm 1$,

(iv) $\begin{pmatrix} 3 & 4 \\ 16 & 21 \end{pmatrix}$.

Then

(i) $\Phi \times \mathrm{id}$, (ii) $\Phi \times \mathrm{id}$,,,

(iii) $\Phi \times \mathrm{id}$,,, (iv) $\Phi \times \mathrm{id}_{A \times B \times C \times D}$

induces an isomorphism as postulated. □

From now on assume that

(1) A and B are indecomposable,

(2) there exists no non-constant holomorphic $A \times B \to T$,

(3) T does not act non-trivially on $A/\mathbf{Z}_p \times B/\mathbf{Z}_q$,

(4) every composition of holomorphic maps $(A \to B \to A)^n$ is constant for $n \gg 0$.

5.4.2 Lemma.

(i) Every A_r *is indecomposable.*

(ii) AB has no non-trivial compact factor. If in addition A or B is compact, then AB is indecomposable.

(iii) $A_r \cong A_{r'}$ *if and only if there exists* $\gamma \in \mathrm{Aut}(A)$ *with* $\gamma \circ \alpha^r = \alpha^{\pm r'} \circ \gamma$*. In particular, if* $\mathbf{Z}_p$ *is central in* Aut (A)*, then* $A_r \not\cong A_{r'}$ *for* $r \not\equiv \pm r'$ (mod p).

Proof. Let $S \in \{A_r, AB\}$ with $S \cong S_1 \times S_2$. By 2.4.2.b, we may assume that there exists $(S \xrightarrow{\pi_1} T_1) \in \mathcal{T}$ with some fibre S_1' and with T_1 isogenous to T. By (2) and (3), every isomorphisms $S \to S_1 \times S_2$ is a $\mathcal{T}$-morphism $\pi \to \pi_1 \circ p_{S_1}$ (where π denotes the given torsion bundle $A_r \to T/\mathbf{Z}_p$ resp. $AB \to T/\mathbf{Z}_{pq}$).

(i) Clearly, A_r cannot be $\mathcal{T}$-isomorphic to $T \times A$. Thus $S_1' \neq \mathbf{C}^0$, and we conclude that $S_2 = \mathbf{C}^0$, since $A \cong S_1' \times S_2$ is indecomposable.

(ii) Again, AB is not $\mathcal{T}$-isomorphic to $T \times A \times B$, whence $S_1' \neq \mathbf{C}^0$. By (4), the isomorphism $A \times B \to S_1' \times S_2$ between the fibres degenerates and therefore induces a simultaneous subdecomposition, if A, B, S_1 or S_2 is compact (see 5.1.5). Denote this isomorphism by f, and assume that S_1 or S_2 is compact with $S_2 \neq \mathbf{C}^0$. Then either all partial maps $rf(a, .) : B \to S_2$ or all $rf(., b) : A \to S_2$ are biholomorphic by (1). On the other hand, it is evident that $rf(\alpha(a), \beta(b)) = rf(a, b)$ for all (a, b); in particular, $rf(a, \beta^2(b)) = rf(a, b) = rf(\alpha^3(a), b)$, a contradiction. Thus $S_2 = \mathbf{C}^0$.

Assertion (iii) is obvious, since every $A_r \xrightarrow{\cong} A_r$ is a $\mathcal{T}$-morphism. ◊

For $k \geq 2$ let $\varepsilon_k := \exp\left(\frac{2\pi i}{k}\right)$ and let $\mathbf{Z}_k$ act on $\mathbf{P}_n$ via $(\kappa, x) \mapsto (\varepsilon_k^\kappa x_0 : x_1 : \ldots : x_n)$. If we want to indicate this action, we let $Z(k) := \mathbf{P}_1$ in what follows. By *blowing up* $x \in \mathbf{P}_2$ $l+1$ *times*, where $l \geq 1$, we mean blow up x l times and then blow up (once) any point in the exceptional curve.

Let X(k) be the manifold that arises from $\mathbf{P}_2$ by blowing up (once) every $(\varepsilon_k^\kappa x_0 : 1 : 0)$, $1 \leq \kappa \leq k$, by blowing up $l+2$ times the points $(0 : 1 : 1)$ for $0 \leq l \leq 2$ and by blowing up five times the point $(1 : 0 : 0)$. The $\mathbf{Z}_k$-action on $\mathbf{P}_2$ lifts to X(k) and also restricts to the complement $U(k) \subset X(k)$ of the inverse image of $(1 : 0 : 0)$. It is easy to see that $\mathbf{Z}_k = \mathrm{Aut}(X(k))$ and $\mathbf{Z}_k = \mathrm{Aut}(U(k))$. Thus, by 5.4.2.(iii), $A(p)_r \cong A(p)_{r'}$ (where $A \in \{X, U\}$), if and only if $r \equiv r'$ (mod p).

Clearly, every pair $(A(p), B(q))$ with $A, B \in \{U, X, Z\}$ satisfies the conditions (1) - (4).

