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THE FIRST CHERN CLASS OF RIEMANNIAN 3-SYMMETRIC SPACES: THE CLASSICAL CASE

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Abstract. *The existence of Einstein metrics compatible with J on a compact connected almost complex manifold (M, J) is deeply concerned with its characteristic classes. Using the method of A. Borel and F. Hirzebruch, we prove that an irreducible simply connected (non- Kähler) compact Riemannian 3-symmetric space $(G/K, J, \langle , \rangle)$ is Einstein if and only if the first Chern class of $(G/K, J)$ vanishes.*

1. INTRODUCTION

Let G be a compact connected Lie group, K a closed subgroup of G, S a maximal torus of K and $\xi = (E_\xi, B_\xi, G)$ a principal G-bundle with the total space E_ξ and the base space B_ξ. A. Borel and F. Hirzebruch ([4]) gave the method to calculate the characteristic classes of the bundle along the fibres of the bundle $(E_\xi/K, B_\xi, G/K)$ in terms of the roots of G relative to S, and concretely calculated the characteristic classes of compact Hermitian symmetric spaces. A. Gray ([6]) introduced the notion of Riemannian 3-symmetric spaces, including Hermitian symmetric spaces, and showed that every Riemannian 3-symmetric space is a homogeneous almost Hermitian manifold with the canonical almost complex structure, and that some of Riemannian 3-symmetric spaces are nearly Kähler manifolds. J.A. Wolf and A. Gray ([18]) gave the complete classification table of simply connected irreducible Riemannian 3-symmetric spaces M such that the group of pseudo-holomorphic isometries of M is a reductive Lie group.

Let $M = (M, J)$ be a compact connected almost complex manifold with the almost complex structure J. The existence of Einstein metrics on M compatible with J is deeply concerned with the characteristic classes. (For example, cf. [2] p. 322). M. Matsumoto ([11]) proved that any 6-dimensional (non-Kähler) nearly Kähler manifold is Einstein, and Y. Watanabe and K. Takamatsu ([17]) generalized the above result. Furthermore, the following results are known.

Theorem 1.1. *([16]). In a 6-dimensional (non-Kähler) nearly Kähler Einstein manifold, the generalized first Chern form vanishes.*

Theorem 1.2. *([17]). An irreducible (non-Kähler) nearly Kähler manifold with vanishing generalized first Chern form is Einstein.*

Taking account of these results and the results in [12], K. Sekigawa has suggested me that a compact irreducible (non-Kähler) nearly Kähler manifold M is Einstein if the first Chern class of M vanishes. Then, we shall prove the following theorem.

Main Theorem. *Let $(G/K, J, \langle , \rangle)$ be a simply connected irreducible (non-Kähler) compact Riemannian 3-symmetric space. Then $(G/K, J, \langle , \rangle)$ is Einstein if and only if the first Chern class of $(G/K, J)$ vanishes, where J is the canonical almost complex structure and $\langle , \rangle$ is the G-invariant Riemannian metric on G/K induced by a biinvariant Riemannian metric on G.*

As the author has already proved in [9] that the above result is valid in the case G is exceptional, we shall prove the Main Theorem only in the case G is classical.

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2. PRELIMINARIES

In this section, we shall recall the Lie algebras of classical compact simple Lie groups. In the sequel, we denote by $\mathbf{R}$, $\mathbb{C}$ and $\mathbb{H}$ the set of all real numbers, complex numbers and quaternionic numbers, respectively, and furthermore by $gl(N, \mathbf{R})$, $gl(N, \mathbb{C})$ and $gl(N, \mathbb{H})$ the set of all $N \times N$ real matrices, complex matrices and quaternionic matrices, respectively. We denote by $E_{\lambda\mu} \in gl(N, \mathbf{R}) \subset gl(N, \mathbb{C}) \subset gl(N, \mathbb{H}) (1 \leq \lambda, \mu \leq N)$ the matrix whose r-th row and s-th column is given by $\delta_{r\lambda}\delta_{s\mu}$.

$$(A_n)g = su(n+1) = \{X \in gl(n+1, \mathbb{C}) | X + {}^t\overline{X} = 0, Trace X = 0\}.$$

We put

$$U_{\lambda\mu} = E_{\lambda\mu} - E_{\mu\lambda},$$

$$U'_{\lambda\mu} = i(E_{\lambda\mu} + E_{\mu\lambda}), \quad (1 \leq \lambda < \mu \leq n+1),$$

$$t_\nu = i(E_{\nu\nu} - E_{\nu+1\,\nu+1}), \quad (1 \leq \nu \leq n).$$

Then $\left\{\sqrt{2/(\lambda^2+\lambda)} \sum_{\nu=1}^{\lambda} \nu t_\nu \ (1 \leq \lambda \leq n), U_{\lambda\mu}, U'_{\lambda\mu} \ (1 \leq \lambda < \mu \leq n+1)\right\}$ forms an orthonormal basis for $su(n+1)$ with respect to the inner product $\langle , \rangle$ on $su(n+1)$ defined by $\langle X, Y\rangle = -(1/2) Trace XY$ for $X, Y \in su(n+1)$. Then the inner product $\langle , \rangle$ on $su(n+1)$ induces a biinvariant Riemannian metric on the Lie group $G = SU(n+1)$.

