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ON COLLINEARITY, PARALLELISM AND SPHERICITY FOR PAIRS OF CURVES

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To Professor S.A. Robertson on his 55th birthday

1. The relationship between the notions of collinearity and equichordality [1] is similar to the one between the notions of parallelism and self-parallelism [2]. In [1] some results concerning self-parallelism, equichordality and sphericity were proved. It is therefore natural to look for analogous results but now relating the ideas of parallelism, collinearity and sphericity. This is what we aim at in section 3 of this short note. For simplicity we shall consider only embeddings of S^1 into R^n but the proofs work equally well if we replace S^1 by a compact, connected, smooth manifold. In section 4 we deal with collinear equichordal embeddings and make a few simple considerations on lengths and chordal areas.

2. Let $f, g : S^1 \to R^n$ be smooth ($= C^1$) embeddings. We say that

- f and g are *parallel* if, for every $x \in S^1, N_f(x) = N_g(x)$, where $N_f(x)$ and $N_g(x)$ denote the normal (affine) hyperplanes to f and g.
- f is *self-parallel* if there is a non-trivial diffeomorphism $\delta : S^1 \to S^1$ such that f and $f \circ \delta$ are parallel.
- f and g are *collinear with respect to* p if, for every $x \in S^1, f(x), g(x)$ and p are distinct, collinear and $||f(x) - g(x)||$ does not depend on x.
- f is δ*-equichordal with respect to* p if there is a diffeomorphism $\delta : S^1 \to S^1$ such that f and $f \circ \delta$ are collinear with respect to p.
- f is *spherical with centre* p if $f(S^1)$ is contained in a round $(n-1)$-sphere with centre p.

We remark that if f is δ-equichordal then δ is an involution. For results on parallelism and equichordality we refer the reader to [2] and [1] respectively.

3. Let $f, g : S^1 \to R^n$ be collinear smooth embeddings with respect to p. Throughout this note, unless otherwise stated, we shall assume that p is the origin 0. Therefore $f(x) = \lambda(x)g(x)$, where $\lambda : S^1 \to R$ is smooth and $\lambda(x) > 1$ for every $x \in S^1$, or $0 < \lambda(x) < 1$ for every $x \in S^1$, or $\lambda(x) < 0$ for every $x \in S^1$.

A question which arises naturally when $n = 2$ concerns the position of 0 relatively to $f(S^1)$ and $g(S^1)$.

Theorem 3.1. *Let $f, g : S^1 \to R^2$ be smooth embeddings which are collinear with respect to 0. Then 0 is either inside both $f(S^1)$ and $g(S^1)$ or outside both $f(S^1)$ and $g(S^1)$.*

Proof. The proof is straightforward. Let $\lambda_f : S^1 \to S^1$ (resp. $\lambda_g : S^1 \to S^1$) be given by $\lambda_f(x) = f(x)/\|f(x)\|$ (resp. $\lambda_g(x) = g(x)/\|g(x)\|$). Since f and g are collinear we have either $\lambda_f = \lambda_g$ or $\lambda_f = A \circ \lambda_f$, where A is the antipodal map. Hence degree λ_f = degree λ_g and the result follows from the standard characterization of inside and outside of an embedding in terms of degrees [3].

It is not difficult to produce examples for both cases. In particular, for an example where 0 is outside do as follows. Let f be the standard embedding of S^1 as a circle in $R^+ \times R^+$ with radius r and centre a. Let c be greater than $\|a\| + r$. Then take $g : S^1 \to R^2$ given by $g(x) = (1 - c/\|f(x)\|)f(x)$.

Let us return to the case of embeddings of S^1 into R^n. We have

Theorem 3.2. *Let $f, g : S^1 \to R^n$ be smooth embeddings.*

a) If f and g are parallel and collinear with respect to 0 then f and g are spherical with centre 0.

b) If f and g are spherical with centre p and collinear with respect to 0 then they are parallel.

Proof. Case a) follows at once since if f and g are parallel and collinear with respect to 0 then every normal hyperplane to f passes through 0 and the same happens with every normal hyperplane to g. This implies that the distance-squared functions $D_f, D_g : S^1 \to R$ given by $D_f(x) = \|f(x)\|^2$ and $D_g(x) = \|g(x)\|^2$ are constant. Therefore $f(S^1)$ and $g(S^1)$ are contained in $(n-1)$-spheres with centre 0.