5.43. Examples.

a) There exist indecomposable connected complex spaces X, U, U' with X compact, and with U, U' having no compact factor $\neq \mathbf{C}^0$, such that $U \ncong U'$ and $X \times U \cong X \times U'$: $X := T$, $U := U(q)_1$, $U' := U(q)_2$ with $q \geq 5$ (see 5.4.1.(i), 5.4.2.(i)).

b) There exist indecomposable connected complex spaces X, X', U with X, X' compact, and with U having non compact factor $\neq \mathbf{C}^0$, such that $X \ncong X'$ and $X \times U \cong X' \times U$: $U := U(q)$, $X := X(p)_1$, $X' := X(p)_2$ with $p \geq 5$ (see 5.4.1.(ii), 5.4.2.(i)).

c) There exist indecomposable connected complex spaces X, X', U, U' with X, X' compact, and with U, U' admitting no compact factor $\neq \mathbf{C}^0$, such that $X \not\cong X'$, $U \not\cong U'$, and $X \times X \cong X' \times X', U \times U \cong U' \times U'$:

$X := X(5)_1, X' := X(5)_2, U := U(5)_1, U' := U(5)_2$ (see 5.4.1.(iii), 5.4.2.(i)).

d) There exist connected complex spaces X, U, V, W, with $X \neq \mathbf{C}^0$ indecomposable and compact, and with U, V, W having no compact factor $\neq \mathbf{C}^0$, such that $X \times U \cong V \times W$: $X := X(2)Z(3), U := U(2)U(3), V := X(2)U(3), W := U(2)Z(3)$ (see 5.4.1.(iv), 5.4.2.(ii)).

e) There exist X, Y, U, V with $X \neq \mathbf{C}^0 \neq Y$ compact, and with U, V admitting no compact factor $\neq \mathbf{C}^0$, such that $\dim X \neq \dim Y$ and $X \times U \cong Y \times V$:

$X := X(2)Z(3), Y := X(2)X(3), U := U(2)X(3), V := U(2)Z(3)$ (see 5.4.1.(iv), 5.4.2.(ii)).

In particular, we see that for general U, there is no possibility of introducing a reasonable notion of a unique maximal compact factor.

Choosing A, B, C appropriately, one can show in a similar way that a general $X \in \mathscr{C}$ does not admit a unique maximal factor in any of the $\mathscr{P}$-categories listed in 5.3.1.a(ii) other than $\mathscr{C}_0$ or $\mathscr{C} \setminus \mathscr{T}$.

6. AUTOMORPHISMS OF PRODUCTS

Let U be a connected complex space with standard decomposition $U \cong U_d \times U' \cong X_1' \times \dots \times X_l' \times U'$ (compare 5.3.4.a), and let $\phi \in \mathrm{Aut}(U)$. By 5.3.2 and 5.3.4, every partial map $U_c \to U_c, X_\lambda' \to X_\lambda', U' \to U'$, given by ϕ or ϕ^{-1} is biholomorphic. In general, however. ϕ need not be a product of isomorphisms between the individual factors. For every ϕ to be a product of automorphisms of U_c and U', it is necessary that there exist no non-constant holomorphic mappings $U' \to \mathrm{Aut}(U_c)$, $U_d \to \mathrm{Aut}(U')$. In the reduced case, this condition is easily seen to be sufficient as wcll; in general, it is not.

If U is reduced and compact with $A(U) = 0$, then evidently all $\phi \in \mathrm{Aut}(U)$ are products of isomorphisms between the indecomposable factors of U. This assertion does no longer hold for non-reduced U; for instance, the automorphism of $P_2 \times P_2$ ($P_2 \hookrightarrow C$ the double point) given by $(x, y) \mapsto (x + xy, y + sy)$ is not a product.

In view of these difficulties, wc henceforth restrict our attention to the compact reduced case.

6.1. Decomposition-preserving automorphisms

Let X be a reduced compact complex space with a decomposition $f: X \to Y_1 \times \dots Y_n$.

6.1.1. Definition. An automorphism ϕ of X *preserves the decomposition f*, if all partial maps $Y_\nu \to Y_\nu$ ($1 \leq \nu \leq n$) given by ϕ and ϕ^{-1} are biholomorphic. We let $\mathrm{Aut}_f(X) := \{\phi \in \mathrm{Aut}(X) : \phi \text{ preserves } f\}$.

6.1.1.a Remarks. From 5.3.2 and 5.3.4, we infer:

(i) If Y_ν and Y_μ have no positive-dimensional common factor for all $1 \le \mu, \nu \le n$, $\mu \ne \nu$, and if at most one Y_ν is contained in $\mathscr{T}$, then Aut (X) = Aut_f(X) .

(ii) If $Y_1 \cong \ldots \cong Y_n \notin \mathscr{T}$ are indecomposable, then Aut (X) = $\bigcup_{\sigma \in S(n)} J_\sigma \circ \mathrm{Aut}_f(X)$,

where $\mathscr{S}(n)$ denotes the group of all permutations of {1, . . . , n}) .

We shall now – in the case n = 2 – demonstrate how to construct Aut_f(X) from $\mathrm{Aut}(Y_1) \times \mathrm{Aut}(Y_2)$. Then, using the above remarks, one can build up successively Aut (X) from $\prod_{\lambda=1}^{l} \mathrm{Aut}(X_\lambda)^{n_\lambda} \times \mathrm{Aut}(X')$, where $X \cong \left(\prod_{\lambda=1}^{l} X_\lambda^{n_\lambda}\right) \times X'$ is a standard decomposition of x .

To simplify the notation, we consider reduced compact complex spaces Y, Z with $Y \notin \mathscr{T}$, and we let $\mathrm{Aut}^+(Y \times Z) := \mathrm{Aut}_{\mathrm{id}_{Y\times Z}}(Y \times Z)$. Then, by 5.1.2, every $\phi \in \mathrm{Aut}^+(Y \times Z)$ degenerates.