$$(B_n)g = so(2n+1) = \{X \in gl(2n+1, \mathbf{R}) | X + {}^tX = 0\}.$$

We put $u_{rs} = E_{rs} - E_{sr}$ $(1 \leq r < s \leq 2n+1)$, and

$$U_{\lambda\mu} = (1/\sqrt{2})(u_{2\lambda-1\,2\mu-1} - u_{2\lambda\,2\mu}),$$

$$U'_{\lambda\mu} = (1/\sqrt{2})(u_{2\lambda-1\,2\mu} + u_{2\lambda\,2\mu-1}),$$

$$V_{\lambda\mu} = (1/\sqrt{2})(u_{2\lambda-1\,2\mu-1} + u_{2\lambda\,2\mu}),$$

$$V'_{\lambda\mu} = (1/\sqrt{2})(-u_{2\lambda-1\,2\mu} + u_{2\lambda\,2\mu-1}),\ (1 \leq \lambda < \mu \leq n),$$

$$U_{\nu} = (1/\sqrt{2})(-u_{2\nu-1\,2n+1} - u_{2\nu\,2n+1}),$$

$$U'_{\nu} = (1/\sqrt{2})(-u_{2\nu-1\,2n+1} + u_{2\nu\,2n+1}),$$

$$t_{\nu} = -u_{2\nu-1\,2\nu},\ (1 \leq \nu \leq n).$$

Then $\{t_\nu(1 \leq \nu \leq n), U_{\lambda\mu}, U'_{\lambda\mu}, V_{\lambda\mu}, V'_{\lambda\mu}(1 \leq \lambda < \mu \leq n), U_\nu, U'_\nu(1 \leq \nu \leq n)\}$ forms an orthonormal basis for $so(2n+1)$ with respect to the inner product $\langle , \rangle$ on $so(2n+1)$ defined by $\langle X, Y\rangle = -(1/2)Trace XY$ for $X, Y \in so(2n+1)$.

$$(C_n) g = sp(n) = \{X \in gl(n, \mathbb{H}) | X + {}^t\overline{X} = 0\}.$$

The quaternions $\mathbb{H}$ is generated by $\{1, i, j, k\}$, where i, j and k satisfy $i^2 = j^2 = k^2 = -1, ij = -ji = k, jk = -kj = i$ and $ki = -ik = j$. We put

$$t_\nu = iE_{\nu\nu},$$

$$W_\nu = jE_{\nu\nu},$$

$$W'_\nu = kE_{\nu\nu},\ (1 \leq \nu \leq n),$$

$$W_{\lambda\mu} = (1/\sqrt{2})j(E_{\lambda\mu} + E_{\mu\lambda}),$$

$$W'_{\lambda\mu} = (1/\sqrt{2})k(E_{\lambda\mu} + E_{\mu\lambda}),$$

$$U_{\lambda\mu} = (1/\sqrt{2})(E_{\lambda\mu} - E_{\mu\lambda}),$$

$$U'_{\lambda\mu} = (1/\sqrt{2})i(E_{\lambda\mu} + E_{\mu\lambda}),\ (1 \leq \lambda < \mu \leq n).$$

Then $\{t_\nu, W_\nu, W'_\nu(1 \leq \nu \leq n), W_{\lambda\mu}, W'_{\lambda\mu}, U_{\lambda\mu}, U'_{\lambda\mu}(1 \leq \lambda < \mu \leq n)\}$ forms an orthonormal basis for $sp(n)$ with respect to the inner product $\langle , \rangle$ on $sp(n)$ defined by $\langle X, Y\rangle = -(1/2)Trace(XY + YX)$ for $X, Y \in sp(n)$.

$$(D_n) g = so(2n) = \{X \in gl(2n, \mathbf{R}) | X + {}^tX = 0\}.$$

We put $u_{rs} = E_{rs} - E_{sr}(1 \leq r < s \leq 2n)$, and

$$U_{\lambda\mu} = (1/\sqrt{2})(u_{2\lambda-1\,2\mu-1} - u_{2\lambda\,2\mu}),$$

$$U'_{\lambda\mu} = (1/\sqrt{2})(u_{2\lambda-1\,2\mu} + u_{2\lambda\,2\mu-1}),$$

$$V_{\lambda\mu} = (1/\sqrt{2})(u_{2\lambda-1\,2\mu-1} + u_{2\lambda\,2\mu}),$$

$$V'_{\lambda\mu} = (1/\sqrt{2})(-u_{2\lambda-1\,2\mu} + u_{2\lambda\,2\mu-1}),\ (1 \leq \lambda < \mu \leq n),$$

$$t_\nu = -u_{2\nu-1\,2\nu},\ (1 \leq \nu \leq n).$$

Then $\{t_\nu(1 \leq \nu \leq n), U_{\lambda\mu}, U'_{\lambda\mu}, V_{\lambda\mu}, V'_{\lambda\mu}(1 \leq \lambda < \mu \leq n)\}$ forms an orthonormal basis for $so(2n)$ with respect to the inner product $\langle , \rangle$ on $so(2n)$ defined by $\langle X, Y\rangle = -(1/2)TraceXY$ for $X, Y \in so(2n)$.

3. CHARACTERISTIC CLASSES OF COMPACT HOMOGENEOUS SPACES

We recall here the method to calculate the characteristic classes of compact homogeneous spaces which has been shown by A. Borel and F. Hirzebruch ([4]).

First of all, we recall the roots of an invariant almost complex structure of a compact homogeneous almost complex manifold. Let G, K and T a compact connected semi-simple Lie group, a proper closed connected subgroup of G of the same rank and a maximal torus of K, respectively. Assume now that the homogeneous space G/K has been endowed with an invariant almost complex structure J and let $\pm b_j$ $(1 \leq j \leq n)$ be the roots of G relative to T complementary to those of K (we call them the complementary roots). The invariant almost complex structure J is induced from a linear endomorphism (which we denote by the same letter J) of the subspace m of g (where $g = k \oplus m, k$ denotes the Lie algebra of K) such that $J^2 = -id_m$ and $J \circ Ad_g a|_m = Ad_g a|_m \circ J$ for any $a \in K$. We decompose m into the 2-dimensional subspaces ℓ_j corresponding to the roots $b_j(1 \leq j \leq n)$: $m = \ell_1 \oplus \ldots \oplus \ell_n$. Each ℓ_j is J-invariant. To each b_j, we attach a sign $\epsilon_j = +1$ or -1 in the following way: take any non-zero element $e_j \in \ell_j$. Then $e_j - \sqrt{-1}Je_j$ is a non-zero element of $\ell_j^{\mathbb{C}} = \ell_j \otimes_{\mathbb{R}} \mathbb{C}$, and is an eigenvector corresponding to either the eigenvalue $e^{2\pi\sqrt{-1}b_j(x)}$ or the eigenvalue $e^{2\pi\sqrt{-1}(-b_j)(x)}$ of $Ad_g(expx)$ for any $x \in t$. In the formar case, we define $\epsilon_j = +1$, and in the latter case, $\epsilon_j = -1$. That is to say, we determine a sign ϵ_j so that the equality