As to case b) we shall begin by assuming that $p = 0$. Then if f and g are collinear and spherical with centre 0 it follows that $f = cg$, where c is a non-zero constant. Therefore $g_{*_x}(T_xS^1) = f_{*_x}(T_xS^1)$, where f_{*_x} and g_{*_x} denote the linear maps between tangent spaces induced by f and g. Since f and g are spherical we have, for every $x \in S^1, N_f(x) = N_g(x)$.

Suppose next that $0 \neq p$. Then either we have a situation as shown in Figure 1a, for all but possibly two points in S^1, or a situation as shown in Figure 1b, again for all but possibly two points in S^1.

Recall that $\|f(x) - p\|, \|f(x) - g(x)\|$ and $\|g(x) - p\|$ do not depend on x, that is the lengths of the sides of the triangles $\{p, g(x), f(x)\}$ are constant. Arguments from elementary plane trigonometry and continuity make it possible to conclude that $\|f(x)\|$ and $\|g(x)\|$ are constant. Consequently f and g are spherical with centre 0 and the result follows as in the first part of the proof.

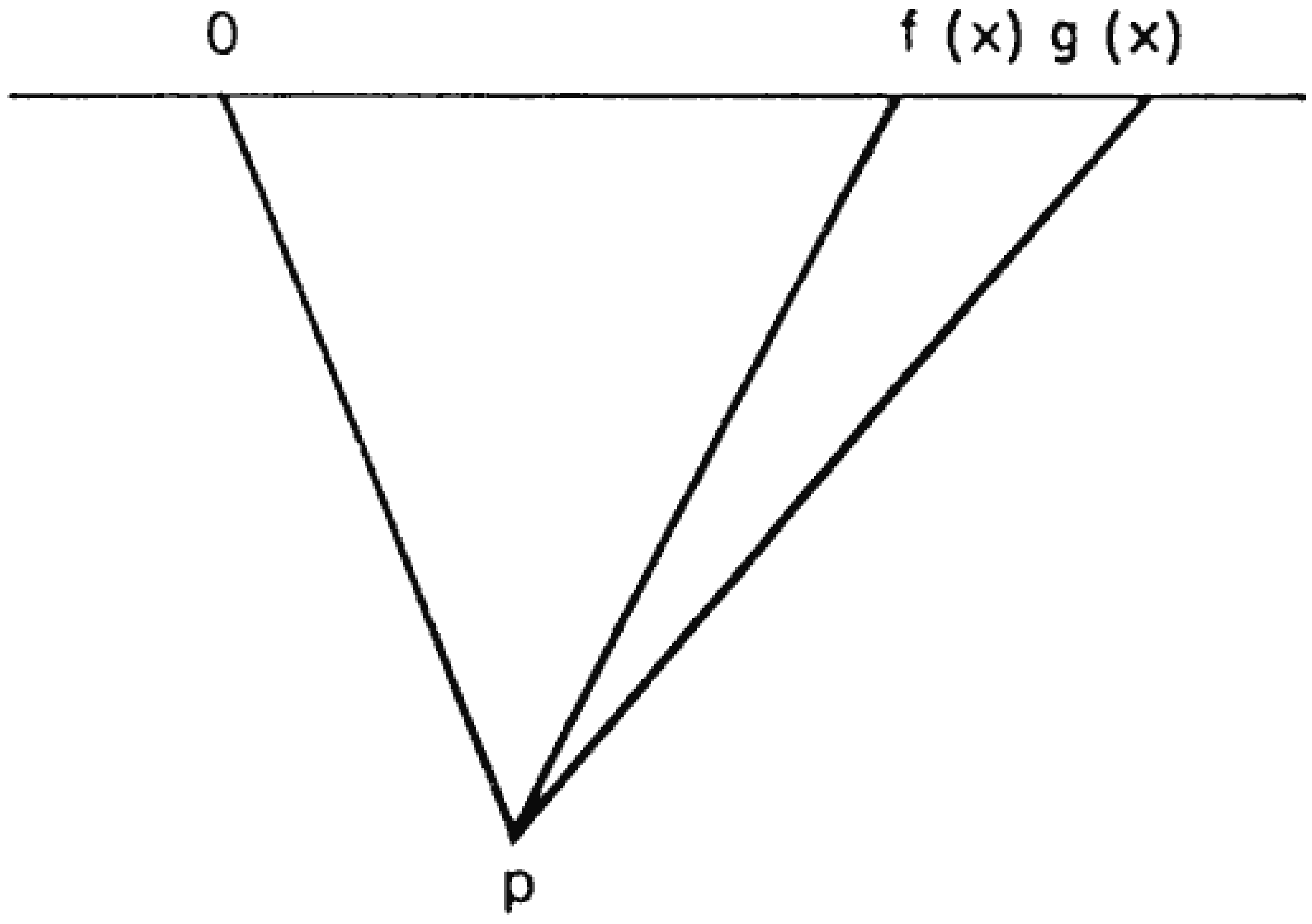

Figure 1a

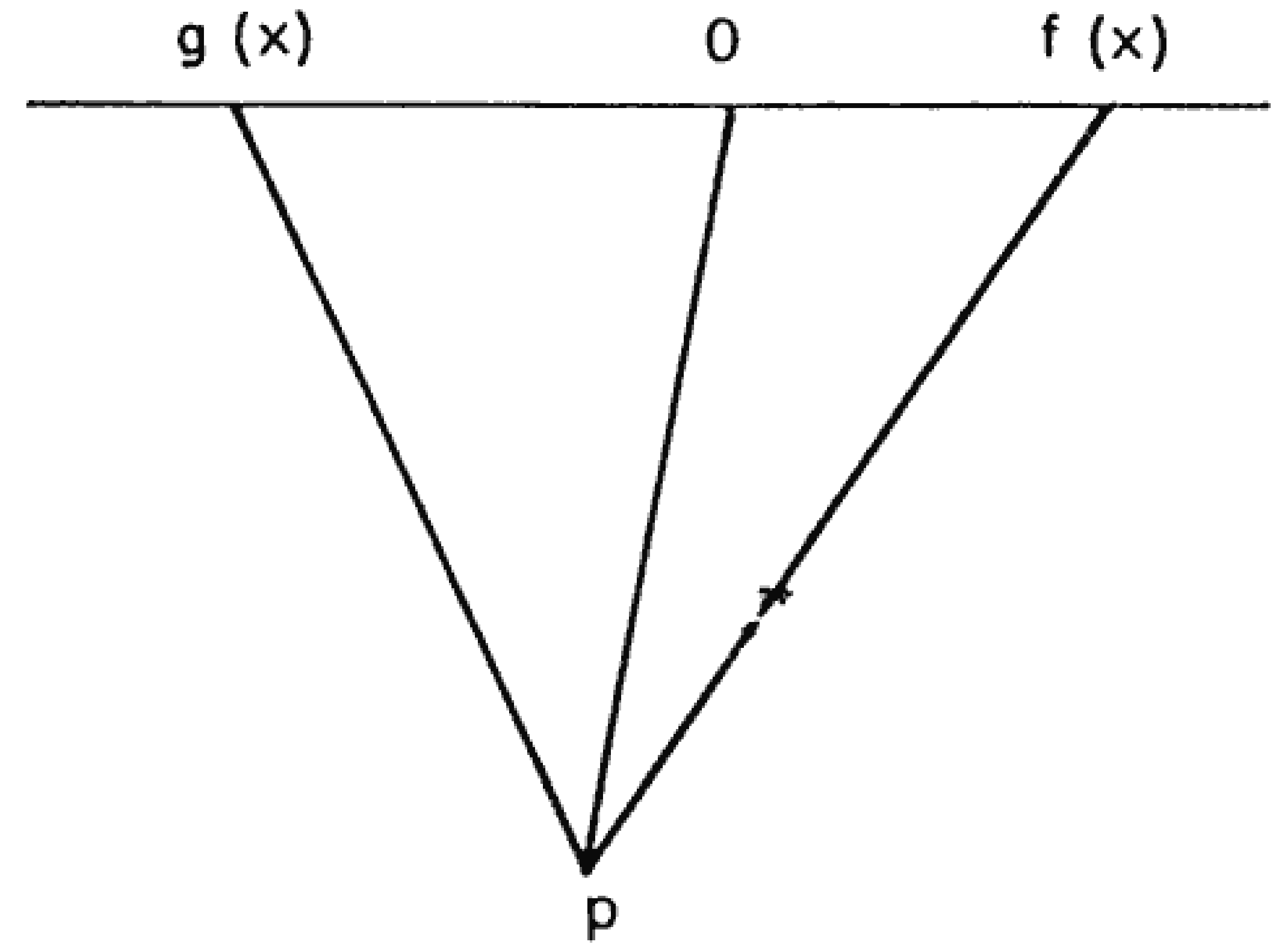

Figure 1b

The proof of case b) shows that for $p \neq 0, n = 3$, spherical collinear curves are not particularly interesting. Either they have concentric circles in a plane as images or their images are circles lying in distinct parallel planes, the straight line determined by the centres being normal to both.