Let $\phi \in \mathrm{Aut}^+(Y \times Z)$, and fix some $(y_0, z_0) \in Y \times Z$. By Theorem 5.3.2, there exist $(\alpha, \delta) \in \mathrm{Aut}(Y) \times \mathrm{Aut}(Z)$, and $\beta' \in \mathrm{Hol}(Z, A(Y)), \gamma' \in \mathrm{Hol}(Y, A(Z))$ with $\beta'(z_0) = \mathrm{id}_Y, \gamma'(y_0) = \mathrm{id}_Z$, such that $\phi(y,z) = (\beta'(z)(\alpha(y)), \gamma'(y)(\delta(z)))$ for all (y, z). As $A(Y \times Z)$ is normal in $\mathrm{Aut}(Y \times Z)$, there exist $\beta \in \mathrm{Hol}(Z, A(Y))$ with $\beta(z_0) = id_Y$ and $\gamma \in \mathrm{Hol}(Y, A(Z))$ with $\gamma(y_0) = id_Z$ such that $\alpha \circ \beta(z) = \beta'(z) \circ \alpha$ and $\delta \circ \gamma(y) = \gamma'(y) \circ \delta$ for all $y \in Y$, $z \in Z$. Evidently, the quadruple $(\alpha, \beta, \gamma, \delta)$ is uniquely determined by these properties.

We shall now derive a necessary and sufficient criterion for such a quadruple $(\alpha, \beta, \gamma, \delta)$ to define $\phi \in \mathrm{Aut}^+(Y \times Z)$ in the way described above. For $(\beta, \gamma) \in \mathrm{Hol}(Z, A(Y)) \times Hol(Y, A(Z))$ define $\langle \beta, \gamma \rangle : Y \times Z \to Y \times Z$ by $(y, z) \mapsto (\beta(z)(y), \gamma(y)(z))$. Evidently, it suffices to find out under which conditions $\langle \beta, \gamma \rangle \in \mathrm{Aut}(Y \times Z)$.

To begin with, we reduce the situauon to the case where Y, Z are tori:

6.1.2. Lemma and Definition. *The functor* $\mathscr{C}_{red} \to \mathscr{E}ns, Z \mapsto \cup\{\mathrm{Hol}(Z, T) : T$ *a torus*$\}$, is *represented by* $\mathrm{alb}^0 : Z \mapsto (\mathrm{alb}^0_Z : Z \to \mathrm{Alb}^0(Z))$.

alb^0_Z *is called the* weak Albanese map of Z.

The proof can be copied word for word from the corresponding one for smooth varieties. Note that $\mathrm{alb}^0_Z = \mathrm{alb}_Z$ if Z is smooth.

Let $(x_0, y_0) \in Y \times Z$ with $\mathrm{alb}^0(x_0, y_0) = 0$ and let $(\beta, \gamma) \in \mathrm{Hol}(Z, A(Y)) \times \mathrm{Hol}(Y, A(Z))$ with $\beta(z_0) = \mathrm{id}_Y, \gamma(y_0) = \mathrm{id}_Z$. Then $\mathrm{alb}^0(\langle \beta, \gamma \rangle) : \mathrm{Alb}^0(Y \times Z) \to \mathrm{Alb}^0(Y \times Z)$ is a holomorphic homomorphism. Moreover, if we let $\bar\beta$ be the composition $(\mathrm{Alb}^0(Z) \overset{\mathrm{alb}^0(\beta)}{\to} \mathrm{Alb}^0(A(Y)) = A(Y) \overset{\mathrm{alb}^0_*}{\to} A(\mathrm{Alb}^0(Y)) = \mathrm{Alb}a^0(Y))$, and

$\overline{\gamma} : \mathrm{Alb}^0(Y) \to \mathrm{Alb}^0(Z)$ accordingly, then $\mathrm{alb}^0(\langle\beta,\gamma\rangle) = \langle\overline{\beta},\overline{\gamma}\rangle$.

6.1.3 Lemma. *The map $\langle\beta,\gamma\rangle$ is biholomorphic, if and only if so is $\langle\overline{\beta},\overline{\gamma}\rangle$.*

Proof. Let $\langle\overline{\beta},\overline{\gamma}\rangle$ be biholomorphic (the other implication is trivial). It suffices to show that $\langle\beta,$ 7) is injective; for this, in turn, we need only show that $\langle\beta,\gamma\rangle$ is injective on every fibre of $\mathrm{alb}^0_{Y\times Z}$.

Let $Y_0 \times Z_0$ be some fibre of $\mathrm{alb}^0_{Y\times Z} = \mathrm{alb}^0_Y \times \mathrm{alb}^0_Z$. Then $\beta|_{Z_0} = [\beta(z_1)], \gamma|_{Y_0} = [\gamma(y_1)]$ for any $z_1 \in Z_0, y_1 \in Y_0$, and therefore $\langle\beta,\gamma\rangle|_{Y_0\times Z_0} = \beta(z_1) \times \gamma(y_1)|_{Y_0\times Z_0}$ is injective. ◊

Let now $\overline{Y} := \mathrm{Alb}^0(Y), \overline{Z} := \mathrm{Alb}^0(Z)$.