$$Ad_g(expx)\{e_j - \sqrt{-1}Je_j\} = e^{2\pi\sqrt{-1}\epsilon_j b_j(x)}\{e_j - \sqrt{-1}Je_j\}$$

holds for any $x \in t$. Linear forms $\epsilon_j b_j$ $(1 \leq j \leq n)$ are called *the roots of the invariant almost complex structure* J.

The characteristic classes of a compact homogeneous spaces may be calculated by the following theorem.

Theorem 3.1. *([4]). Let G, K and S a compact connected Lie group, a closed subgroup of G and a maximal torus of K, respectively, and $\pm b_j$ $(1 \leq j \leq k)$ the roots of G relative to S complementary to those of K. Let ρ be the projection from G/S onto G/K, and τ the transgression in the principal S-bundle $(G, G/S, S)$. Then the Pontrjagin classes are given by*

$$\rho^*(\tilde{p}(G/K)) = \prod_{j=1}^{k} (1 + (\tau(b_j))^2). \tag{3.1}$$

If, moreover, the dimension of G/K is even, then the Euler class is given by

$$\rho^*(e(G/K)) = \pm \prod_{j=1}^{k} (-\tau(b_j)). \tag{3.2}$$

And if, moreover, G is semi-simple, K is connected subgroup of the same rank and G/K has an invariant almost complex structure J, then, if we denote by $\epsilon_j b_j$ $(1 \leq j \leq k)$ the roots of J, the total Chern class is given by

$$\rho^*(c(G/K)) = \prod_{j=1}^{k} (1 - \tau(\epsilon_j b_j)). \tag{3.3}$$

In order to calculate the characteristic classes of a compact homogeneous space G/K from Theorem 3.1, we need to know the cohomology ring of G/K. The real cohomology ring of G/K is completely calculated ([3]): let G be a compact Lie group and K a closed subgroup of G. We denote by B_G and $\rho(K, G)$ the classifying space of G and the projection from B_K onto B_G respectively. A. Borel ([3]) has proved the followings.

Theorem 3.2. *Let G be a compact Lie group and K a closed subgroup of the same rank. Then*

(a) $\rho^*(K, G) : H^*(B_G, \mathbf{R}) \to H^*(B_K, \mathbf{R})$, is injective.

(b) If G is connected, then

$$H^*(G/K, \mathbf{R}) \simeq H^*(B_K, \mathbf{R})/\rho^*(K, G)(H^+(B_G, \mathbf{R})),$$

where $H^+(B_G, \mathbf{R})$ denotes the subalgebra formed by the elements of positive degree of $H^*(B_G, \mathbf{R})$. Furthermore, if we denote by $\phi : G/K \to B_K$ the characteristic map, then

$$H^*(G/K, \mathbf{R}) = \phi^*(H^*(B_K, \mathbf{R})).$$

Theorem 3.3. *Let G be a compact Lie group and T a maximal torus of G. Then the subalgebra $\rho^*(T,G)(H^*(B_G,\mathbf{R}))$ of $H^*(B_T,\mathbf{R})$ is the subalgebra I_G formed by the elements of $H^*(B_T,\mathbf{R})$ which are invariant by the action of the Weyl group $W(G)$ of G.*

From the above two theorems, we have

$$H^*(G/K,\mathbf{R}) = I_K/I_G^+. \tag{3.4}$$

Remark. Let $\{x_1,\dots,x_l\}$ be a basis of $H^1(T,\mathbf{R})$, and τ' the transgression in the universal bundle (E_T,B_T,T) for T. We denote by the same symbol x_j, the corresponding integral linear form on t and $-\tau'(x_j)\in H^2(B_T,\mathbf{R})$. Then we may see that $H^*(B_T,\mathbf{R}) = \mathbf{R}[x_1,\dots,x_l]$, the ring of all polynomials in the x_j's (see [2]).

In the sequel, we denote by $S\{x_1,\dots,x_l\}$ the ring of all symmetric formal power series in the x_j's, with respect to a ring of coefficients which the context will make precise, and by $\mathbf{R}\{x\}$ the ring of all formal power series in x.

4. RIEMANNIAN 3-SYMMETRIC SPACES

Let $(M,\langle,\rangle)$ be a connected Riemannian manifold. Now we suppose that $(M,\langle,\rangle)$ admits an isometry θ_p of $(M,\langle,\rangle)$ for each point $p\in M$ such that

(4.1) $\theta_p^3 = id_M$ for each $p\in M$,

(4.2) for each $p\in M, p$ is an isolated fixed point of θ_p,

(4.3) the tensor field Θ defined by $\Theta_q = (d\theta_q)_q$ for each $q\in M$ is of class C^∞.

Then we define the canonical almost complex structure J by

$$\Theta_p = -\frac{1}{2}I_p + \frac{\sqrt{3}}{2}J_p, \ for\ each\ p\in M,$$

where I_p denotes the identity transformation of T_pM.