We point out that case b) generalizes theorem 4.3 of [1].

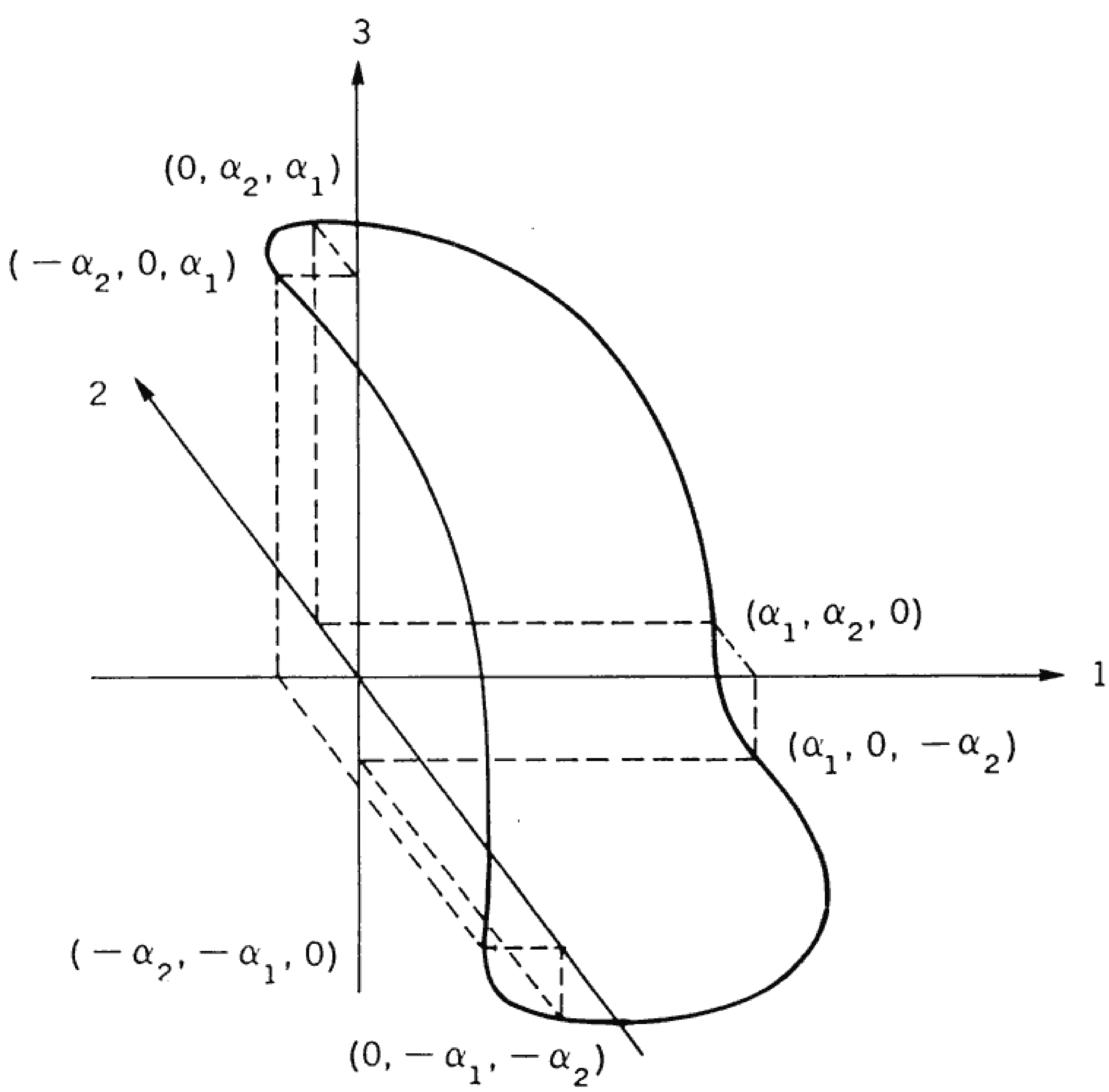

Figure 2

It is not true that if $f, g : S^1 \to R^n$ are parallel and spherical then they are collinear. To see this we take the smooth embedding $f : S^1 \to R^3$ whose image is shown in Figure 2.