6.1.4 Lemma. *$\langle\overline{\beta},\overline{\gamma}\rangle$ is biholomorphic, if and only if $(\overline{\beta} \circ \overline{\gamma})^n = 0$ for n w 0.*

Proof. Let $\sigma := \overline{\beta}\overline{\gamma}, \tau := \overline{\gamma}\overline{\beta}$; then σ is nilpotent, if and only if so is τ. The homomorphism $\langle\overline{\beta},\overline{\gamma}\rangle$ is an isomorphism, if and only if there exists an endomorphism of $\overline{Y} \times \overline{Z}$ given by a matrix $\begin{pmatrix} \alpha' & \beta' \\ \gamma' & \delta' \end{pmatrix}$ such that $\begin{pmatrix} \alpha' + \beta'\overline{\gamma} & \alpha'\overline{\beta} + \beta' \\ \gamma' + \delta'\overline{\gamma} & \gamma'\overline{\beta} + \delta' \end{pmatrix} = \begin{pmatrix} id & 0 \\ 0 & id \end{pmatrix}$. If σ is nilpotent, then $id - \sigma$ and $id - \tau$ are invertible, and a simple computation shows that the matrix $\begin{pmatrix} \alpha' & \beta' \\ 7' & \delta' \end{pmatrix}$, given by $\alpha' = (id-\sigma)^{-1}, \delta' = (id-\tau)^{-1}, \beta' = -\alpha'\overline{\beta}$, 7' $= -\delta'\overline{\gamma}$ defines an inverse of $\langle\overline{\beta},\overline{\gamma}\rangle$.

Conversely, if $\langle\overline{\beta},\overline{\gamma}\rangle$ is invertible, then so is $\langle\beta,\gamma\rangle$ and $\langle\beta,\gamma\rangle$ degenerates, since $U \notin \mathscr{T}$. By 3.1.1.b, $\langle\overline{\beta},\overline{\gamma}\rangle$ degenerates as well, i.e., if $(\langle\overline{\beta},\overline{\gamma}\rangle)^{-1}$ is given by $\begin{pmatrix} \alpha' & \beta' \\ 7' & 6' \end{pmatrix}$, then $(\overline{Y} \xrightarrow{id} \overline{Y} \xrightarrow{\gamma'} \overline{Z} \xrightarrow{id} \overline{Z} \xrightarrow{\beta'} \overline{Y})^n = 0$ for $n \gg 0$. Now $\beta' = -\alpha'\overline{\beta} = -\overline{\beta}\delta', \gamma' = -\delta'\overline{\gamma} = -\overline{\gamma}\alpha'$, and we conclude that $\alpha'\alpha'\overline{\beta}\overline{\gamma} = \alpha'\overline{\beta}\delta'\overline{\gamma} = \beta'\gamma' = \overline{\beta}\delta'\overline{\gamma}\alpha' = \overline{\beta}\overline{\gamma}\alpha'\alpha'$; thus $\beta'\gamma'$ is nilpotent, if and only if so is $\overline{\beta}\overline{\gamma}$. ◊

For $(\widehat{\beta},\widehat{\gamma}) \in \mathrm{Hom}(\overline{Z}, A(Y)) \times \mathrm{Hom}(\overline{Y}, A(Z))$, let $\beta \times \gamma := (\widehat{\beta} \circ \mathrm{alb}^0_Z) \times (\widehat{\gamma} \circ \mathrm{alb}^0_Y) : Z \times Y \to A(Y) \times A(Z)$, and $\overline{\beta} \times \overline{\gamma} := (\mathrm{alb}^0_\bullet \circ \widehat{\beta}) \times (\mathrm{alb}^0_\bullet \circ \widehat{\gamma}) : \overline{Z} \times \overline{Y} \to A(\overline{Y}) \times A(\overline{Z}) = \overline{Y} \times \overline{Z}$.

Summing up, we obtain:

6.1.5 Theorem. *Let Y, Z be reduced connected compact complex spaces with $Y \notin \mathscr{T}$, and let $\mathscr{K}(Y \times Z) := \{(\alpha, \widehat{\beta}, \widehat{\gamma}, \delta) \in \mathrm{Aut}(Y) \times \mathrm{Hol}(\overline{Z}, A(Y)) \times \mathrm{Hol}(\overline{Y}, A(Z)) \times \mathrm{Aut}(Z) : \overline{\beta}\overline{\gamma}$ nilpotent$\}$.*

Then the map $\mathcal{H}(Y \times Z) \to \mathrm{Aut}_+(Y \rtimes Z)$ *given by* $(\alpha, \widehat{\beta}, \widehat{\gamma}, \delta) \mapsto (\alpha \rtimes \delta) \circ \langle \beta, \gamma \rangle$ *is well-defined and bijective.*

6.2. Automorphisms of projective varieties

6.2.1 Lemma. *Let* U *be a connected complex space, and let* T *be a connected compact complex subgroup of* $A(U)$. *Assume there exists a line bundle* L *on* U *that is ample on some orbit* Tu_0 *of* T.

Then there exists $(U \to T) \in \mathcal{T}_l$, *where* $l := \dim T$.