Definition. *A Riemannian manifold $(M,\langle,\rangle)$ is called a Riemannian 3-symmetric space if it admits a family of isometries $\{\theta_p\}_{p\in M}$ of $(M,\langle,\rangle)$ satisfying the conditions (4.1)-(4.3) and furthermore the condition*

$$d\theta_p\circ J = J\circ d\theta_p \ on\ M\ for\ each\ p\in M, \tag{4.4}$$

where J is the canonical almost complex structure.

A. Gray ([6]) showed that a Riemannin 3-symmetric space is characterized by a triple (G,K,σ) satisfying the following conditions (1)-(3):

(1) G is a connected Lie group and σ is an automorphism of G of order 3,

(2) K is a closed subgroup of G such that $G_0^\sigma \subset K \subset G^\sigma$, where $G^\sigma = \{g \in G | \sigma(g) = g\}$ and G_0^σ denotes the identity component of G^σ.

Let g and k be the Lie algebras of G and K respectively. Then we have the direct sum decomposition

$$g = k \oplus m, \quad Ad(K)m = m. \tag{4.5}$$

(3) There exists a positive-definite inner product $\langle , \rangle$ on m which is both $Ad(K)$-invariant and σ-invariant.

The inner product $\langle , \rangle$ on m in (3) induces a G-invariant Riemannian metric $\langle , \rangle$ on the homogeneous space G/K, and $(G/K, \langle , \rangle)$ becomes a Riemannian 3-symmetric space. The canonical almost complex structure J on G/K is given by

$$\sigma|_m = -\frac{1}{2}\, id_m + \frac{\sqrt{3}}{2}\, J \text{ at the origin } eK \in G/K. \tag{4.6}$$

A. Gray ([6]) also showed that the corresponding almost Hermitian manifold $(G/K, J, \langle , \rangle)$ is a quasi-Kähler manifold (also known as O^*-space), and that $(G/K, J, \langle , \rangle)$ is a nearly Kähler manifold (i.e., by definition, $(\nabla_X J)X = 0$ for any differentiable vector field X on G/K, where ∇ denotes the Riemannian connection of $(G/K, \langle , \rangle)$) if and only if $(G/K, \langle , \rangle)$ is a naturally reductive Riemannian homogeneous space with respect to the decomposition (4.5).

J.A. Wolf and A. Gray ([18]) have obtained the complete classification table of irreducible Riemannian 3-symmetric spaces. Let $(G, K = G^\sigma, \sigma)$ be a triple such that G is a compact connected classical simple Lie group and σ is an inner automorphism of G of order 3, and $\langle , \rangle$ be the G-invariant Riemannian metric on the homogeneous space G/K induced by a biinvariant Riemannian metric on G. Then, we may easily see that the corresponding compact Riemannian 3-symmetric space $(G/K, J, \langle , \rangle)$ with the canonical almost complex structure J is a nearly Kähler manifold. From the classification table in [18], we see that if $(G/K, J, \langle , \rangle)$ is not Kähler, then the corresponding triple $(G, K = G^\sigma, \sigma = Ad(exp(2\pi v)))$ is listed in Table 1.

We denote by ∇, R, ρ and ρ^* the Riemannian connection, the Riemannian curvature tensor, the Ricci tensor and the Ricci $*$-tensor respectively. The first Chern class of a nearly Kähler manifold is represented by the 2-form γ_1 (known as the generalized first Chern form) defined by

$$8\pi\gamma_1(X, Y) = 5\rho^*(JX, Y) - \rho(JX, Y)$$

for $X, Y \in \mathfrak{X}(M)$ ($\mathfrak{X}(M)$ denotes the Lie algebra of all differentiable vector fields on M).

The following theorem has been proved by K. Sekigawa and J. Watanabe ([14]).

Table 1

G	v	$K = G^{\sigma}$
SU(n+1)	$\frac{i}{3(n+1)}\left((2n+2-h-m)\sum_{\alpha=1}^{h} E_{\alpha\alpha} + (n+1-h-m)\sum_{\alpha=h+1}^{m} E_{\alpha\alpha} - (h+m)\sum_{\alpha=m+1}^{n+1} E_{\alpha\alpha}\right)$ $(1 \leq h < m \leq n)$	S(U(h)×U(m−h)× U(n−m+1))
SO(2n+1)	$-\frac{1}{3}\sum_{\alpha=1}^{m} u_{2\alpha-1\ 2\alpha}$ $(2 \leq m \leq n)$	U(m)×SO(2n−2m+1)
Sp(n)	$\frac{i}{3}\sum_{\alpha=1}^{m} E_{\alpha\alpha}$ $(1 \leq m \leq n-1)$	U(m)×Sp(n−m)
SO(2n)	$-\frac{1}{3}\sum_{\alpha=1}^{m} u_{2\alpha-1\ 2\alpha}$ $(2 \leq m \leq n-1,\ 4 \leq n)$	U(m)×SO(2n−2m)

Theorem 4.1. *Let $(G, K = G^{\sigma}, \sigma = Ad(exp(2\pi v)))$ be any one of the triples in Table 1 and $\langle , \rangle$ be the G-invariant Riemannian metric on the homogeneous space G/K which is induced by a biinvariant Riemannian metric on G. Then the corresponding Riemannian 3-symmetric space $(G/K, J, \langle , \rangle)$ is irreducible and not locally symmetric, and furthermore is Einstein if and only if G/K is one of the followings:*

(1) $SU(3m)/S(U(m) \times U(m) \times U(m)), \quad m \geq 1,$

(2) $SO(3m-1)/(U(m) \times SO(m-1)), \quad m \geq 2,$

(3) $Sp(3m-1)/(U(2m-1) \times Sp(m)), \quad m \geq 1.$

If G/K is one of the spaces in (1)-(3), then $\rho - 5\rho^{*} = 0$ holds on G/K, and hence the generalized first Chern form of the corresponding nearly Kähler manifold $(G/K, J, \langle , \rangle)$ vanishes, where J denotes the canonfcal almost complex structure.