Such an embedding appears in [4] and it is spherical and self-parallel. There is in fact just one non-trivial diffeomorphism $\delta : S^1 \to S^1$ such that f and $f \circ \delta$ are parallel. If f and $f \circ \delta$ were collinear with respect to p that would mean that f was δ-equichordal with respect to p. Hence p would belong to all the line segments $f(x) f(\delta(x))$ [1]. It is clear that no point with such a property exists for f.

4. Let $f, g : S^1 \to R^n$ be smooth embeddings and assume that f and g are collinear with respect to 0 and that f is δ-equichordal with respect to p. One might be tempted to think that 0 and p must coincide and that g would also be δ-equichordal with respect to p. This is not the case as the example following Theorem 3.1 shows. In fact in that example f is antipodally equichordal with respect to a and f and g are collinear with respect to 0. Obviously g cannot also be equichordal with respect to a since a is outside $g(S^1)$ [1]. However if $p = 0$ we have

Proposition 4.1. *If f and g are collinear with respect to 0 and f is δ-equichordal with respect to 0 then g is also δ-equichordal with respect to 0.*

Proof. Write $g(x) = f(x) + \lambda(x)(f(x) - f(\delta(x)))/r$, with $r = \|f(x) - f(\delta(x))\|$. Then $\|g(x) - f(x)\| = |\lambda(x)|$ and it follows that $\lambda : S^1 \to R$ is a constant map. We denote $\lambda(x)$ by c. It is not difficult now to obtain $\|g(x) - g(\delta(x))\| = |r + 2c|$.

Theorem 4.1. *Let f and g be collinear with respect to 0 and assume that f is δ-equichordal with respect to 0. Then*

$$|r + 2c|\pi \leq \mathit{length}\ g \leq (|1 + c/r| + |c|/r)\,\mathit{length}\ f$$

where r and c are as above.

Proof. We shall regard f and g as periodic maps from R into R^n assuming f to be parametrized by arc-length. Accordingly δ will be taken as a lift of $\delta : S^1 \to S^1$ which since this last map is orientation-preserving will have positive derivative. Moreover $\delta(x + \ell) = \delta(x) + \ell$, with $\ell = \text{length} f$. Take $g(x) = f(x) + (c/r)(f(x) - f(\delta(x)))$. Then $g'(x) = f'(x) + (c/r)f'(x) - (c/r)\delta'(x)f'(\delta(x))$ and $\|g'(x)\| \leq |1 + c/r| + (|c|/r)\delta'(x)$.

Therefore length $g \leq (|1 + c/r| + |c|/r)$ length f.

The other inequality follows from the proof of proposition 4.1 and theorem 5.1 of [1].

We shall go on assuming that f and g are collinear and that f is δ-equichordal with respect to 0. Hence g is also δ-equichordal with respect to 0. Moreover the map λ_f defined in the proof of theorem 3.1 will be assumed to be an embedding. Recall that $\lambda_f = \lambda_g$ or

$\lambda_f = A \circ \lambda_g$. Also f, g, λ_f and λ_g will be regarded as maps from R into R^n with period 1. Let us denote by μ a change of parameter such that $\lambda_f \circ \mu$ and $\lambda_g \circ \mu$ are parametrized by arc-length. We can then speak of the chordal areas $A(f)$ and $A(g)$. They are given by

$$A(f) = (1/2)\int_0^L ||(f \circ \mu)(t)||^2 dt, A(g) = (1/2)\int_0^L ||(g \circ \mu)(t)||^2 dt$$

where L is the length of λ_f and λ_g (see [1] for details).

Theorem 4.2. *Let f and g be collinear with respect to 0 with f δ-equichordal with respect to 0 and such that λ_f is an embedding. Then*

$$(1/4)\pi|r + 2c|^2 \leq A(g) \leq A(f) + (1/2)L(|c|^2 + r|c|)$$

where r, c and L are as above.

Proof. The left-hand side inequality follows from the proof of proposition 4.1 and theorem 5.3 of [1].

As to the right-hand side inequality we start from $g(x) = f(x) + (c/r)(f(x) - f(\delta(x)))$. Then $||g(\mu(x))||^2 \leq (||f(\mu(x))|| + |c|)^2$ and consequently $A(g) \leq A(f) + (1/2)|c|^2 L + |c|\int_0^L ||f(\mu(x))||dx$. Since $||f(\mu(x))|| + ||f(\mu(x + (1/2)L))|| = r[1]$ we conclude that $\int_0^L ||f(\mu(x))||dx = (1/2)rL$ and the results follows.

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Thanks are also due to the referee whose comments led to a sharper version of our precious theorem 4.2.

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