Proof. Denote by $\widehat{L}$ the line bundle $(E_U^* L) \otimes ((\cdot u_0 \circ p_T)^* L)^{-1}$ on $T \times U$ (where $E = E_U$ denotes the evaluation map). Evidently, $\widehat{L}|_{T\times\{u\}}$ is topologically trivial for all $u \in U$; thus $u \mapsto j_u^* \widehat{L}$ defines a holomorphic map $\tau : U \to \mathrm{Pic}_0(T)$. Let T_0 denote the connected component of $\tau^{-1}\tau(u_0) \cap Tu_0$ that contains u_0. As $\widehat{L}$ is trivial along every $T \times \{u\}, u \in T_0$, there exists a line bundle L_1 on T_0 with $\widehat{L}_{|T\times T_0} = p_{T_0}^* L_1$; thus $E^*L|T \times T_0 = p_{T_0}^* L_1 \otimes (\cdot u_0 \circ p_T)^* L$. We conclude that L_1 is ample, since so is $E^*L|_{\{u_0\}\times T_0} \cong p_{T_0}^* L_1|_{\{u_0\}\times T_0}$. Thus $E^*L|_{T\times T_0}$ is ample, i.e. $E|T \times T_0 \to E(T \times T_0) = Tu_0$ is finite, whence $T_0 = \{u_0\}$. This shows that $\tau|_{Tu_0}$ is finite and hence surjective. In particular, there exists some finite holomorphic homomorphism $\beta : \mathrm{Pic}_0 T \to T$ such that $\alpha := \beta \circ \tau$ satisfies the condition of Lemma 2.4.2. $\square$

6.2.1.a Corollary. *Let X be a projective variety. Then there exists* $(X \to A(X)) \in \mathcal{T}_a$, *where* $a := \dim A(X)$.

6.2.1.b Corollary. *Let X be a projective variety with standard decomposition* $X \cong X_c \times X' \cong X_1' \times \dots \times X_l' \times X' = X_1^{n_1} \times \dots \times X_l^{n_l} \times X'$ *(compare 5.3.4.a).*

Then $\mathrm{Aut}(X) \cong \left(\prod_{\lambda=1}^{l} \mathrm{Aut}(X_\lambda')\right) \times \mathrm{Aut}(X') \times \mathrm{Hol}(\mathrm{Alb}^0(X_c), A(X'))$ *(where the isomorphism is given by 6.1.5), and* $\mathrm{Aut}(X_\lambda') \cong \bigcup_{\sigma \in S(n_\lambda)} J_\sigma \circ (\mathrm{Aut}(X_\lambda))^{n_\lambda}$ *(compare 6.1.1.a(ii)).*

6.2.1.c Example. Let T_1 be a two-dimensional torus of algebraic dimension 1, and let $\tau : T_1 \to T$ denote its equivariant algebraic reduction. Let $C \to T$ be a surjective holomorphic map from a compact Riemann surface of genus ≥ 2 onto T, and let $X := T_1 \times_T C$. Then X is a two-dimensional compact Kähler manifold, $A(X) \cong \mathrm{Ker}\,\tau$ is one-dimensional, and $X \notin \mathcal{T}$.

7. ISOGENY DECOMPOSITIONS

In Shioda's as well as in Parigi's examples for X x Y $\cong$ X x Z, the varieties Y, Z always admit coverings S $\to Y$, S $\to Z$ (with the same S) and thus are still closely related to each other. We shall now see that this fact is not accidental.

7.1. Isogenous products

7.1.1 Definition. Let S_1, S_2 be connected complex spaces.

(i) S_1 and S_2 are *isogenous*, if there exist coverings (i.e. locally biholomorphic finite mappings with connected domain) S $\to S_1$, S $\to S_2$.

Notation: $S_1 \sim S_2$. A diagram S_1 t S $\to S_2$ of coverings is called an isogenybetween S_1 and S_2.

(ii) S_1 is an ***isogeny** factor* of S_2, if $S_2 \sim S_1$ x S_1' with suitable S_1'. S_2 is ***strongly** indecomposable*, if it admits no isogeny factor $\neq$ C ^{0}, S_2.

7.1.1.a Remarks. (i) $\sim$ is an equivalence relation.

(ii) If $(\pi : U \to T) \in \mathscr{T}$, then T is an isogeny factor of U.

7.1.2 Lemma. ***Ler** ϕ **: S** $\to$ **X x Y be a covering.***

***Then** there **exist coverings** $\alpha : X' \to$ **X,** $\beta : Y' \to$ **Y with the following** properties:*

(i) α x β factors through ϕ.

(ii) If 7 : $X'' \to$ X, $\delta : Y'' \to$ Y are coverings such that 7 x δ factors through ϕ, then 7 factors through α and δ factors through β.

(iii) If ϕ is biholomorphic, then α = id, and β = id,.

Proof. Let $\widehat{\alpha} : \widetilde{X} \to X, \widehat{\beta} : \widetilde{Y} \to$ Y be the universa] coverings with deck transformation groups $G \cong \pi_1(X), H \cong \pi_1(Y)$. Then G' := $G \cap \pi_1(S), H'$:= $H \cap \pi_1(S)$ havefinite index in G, H, respectively, and G' x H' is a subgroup of π_1 (S) . Thus there exist factorizations $\widehat{\alpha}$ = $(\widetilde{X} \to \widetilde{X}/G' \xrightarrow{\alpha} X)$, $\widehat{\beta}$ = $(\widetilde{Y} \to \widetilde{Y}/H' \xrightarrow{\beta}$ Y) , and the assertion follows with X' = $\widetilde{X}/G'$, Y' = $\widetilde{Y}/H'$.