5. THE PROOF OF THE MAIN THEOREM

For our aim, we shall calculate the first Chern classes of the respective homogeneous spaces listed in Table 1 on the basis of the facts in the previous sections.

Case(1) $\qquad G/K = SU(n+1)/S(U(h) \times U(m-h) \times U(n-m+1)).$

As a maximal torus of G, we may take

$$T = \left\{ \sum_{\alpha=1}^{n+1} e^{2\pi i\theta_{\alpha}} E_{\alpha\alpha} \in SU(n+1) \mid \theta_{\alpha} \in \mathbf{R}(1 \leq \alpha \leq n+1), \sum_{\alpha=1}^{n+1} \theta_{\alpha} = 0 \right\}.$$

We define linear forms x_λ $(1 \leq \lambda \leq n+1)$ on the Lie algebra t of T by

$$x_\lambda \left(2\pi i \sum_{\alpha=1}^{n+1} \theta_\alpha E_{\alpha\alpha} \right) = \theta_\lambda, \; for \; 2\pi i \sum_{\alpha=1}^{n+1} \theta_\alpha E_{\alpha\alpha} \in t,$$

then x_λ's are integral linear forms. We may easily see that the roots fo G relative to T are $\pm(x_\lambda - x_\mu)(1 \leq \lambda < \mu \leq n+1)$, the roots of K are $\pm(x_\lambda - x_\mu)(1 \leq \lambda < \mu \leq h$ or $h+1 \leq \lambda < \mu \leq m$ or $m+1 \leq \lambda < \mu \leq n+1)$, and that the complementary roots are $\pm(x_\lambda - x_\mu)$ $(1 \leq \lambda \leq h < \mu \leq n+1$ or $h+1 \leq \lambda \leq m < \mu \leq n+1)$.

The subspace m of g in the decomposition (4.5) may be decomposed into the $Ad(K)$-invariant subspaces m_1, m_2 and m_3 such that the linear isotropy representation of K on each m_s is irreducible $(s = 1, 2, 3)$. m_1, m_2 and m_3 are given respectively by

$$m_1 = span_{\mathbb{R}} \{U_{\lambda\mu}, U'_{\lambda\mu} \; (1 \leq \lambda \leq h < \mu \leq m)\},$$

$$m_2 = span_{\mathbb{R}} \{U_{\lambda\mu}, U'_{\lambda\mu} \; (1 \leq \lambda \leq h, \; m+1 \leq \mu \leq n+1)\},$$

$$m_3 = span_{\mathbb{R}} \{U_{\lambda\mu}, U'_{\lambda\mu} \; (h+1 \leq \lambda \leq m < \mu \leq n+1)\}.$$

From (4.6) and Table 1, we may see that the canonical almost complex structure J is given by

$$JU_{\lambda\mu} = U'_{\lambda\mu}, JU'_{\lambda\mu} = -U_{\lambda\mu}, \; for \; U_{\lambda\mu}, U'_{\lambda\mu} \in m_1,$$

$$JU_{\lambda\mu} = -U'_{\lambda\mu}, JU'_{\lambda\mu} = U_{\lambda\mu}, \; for \; U_{\lambda\mu}, U'_{\lambda\mu} \in m_2,$$

$$JU_{\lambda\mu} = U'_{\lambda\mu}, JU'_{\lambda\mu} = -U_{\lambda\mu}, \; for \; U_{\lambda\mu}, U'_{\lambda\mu} \in m_3.$$

The roots of J are given by

$$x_\lambda - x_\mu, \quad (1 \leq \lambda \leq h < \mu \leq m \; or \; h+1 \leq \lambda \leq m < \mu \leq n+1),$$

$$-(x_\lambda - x_\mu), \quad (1 \leq \lambda \leq h, \; m+1 \leq \mu \leq n+1).$$

The Weyl group $W(G)$ of G is isomorphic to the group of all permutations of $x_1, \ldots, x_{n+1}$. Hence $I_G = S\{x_1, \ldots, x_{n+1}\}$. And we may see that $I_K = S\{x_1, \ldots, x_h\} \otimes S\{x_{h+1}, \ldots, x_m\} \otimes S\{x_{m+1}, \ldots, x_{n+1}\}$. Hence

$$H^*(G/K, \mathbb{R}) \simeq S\{x_1, \ldots, x_h\} \otimes S\{x_{h+1}, \ldots, x_m\} \otimes S\{x_{m+1}, \ldots, x_{n+1}\} / I_G^+,$$

and any element of $H^*(G/K,\mathbf{R})$ may be expressed by the elements of $S\{x_1,\dots,x_h\}$ and $S\{x_{h+1},\dots,x_m\}$. Note that G/K is torsion free. From (3.3), we have

$$c(G/K,J)=\prod_{\substack{1\le\lambda\le h\\ h+1\le\mu\le m}}(1+x_\lambda-x_\mu)\prod_{\substack{1\le\lambda\le h\\ m+1\le\nu\le n+1}}(1-x_\lambda+x_\nu)\prod_{\substack{h+1\le\mu\le m\\ m+1\le\nu\le n+1}}(1+x_\mu-x_\nu)\ mod\ I_G^+.$$