Let now X x Y $\xleftarrow{\phi}$ S $\xrightarrow{\psi} U$ x V be an isogeny (between connected complex spaces), and construct the triangle

$$\begin{array}{ccc} X' \times Y' & \xrightarrow{\phi'} & S \\ & \searrow_{\alpha\times\beta} & \downarrow\phi \\ & & X \times Y \end{array}$$

as above. Let $(f_1 : X_2 \times Y_2 \to U_1 \times V_1) := (\psi \circ \phi' : X' \times Y' \to U \times V)$, and apply the same construction to f_1, thus obtaining

$$\begin{array}{ccc} U_3 \times V_3 & \stackrel{f_2}{\rightarrow} & X_2 \times Y_2 \\ & \searrow_{\alpha_1 \times \beta_1} & \downarrow f_1 \\ & & U_1 \times V_1 \end{array}$$

Iterating this procedure, we ai-rive at

$$\begin{array}{ccccccccc} \dots & \rightarrow & X_{2n+2} \times Y_{2n+2} & & \stackrel{\alpha_{2n} \times \beta_{2n}}{\longrightarrow} & & X_{2n} \times Y_{2n} & \dots \\ & \nearrow f_{2n+2} & & \searrow f_{2n+1} & & \nearrow f_{2n} & & \\ \dots & U_{2n+3} \times V_{2n+3} & & \stackrel{\alpha_{2n+1} \times \beta_{2n+1}}{\longrightarrow} & & U_{2n+1} \times V_{2n+1} & \rightarrow & \dots \end{array}$$

By construction, if some f_n is biholomotphic, then so are all f_m for $m \geq n$, and f_m and f_{m+1} are then inverse to each other.

Let $((x_{2n}, y_{2n})) \in \prod_{n \geq 1} (X_{2n} \times Y_{2n})$ with $\alpha_{2n}(x_{2n+2}) = x_{2n}, \beta(y_{2n+2}) = y_{2n}$, and let $(u_{2n+1}, v_{2n+1}) := f_{2n+1}(x_{2n+2}, y_{2n+2})$. Consider the sequence

$$(*) \dots \rightarrow X_{2n+4} \stackrel{\overleftarrow{y}_{2n+4}}{\rightarrow} U_{2n+3} \stackrel{\overrightarrow{v}_{2n+3}}{\rightarrow} Y_{2n+2} \stackrel{\overrightarrow{x}_{2n+2}}{\rightarrow} V_{2n+1} \stackrel{\overleftarrow{u}_{2n+1}}{\rightarrow} X_{2n} \rightarrow \dots$$

and denote by $R_{n+l} \stackrel{(*)}{\rightarrow} R'_n$ the map given by a subsequence of length 1, where ***R, R'*** $\in$ {X, Y, U, V}, appropriately.

7.13 Definition. The isogeny X x Y ⟵ S ⟶ U x ***V degenerates*** (with respect to the family $((x_{2n}, y_{2n})))$, if the reduction of $R_{n+l} \stackrel{(*)}{\rightarrow} R_n$ is constant for 1 ≫ 0 and all n.

7.1.3.a Remark. If $(X_{2+4k} \stackrel{(*)}{\rightarrow} X_2)_{red}$ is constant, then so is $(R_{n+l} \stackrel{(*)}{\rightarrow} R'_n)_{red}$ for all n and all $l \geq 4k + 6$.

Proof. Clearly, $\alpha_{2n+2} \circ \overleftarrow{y}_{2n+4} = \overleftarrow{y}_{2n+2} \circ \alpha_{2n+3}$, and cormsponding relations hold for x, u, v. Thus $(X_{4k+2} \stackrel{(*)}{\rightarrow} X_2) \circ \alpha_{4k+2} \circ \dots \circ \alpha_{2n+4k} = \alpha_2 \circ \dots \circ \alpha_{2n} \circ (X_{2n+4k+2} \stackrel{(*)}{\rightarrow} X_{2n+2})$, whence $X_{4k+2} \stackrel{(*)}{\rightarrow} X_2$ is constant, if and only if so is $X_{2n+4k+2} \stackrel{(*)}{\rightarrow} X_{2n+2}$. Furthermore, every subsequence of (*) of length $\geq 4k + 6$ contains some $X_{2n+4k+2} \stackrel{(*)}{\rightarrow} X_{2n+2}$. ◊

From now on assume that X is compact.

7.1.4 Lemma. *(compare 5.1.1).*

(i) If $l \geq 2$, then $R_{n+l} \overset{(*)}{\to} R'_n$ *factors (set-theoretically) through* $Hol(R'_{n+2}, R'_n) \overset{\cdot r'_{n+2}}{\to} R'_n$ *with* $r_{n+l} \mapsto \gamma_n$, *where* $\gamma \in \{\alpha, \beta\}$ *according as* $R' \in \{U, X\}$ *or* $R' \in \{V, Y\}$ *with corresponding* $r, r' \in \{x, y, u, v\}$.

(ii) If $n \gg 0$ and if $(R_{n+l} \overset{(*)}{\to} R'_n)$ *contains* $(X_{m+10} \overset{(*)}{\to} Y_m)$, *then* $R_{n+l} \overset{(*)}{\to} R'_n$ *factors holomorphically through* $A(R'_n) \overset{\cdot r'_n}{\to} R'_n$ *with* $r_{n+l} \mapsto \mathrm{id}_{R'_n}$.

Proof. The proof of (i) does not require X to be compact; thus we may assume $R'_n = X_n$ for symmeuy reasons.

Let $\phi := lf_n \circ (\overleftarrow{y}_{n+2} \circ p_{X_{n+2}}, rf_{n+1}) : X_{n+2} \times Y_{n+2} \to X_n$; then $\phi(x_{n+2}, .) = \overleftarrow{u}_{n+1} \overrightarrow{x}_{n+2}$ and $\phi(., y_{n+2}) = lf_n \circ f_{n+1}(., y_{n+2}) = \alpha_n$.

Thus we obtain a commutative diagram

$$\begin{array}{ccccc} r_{n+l} & \in & R_{n+l} & & \\ \downarrow & & \downarrow (*) & \searrow (*) & \\ y_{n+2} & \in & Y_{n+2} & \overset{\overleftarrow{u}_{n+1}\overrightarrow{x}_{n+2}}{\longrightarrow} & X_n \\ \downarrow & & \downarrow \rho_\phi & \nearrow \cdot x_{n+2} & \\ \alpha_n & \in & Hol(X_{n+2}, X_n) & & \end{array}$$

which proves (i).

Consider now $\psi := (X_{n+2} \times X_{n+4} \overset{\mathrm{id} \times (*)}{\longrightarrow} X_{n+2} \times Y_{n+2} \overset{\phi}{\to} X_n)$, let $\widetilde{W}_n := \rho_\psi[X_{n+4}]$ (compare 2.2.2), and denote by W_n the weak normalization of $(\widehat{W}_n)_{red}$. Applying 2.3.2 to the sequence $\ldots \to X_{n+2} \overset{\alpha_n}{\to} X_n \to \ldots$, we conclude that $|W_n| \subset A(X_n) \circ \alpha_n$, and from 2.3.2.a we infer that the natural map $\widehat{W}_n \to Hol(X_n)$ is holomorphic with image contained in $Aut(X_n)$. This yields a commutative diagram

$$\begin{array}{ccccc} X_{n+8} & \overset{(*)}{} & X_{n+4} & \overset{(*)}{} & X_n \\ \downarrow & \nearrow & \uparrow red & & \uparrow \cdot x_n \\ Aut(X_{n+4}) & & (X_{n+4})_{red} & \to & A(X_n) \end{array}$$

and we conclude that $X_{n+8} \overset{(*)}{\to} X_n$ factors through $\cdot x_n : A(X_n) \to X_n$, since the orbit map $\cdot x_{n+4} : Aut(X_{n+4}) \to X_{n+4}$ factors through $(X_{n+4})_{red} \hookrightarrow X_{n+4}$.

From the commutative diagram

$$\begin{array}{ccccccccc} R_{n+1} & \overset{(*)}{\to} & X_{m+8} & & \overset{(*)}{} & & X_m & \overset{(*)}{\to} & R_{n+2} & & \overset{(*)}{} & & R'_n \\ & & & \searrow & & \nearrow \cdot x_m & & & & \searrow & & \nearrow \cdot r'_{n+2} & \\ & & & & A(X_m) & & & & & & \mathrm{Hol}(R'_{n+2}, R'_n) & & \end{array}$$

we infer that there exists $V_n \underset{(rcc)}{\hookrightarrow} \mathrm{Hol}(R'_{n+2}, R'_n)$ with $\gamma_n \in V_n$ such that $X_{m+8} \overset{(*)}{\to} R'_n$ factors holomorphically through $\cdot r'_{n+2} : V_n \to R'_n$. Now assertion (ii) follows by applying 2.3.2 to the sequence $\ldots \to R'_{n+2} \overset{\gamma_n}{\to} R'_n \to \ldots$ and to the family (V_n). *0*

7.1.5 Proposition. *Let* $l := \lim_{\kappa \to \infty} \dim \mathrm{Im}(X_{2+4k} \overset{(*)}{\to} X_{,})$.

Then there exists an l-dimensional torus T which is an isogeny factor of X, Y, U and V. In particular, if X, Y, U and V do not admit a common torus isogeny factor (of positive dimension), then every isogeny between X x Y and U x V degenerates.

Proof. Evidently, $l = \lim_{\kappa \to \infty} \dim \mathrm{Im}(S_{m+k} \overset{(*)}{\to} S'_m)$ for all m $\in$ N and all S, S' $\in$ *{X,Y, U, V}* (compare 7.1.3.a).

By Lemma 7.1.4.(ii), there exists a commutative diagram

$$\begin{array}{ccccc} S_{m+2k} & \overset{(*)}{} & S'_{m+k} & \overset{(*)}{\to} & S''_m \\ \downarrow & \nearrow \cdot s'_{m+k} & & \searrow & \uparrow \cdot s''_m \\ A(S'_{m+k}) & & & & A(S''_m) \end{array}$$

for m sufficiently large and $k \geq 16$. Increasing k, we may assume that dim Im $(S_{m+2k} \overset{(*)}{\to} S''_m) = 1$; then $\mathrm{Im}(S'_{m+k} \overset{(*)}{\to} S''_m)$ coincides with the image of the orbit $A(S'_{m+k}) s'_{m+k}$ and hence is the orbit of some $T(S''_m) \sqsubseteq A(S''_m)$. Thus, for all m $\gg$ 0 and all $R \in \{X, Y, U, V\}$, there exists an l-dimensional $T(R_,) \sqsubseteq A(R_,)$ such that every $R'_{m+k} \overset{(*)}{\to} R_m$ factors through $\cdot r_m : T(R_,) \to R_m$, if k is sufficiently large. Using Lemma 2.4.2 and 7.1.1.a(ii), we conclude that $T(R_m)$ is an isogeny factor of *R,;* clearly, $T(R_m)$ and $T(R'_n)$ *are* isogenous for all $R, R' \in \{X, Y, U, V\}$. $\Diamond$

7.1.6 Lemma. ***If the isogeny X x Y ← S → U x V degenerates, then f_n is a degenerating isomorphism for n w*** 0.

Proof. By 7.1.2(iii) and 7.1.3.b, it suffices to show that f_n is biholomorphic for n w 0. For this, in turn, we need only show that f_n induces an isomorphism between the corresponding fundamental groups.