In particular, the first Chern class $c_1(G/K,J)$ is given by

$$c_1(G/K,J)=\sum_{\substack{1\le\lambda\le h\\ h+1\le\mu\le m}}(x_\lambda-x_\mu)+\sum_{\substack{1\le\lambda\le h\\ m+1\le\nu\le n+1}}(-x_\lambda+x_\nu)+\sum_{\substack{h+1\le\mu\le m\\ m+1\le\nu\le n+1}}(x_\mu-x_\nu)\ mod\ I_G^+$$

$$=(-n+2m-h-1)\sum_{\lambda=1}^{h}x_\lambda+(n-m-h+1)\sum_{\mu=h+1}^{m}x_\mu+(2h-m)\sum_{\nu=m+1}^{n+1}x_\nu\ modI_G^+$$

$$=(3(m-h)-(n+1))\sum_{\lambda=1}^{h}x_\lambda+(n+1-3h)\sum_{\mu=h+1}^{m}x_\mu\ mod\ I_G^+.$$

Since $p\sum_{\lambda=1}^{h}x_\lambda+q\sum_{\mu=h+1}^{m}x_\mu=0\ mod\ I_G^+(p,q\in\mathbf{R})$ if and only if $p=q=0$, we see that $c_1(G/K,J)=0$ if and only if

$$G/K=SU(3h)/S(U(h)\times U(h)\times U(h))\quad(h\ge 1).$$

Case(2) $\qquad G/K=SO(2n+1)/(U(m)\times SO(2n-2m+1)).$

As a maximal torus of G, we may take

$$T=\left\{\sum_{\alpha=1}^{n}(\cos 2\pi\theta_\alpha(E_{2\alpha-1\,2\alpha-1}+E_{2\alpha\,2\alpha})-\sin 2\pi\theta_\alpha U_{2\alpha-1\,2\alpha})+\right.$$

$$\left.+E_{2n+1\,2n+1}\,|\theta_\alpha\in\mathbf{R}(1\le\alpha\le n)\right\}.$$

We define linear forms x_λ $(1\le\lambda\le n)$ on the Lie algebra t of T by

$$x_\lambda\left(\sum_{\alpha=1}^{n}2\pi\theta_\alpha u_{2\alpha-1\,2\alpha}\right)=\theta_\lambda,\ for\ \sum_{\alpha=1}^{n}2\pi\theta_\alpha u_{2\alpha-1\,2\alpha}\in t,$$

then x_λ's are integral linear forms. We may easily see that the roots of G relative to T are $\pm(x_\lambda \pm x_\mu)$ $(1 \leq \lambda < \mu \leq n)$ and $\pm x_\nu$ $(1 \leq \nu \leq n)$, the roots of K are $\pm(x_\lambda - x_\mu)$ $(1 \leq \lambda < \mu \leq m), \pm(x_\lambda \pm x_\mu)(m+1 \leq \lambda < \mu \leq n)$ and $\pm x_\nu$ $(m+1 \leq \nu \leq n)$, and that the complementary roots are $\pm(x_\lambda + x_\mu)$ $(1 \leq \lambda \leq m, \lambda < \mu \leq n), \pm(x_\lambda - x_\mu)$ $(1 \leq \lambda \leq m < \mu \leq n)$ and $\pm x_\nu (1 \leq \nu \leq m)$.

The subspace m of g in the decomposition (4.5) may be decomposed into the $Ad(K)$-invariant subspaces m_1 and m_2 such that the linear isotropy representation of K on each m_s is irreducible $(s = 1, 2)$. m_1 and m_2 are given respectively by

$$m_1 = span_{\mathbf{R}}\{V_{\lambda\mu}, V'_{\lambda\mu}, U_{\lambda\mu}, U'_{\lambda\mu}(1 \leq \lambda \leq m < \mu \leq n), U_\nu, U'_\nu(1 \leq \nu \leq m)\},$$

$$m_2 = span_{\mathbf{R}}\{U_{\lambda\mu}, U'_{\lambda\mu}\ (1 \leq \lambda < \mu \leq m)\}.$$

From (4.6) and Table 1, we may see that the canonical almost complex structure J is given by

$$JV_{\lambda\mu} = V'_{\lambda\mu}, JV'_{\lambda\mu} = -V_{\lambda\mu},\ for\ V_{\lambda\mu}, V'_{\lambda\mu} \in m_1,$$

$$JU_{\lambda\mu} = U'_{\lambda\mu}, JU'_{\lambda\mu} = -U_{\lambda\mu},\ for\ U_{\lambda\mu}, U'_{\lambda\mu} \in m_1,$$

$$JU_\nu = -U'_\nu, JU'_\nu = U_\nu,\ for\ U_\nu, U'_\nu \in m_1,$$

$$JU_{\lambda\mu} = -U'_{\lambda\mu}, JU'_{\lambda\mu} = U_{\lambda\mu},\ for\ U_{\lambda\mu}, U'_{\lambda\mu} \in m_2.$$

The roots of J are given by

$$x_\lambda \pm x_\mu,\ (1 \leq \lambda \leq m < \mu \leq n),$$

$$-(x_\lambda + x_\mu),\ (1 \leq \lambda < \mu \leq m),$$

$$x_\nu,\ (1 \leq \nu \leq m).$$

We may see that $I_G = S\{X_1^2, \ldots, x_n^2\}$ and $I_K = S\{x_1, \ldots, x_m\} \otimes S\{x_{m+1}^2, \ldots, x_n^2\}$. Therefore, by (3.4), we have

$$H^*(G/K, \mathbf{R}) \simeq S\{x_1, \ldots, x_m\} \otimes S\{x_{m+1}^2, \ldots, x_n^2\}/U_G^+,$$

and any elements of $H^*(G/K, \mathbf{R})$ may be expressed by the elements of $S\{x_1, \ldots, x_m\}$. If we denote by $\sigma_r \in H^{2r}(G/K, \mathbf{R})$ the r-th Chern class of the canonical principal $U(m)$-bundle $(SO(2n+1)/SO(2n-2m+1), G/K, U(m))$, then

$$\sum_{r=0}^{m} \sigma_r = \prod_{\lambda=1}^{m} (1 + x_\lambda)\ mod\ I_G^+.$$

From (3.3), we have

$$c(G/K,J) = \prod_{1\leq\lambda\leq m}(1+x_\lambda)\prod_{\substack{1\leq\lambda\leq m\\ m+1\leq\nu\leq n}}(1+x_\lambda+x_\nu)(1+x_\lambda-x_\nu)\prod_{1\leq\lambda<\mu\leq m}(1-x_\lambda-x_\mu)\, mod\ I_G^+.$$