Let $G_{2n} = \pi_1(X_{2n}), H_{2n} = \pi_1(Y_{2n}), G_{2n+1} = \pi_1(U_{2n+1}), H_{2n+1} = \pi_1(V_{2n+1})$. By construction of the sequence (f_n), the sequence ($G_n \times H_n$) satisfies the condition of Lemma 0.3.3, if the isogeny $X \times Y$ t $S \to U \times V$ degenerates. 0

7.2. Cancellation

7.2.1 Lemma. ***Let U_1, U_2 be connected complex spaces, and let T_1, T_2 be tori such that $T_1 \times U_1 \sim T_2 \times T_2$.***

If there exists no positive-dimensiona1 torus that is an isogeny factor of both U_1 and U_2, then $T_1 \sim T_2$ and $U_1 \sim U_2$.

Proof. It is easily seen (e.g. by using 7.1.2) that every isogeny factor of a torus is isogenous to a torus. Thus, by 7.1.5, every isogenous between $T_1 \times U_1$ and $T_2 \times U_2$ degenerates. By 7.1.6, we may assume that there exists a degenerating isomorphism $f: T_1 \times U_1 \to T_2 \times U_2$, which, by 5.1.5, induces a simultaneous subdecomposition. As neither U_1 nor U_2 admits a positive-dimensional torus factor, we conclude that (with the notations of 3.3.2) $T_1 = T_{1T_2} \cong T_{2T_1} = T_2$ and $U_1 = U_{1U_2} \cong U_{2U_1} = U_2$. 0

For any connected complex space U denote by t(U) the maximal m $\in$ N such that there exists an m -dimensiona1 torus that is an isogeny factor of U. Thus U is isogenous to $T(U) \times U_+$, where $T(U)$ is a $t(U)$-dimensional torus and U_+ is a connected complex space with $t(U_+) = 0$.

7.2.1-a Corollary.

(i) Let $U \sim T \times U'$ with some torus T. If dim ***$T = t(U)$ or if $t(U') = 0$, then $T(U) \sim T$ and $U_+ \sim U'$.***

(ii) $T(U) \times T(V) \sim T(U \times V)$ and $U_+ \times V_+ \sim (U \times V)_+$ for all connected complex spaces U and V.

Proof. The assertion (i) is obvious by 7.2.1. To prove (ii), consider any isogeny between $U_+ \times V_+$ and $T \times Y$, where T is a torus and Y a suitable connected complex space. By 7.1.5, this isogeny degenerates, and using 7.1.6 and 5.1.5, we conclude that U_+ and T or V_+ and T possess a common isogeny factor. Thus dim T = 0 and we can apply 7.2.1 to $T(U \times V) \times (U \times V)_+$ -$(T(U) \times T(V)) \times (U_+ \times V_+)$. 0

7.2.2 Lemma. ***Let T, T_1, T_2 be tori with $T \times T_1 \sim T \times T_2$. Then $T_1 \sim T_2$.***

Proof. We proceed by induction on dim $T \times T_1$. In the induction step, we may assume that dim $T_1 > 0$, and that T_1 and T_2 have no common torus isogeny factor. Then, by 7.1.5, any isogeny between $T \times T_1$ and $T \times T_2$ degenerates, whence, by 7.1.6, we may assume that there exists a degenemting isomorphism $T \times T_1 \to T' \times T_2$ with some torus $T' \sim T$. From 5.1.5 we infer $T \cong T_{T'} \times T_2$ and $T' \cong T_{T'} \times T_1$, since T_1 and T_2 have no positive-dimensional common factor. Thus $T_1 \sim T_2$ by induction hypothesis. 0

7.23 Theorem. *Let X, Y, Z be connected complex spaces, such that X, Y or Z is compact. If $X \times Y$ and $X \times Z$ are isogenous, then so are Y and Z.*

Proof. By 7.2.1.a, wehave $T(X) \times T(Y) \sim T(X \times Y) \sim T(X \times Z) \sim T(X) \times T(Z)$ and $X_+ \times Y_+ \sim (X \times Y)_+ \sim (X \times Z)_+ \sim X_+ \times Z_+$. Thus $T(Y) \sim T(Z)$ by 7.2.2. By 7.1.5, every isogeny between $X_+ \times Y_+$ and $X_+ \times Z_+$ degenerates (note that X_+, Y_+ or Z_+ is compact). Using 7.1.6, we may assume $X_+ \times Y_+ \cong X_+ \times Z_+$, whence $Y_+ \cong Z_+$ by 5.2.1. Thus $Y \sim T(Y) \times Y_+ \sim T(Z) \times Z_+ \sim Z$. 0

7.2.3.a Corollary. *If $X \times Y \cong X \times Z$ with X, Y or Z compact, then Y and Z are isogenous.*

73. Decomposition

73.1 Theorem. *Every connected complex space U admits a unique isogeny decomposition (up to reordering) $U \sim X_1 \times \ldots \times X_n \times T(U) \times U'$ ($n \geq 0$), suchthat*

(i) $T(U)$ is a (possibly zero-dimensionai) torus and U' has no compact isogeny factor $\neq \mathbb{C}^0$,

(ii) every X_ν ($1 \leq \nu \leq n$) is compact, strongly indecomposable, $\neq \mathbb{C}^0$ and not isogenous to any torus.

Proof. Evident by 7.2.1.a, 7.1.6, and 5.3.4. 0

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