In particular, the first Chern class $c_1(G/K,J)$ is given by

$$c_1(G/K,J) = \sum_{\lambda=1}^{m} x_\lambda + \sum_{\nu=m+1}^{n}\sum_{\lambda=1}^{m}(x_\lambda + x_\nu + x_\lambda - x_\nu) + \sum_{1\leq\lambda<\mu\leq m}(-x_\lambda - x_\mu)\ mod\ I_G^+.$$

$$= (2n-3m+2)\sigma_1 .$$

Since $\sigma_1 \neq 0$, we see that $c_1(G/K,J) = 0$ if and only if

$$G/K = SO(3m-1)/(U(m)\times SO(m-1))\ (m\geq 2 : even).$$

Case(3) $\qquad G/K = Sp(n)/(U(m)\times Sp(n-m)).$

As a maximal torus of G, we may take

$$T = \left\{\sum_{\alpha=1}^{n} e^{2\pi i\theta_\alpha}E_{\alpha\alpha} \in Sp(n)|\theta_\alpha \in \mathbf{R}(1\leq\alpha\leq n)\right\}.$$

We define linear forms $x_\lambda(1\leq\lambda\leq n)$ on the Lie algebra t of T by

$$x_\lambda\left(\sum_{\alpha=1}^{n} 2\pi i\theta_\alpha E_{\alpha\alpha}\right) = \theta_\lambda,\ for\ \sum_{\alpha=1}^{n} 2\pi i\theta_\alpha E_{\alpha\alpha}\in t,$$

then x_λ's are integral linear forms. We may easily see that the roots of G relative to T are $\pm(x_\lambda \pm x_\mu)(1\leq\lambda<\mu\leq n)$ and $\pm 2x_\nu(1\leq\nu\leq n)$, the roots of K are $\pm(x_\lambda - x_\mu)(1\leq\lambda<\mu\leq m), \pm(x_\lambda\pm x_\mu)(m+1\leq\lambda<\mu\leq n)$ and $\pm 2x_\nu(m+1\leq\nu\leq n)$, and that the complementary roots are $\pm(x_\lambda+x_\mu)(1\leq\lambda<\mu\leq m), \pm(x_\lambda\pm x_\mu)(1\leq\lambda\leq m<\mu\leq n)$ and $\pm 2x_\nu(1\leq\nu\leq m)$.

The subspace m of g in the decomposition (4.5) may be decomposed into the $Ad(K)$-invariant subspaces m_1 and m_2 such that the linear isotropy representation of K on each m_s is irreducible ($s = 1,2$). m_1 and m_2 are given respectively by

$$m_1 = span_{\mathbf{R}}\{W_\nu, W'_\nu(1\leq\nu\leq m), W_{\lambda\mu}, W'_{\lambda\mu}(1\leq\lambda<\mu\leq m)\},$$

$$m_2 = span_{\mathbf{R}}\{W_{\lambda\mu}, W'_{\lambda\mu}, U_{\lambda\mu}, U'_{\lambda\mu}(1 \leq \lambda \leq m < \mu \leq n)\}.$$

From (4.6) and Table 1, we may see that the canonical almost complex structure J is given by

$$JW_\nu = -W'_\nu, JW'_\nu = W_\nu, \ for \ W_\nu, W'_\nu \in m_1,$$

$$JW_{\lambda\mu} = -W'_{\lambda\mu}, JW'_{\lambda\mu} = W_{\lambda\mu}, \ for \ W_{\lambda\mu}, W'_{\lambda\mu} \in m_1,$$

$$JW_{\lambda\mu} = W'_{\lambda\mu}, JW'_{\lambda\mu} = -W_{\lambda\mu}, \ for \ W_{\lambda\mu}, W'_{\lambda\mu} \in m_2,$$

$$JU_{\lambda\mu} = U'_{\lambda\mu}, JU'_{\lambda\mu} = -U_{\lambda\mu}, \ for \ U_{\lambda\mu}, U'_{\lambda\mu} \in m_2.$$

The roots of J are given by

$$-2x_\nu, \ (1 \leq \nu \leq m),$$

$$-(x_\lambda + x_\mu), \ (1 \leq \lambda < \mu \leq m),$$

$$x_\lambda \pm x_\mu, \ (1 \leq \lambda \leq m < \mu \leq n).$$

We may see that $I_G = S\{x_1^2, \ldots, x_n^2\}$ and $I_K = S\{x_1, \ldots, x_m\} \otimes S\{x_{m+1}^2, \ldots, x_n^2\}$. Therefore, by (3.4), we have

$$H^*(G/K, \mathbf{R}) \simeq S\{x_1, \ldots, x_m\} \otimes S\{x_{m+1}^2, \ldots, x_n^2\}/I_G^+,$$

and any elements of $H^*(G/K, \mathbf{R})$ may be expressed by the elements of $S\{x_1, \ldots, x_m\}$. Note that G/K is torsion free. If we denote by $\sigma_r \in H^{2r}(G/K, \mathbf{R})$ the r-th Chern class of the canonical principal $U(m)$-bundle $(Sp(n)/Sp(n-m), G/K, U(m))$, then

$$\sum_{r=0}^{m} \sigma_r = \prod_{\lambda=1}^{m} (1 + x_\lambda) \ mod \ I_G^+.$$

From (3.3), we have

$$c(G/K, J) = \prod_{\lambda=1}^{m}(1 - 2x_\lambda) \prod_{1 \leq \lambda < \mu \leq m} (1 - x_\lambda - x_\mu) \prod_{1 \leq \lambda \leq m < \mu \leq n} (1 + x_\lambda + x_\mu)(1 + x_\lambda - x_\mu) \quad mod \ I_G^+.$$

In particular, the first Chern class $c_1(G/K, J)$ is given by

$$c_1(G/K, J) = (2n - 3m - 1)\sigma_1.$$

Since $\sigma_1 \neq 0$, we see that $c_1(G/K, J) = 0$ if and only if

$$G/K = Sp(3m-1)/(U(2m-1) \times Sp(m)) \ (m \geq 1).$$

Case(4) $$G/K = SO(2n)/(U(m) \times SO(2n-2m)).$$

As a maximal torus of G, we may take

$$T = \left\{ \sum_{\alpha=1}^{n} (\cos 2\pi\theta_\alpha (E_{2\alpha-1\, 2\alpha-1} + E_{2\alpha\, 2\alpha}) - \sin 2\pi\theta_\alpha u_{2\alpha-1\, 2\alpha}) \right.$$

$$\left. \in SO(2n) | \theta_\alpha \in \mathbf{R} (1 \leq \alpha \leq n) \right\}.$$

We define linear forms $x_\lambda (1 \leq \lambda \leq n)$ on the Lie algebra t of T by

$$x_\lambda \left(\sum_{\alpha=1}^{n} 2\pi\theta_\alpha u_{2\alpha-1\, 2\alpha} \right) = \theta_\lambda, \ for \sum_{\alpha=1}^{n} 2\pi\theta_\alpha u_{2\alpha-1\, 2\alpha} \in t,$$

then x_λ's are integral linear forms. We may easily see that the roots of G relative to T are $\pm(x_\lambda \pm x_\mu)(1 \leq \lambda < \mu \leq n)$, the roots of K are $\pm(x_\lambda - x_\mu)(1 \leq \lambda < \mu \leq m)$ and $\pm(x_\lambda \pm x_\mu)(m+1 \leq \lambda < \mu \leq n)$, and that the complementary roots are $\pm(x_\lambda + x_\mu)(1 \leq \lambda \leq m, \lambda < \mu \leq n)$ and $\pm(x_\lambda - x_\mu)(1 \leq \lambda \leq m < \mu \leq n)$.

The subspace m of g in the decomposition (4.5) may be decomposed into the $Ad(K)$-invariant subspaces m_1 and m_2 such that the linear isotropy representation of K on each m_s is irreducible $(s = 1, 2)$. m_1 and m_2 are given respectively by

$$m_1 = span_{\mathbf{R}} \{ V_{\lambda\mu}, V'_{\lambda\mu}, U_{\lambda\mu}, U'_{\lambda\mu} \ (1 \leq \lambda \leq m < \mu \leq n) \},$$

$$m_2 = span_{\mathbf{R}} \{ U_{\lambda\mu}, U'_{\lambda\mu} \ (1 \leq \lambda < \mu \leq m) \}.$$

From (4.6) and Table 1, we may see that the canonical almost complex structure J is given by

$$JV_{\lambda\mu} = V'_{\lambda\mu}, JV'_{\lambda\mu} = -V_{\lambda\mu}, \ for \ V_{\lambda\mu}, V'_{\lambda\mu} \in m_1,$$

$$JU_{\lambda\mu} = U'_{\lambda\mu}, JU'_{\lambda\mu} = -U_{\lambda\mu}, \ for \ U_{\lambda\mu}, U'_{\lambda\mu} \in m_1,$$

$$JU_{\lambda\mu} = -U'_{\lambda\mu}, JU'_{\lambda\mu} = U_{\lambda\mu}, \ for \ U_{\lambda\mu}, U'_{\lambda\mu} \in m_2.$$

The roots of J are given by

$$x_\lambda \pm x_\mu, \quad (1 \leq \lambda \leq m < \mu \leq n),$$

$$-(x_\lambda + x_\mu), \quad (1 \leq \lambda < \mu \leq m).$$

We may see that $I_G = S\{x_1^2, \ldots, x_n^2\}$ and $I_K = S\{x_1, \ldots, x_m\} \otimes S\{x_{m+1}^2, \ldots, x_n^2\}$. Therefore, by (3.4), we have

$$H^*(G/K, \mathbf{R}) \simeq S\{x_1, \ldots, x_m\} \otimes S\{x_{m+1}^2, \ldots, x_n^2\}/I_G^+,$$

and any elements of $H^*(G/K, \mathbf{R})$ may be expressed by the elements of $S\{x_1, \ldots, x_m\}$. If we denote by $\sigma_r \in H^{2r}(G/K, \mathbf{R})$ the r-th Chern class of the canonical principal $U(m)$-bundle $(SO(2n)/SO(2n-2m), G/K, U(m))$, then

$$\sum_{r=0}^{m} \sigma_r = \prod_{\lambda=1}^{m} (1 + x_\lambda) \ mod\ I_G^+.$$

From (3.3), we have

$$c(G/K, J) = \prod_{1 \leq \lambda < \mu \leq m} (1 - x_\lambda - x_\mu) \prod_{1 \leq \lambda \leq m < \nu \leq n} (1 + x_\lambda - x_\nu)(1 + x_\lambda + x_\nu) \ mod\ I_G^+.$$

In particular, the first Chern class $c_1(G/K, J)$ is given by

$$c_1(G/K, J) = (2n - 3m + 1)\sigma_1.$$

Since $\sigma_1 \neq 0$, we see that $c_1(G/K, J) = 0$ if and only if

$$G/K = SO(3m-1)/(U(m) \times SO(m-1)) \ (m \geq 2; odd).$$

Together with Theorem 4.1, this completes the proof of the Main Theorem for the case G is classical.

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