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ON FAMILIES OF VARIETIES MEASURABLE WITH RESPECT TO THE SIMILARITY GROUP IN THE THREE-DIMENSIONAL SPACE

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Abstract. *In this note all the families of varieties which are measurable with respect to a subgroup of the general similarity transformations group of the space* $\mathbf{R}^3$ *are determined.*

1. Let G_7 be the similarity transformation in the affine space:

$$(1)\qquad \begin{cases} x' = h[(1+l^2-m^2-n^2)x + 2(lm-n)y + 2(ln+m)z] + a \\ y' = h[2(lm+n)x + (1-l^2+m^2-n^2)y + 2(mn-l)z] + b \\ z' = h[2(ln-m)x + 2(mn+l)y + (1-l^2-m^2+n^2)z] + c \end{cases}$$

where $h, l, m, n, a, b, c \in \mathbf{R}$ $h \neq 0$ and $h(1+l^2+m^2+n^2)$ is the homothetic ratio [2]. This group is generated by the infinitesimal transformations

$$X_1 f = x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} + z\frac{\partial f}{\partial z} \quad X_2 f = y\frac{\partial f}{\partial z} - z\frac{\partial f}{\partial y} \quad X_3 f = -x\frac{\partial f}{\partial z} + z\frac{\partial f}{\partial x}$$

$$X_4 f = x\frac{\partial f}{\partial y} - y\frac{\partial f}{\partial x} \quad X_5 f = \frac{\partial f}{\partial x} \quad X_6 f = \frac{\partial f}{\partial y} \quad X_7 f = \frac{\partial f}{\partial z}$$

Remark 1. $X_1 f$ is a generator for the dilatation group, $\langle X_2 f\rangle\langle X_3 f\rangle\langle X_4 f\rangle$ are the groups of rotations around the axes x, y, z respectively and $X_5 f, X_6 f, X_7 f$ generate the translation group.

In the note [3] I have determined all the subgroups of the group (1). These subgroups, when written down through the infinitesimal transformations $X_i f$, are, up to isomorphism:

$$G_1^1(\lambda) = [X_1 f + \lambda X_4 f]_{\lambda\in\mathbf{R}}, \quad G_1^2(\epsilon) = [X_4 f + \epsilon X_7 f]_{\epsilon\in\{0,1\}}, \quad G_1^3 = [X_7 f];$$

$$G_2^1 = [X_1 f, X_4 f], \quad G_2^2(\lambda) = [X_1 f + \lambda X_4 f, X_7 f]_{\lambda\in\mathbf{R}}, \quad G_2^3 = [X_4 f, X_7 f];$$

$$G_2^4 = [X_6 f, X_7 f];$$

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$$G_3^1 = [X_1 f, X_4 f, X_7 f], \quad G_3^2(\lambda) = [X_1 f + \lambda X_4 f, X_5 f, X_6 f]_{\lambda \in \mathbb{R}},$$

$$G_3^3(\epsilon) = [X_4 f + \epsilon X_7 f, X_5 f, X_6 f]_{\epsilon \in \{0,1\}}, \quad G_3^4 = [X_2 f, X_3 f, X_4 f],$$

$$G_3^5 = [X_5 f, X_6 f, X_7 f];$$

$$G_4^1 = [X_1 f, X_2 f, X_3 f, X_4 f], \quad G_4^2 = [X_1 f, X_2 f, X_6 f, X_7 f],$$

$$G_4^3 = [X_1 f, X_5 f, X_6 f, X_7 f], \quad G_4^4 = [X_4 f, X_5 f, X_6 f, X_7 f];$$

$$G_5^1(\lambda, \mu) = [X_1 f, X_2 f + \lambda X_3 f + \mu X_4 f, X_5 f, X_6 f, X_7 f]_{\lambda\mu \in \mathbb{R}};$$

$$G_6^1 = [X_2 f, X_3 f, X_4 f, X_5 f, X_6 f, X_7 f]; \text{(}^1\text{)}$$

From the general theory [3] it is known that, if G is a transformation group in the homogeneous space X with coordinates $(x_1, \dots, x_n)$ and G_r is a subgroup of G depending on r parameters, a family of varieties $\mathscr{F}_q$ depending on the parameters $(\alpha_1, \dots, \alpha_q)$ for which G_r is the maximal invariance group, has an associated group H_r depending on the parameters $(\alpha_1, \dots, \alpha_q)$ isomorphic to G_r. One obtains the equations of these families of varieties as solutions of the system:

$$(2) \qquad \xi_h^i(x) \frac{\partial F(x,\alpha)}{\partial x_i} + \eta_h^k(\alpha) \frac{\partial F(x,\alpha)}{\partial \alpha_k} = 0 \quad i = 1, \dots, n; k = 1, \dots, q \quad (^2)$$

In this note we shall find all the families of varieties admitting the group (1) or one of its subgroups as invariance group and, particularly, the measurable families.

2. FAMILIES OF VARIETIES DEPENDING ON 7 PARAMETERS

We shall present in detail a method for obtaining the families $\mathscr{F}_7$. In order to determine these families in the parameters $\alpha_i (i = 1, \dots, 7)$ let be $H_7(\alpha_1, \dots, \alpha_7)$ a transitive group depending on 7 parameters isomorphic to the group $G_7 = [X_1 f, X_2 f, X_3 f, X_4 f, X_5 f, X_6 f, X_7 f]$ of similitudes in the space A_3; and let be $A_i f = a_i^j \dfrac{\partial f}{\partial \alpha_j} (i = 1, \dots, 7)$ its infinitesimal transformations: these transformations will satisfy the equations

$$(3.a) \qquad \begin{gathered} (A_1, A_2) = (A_1, A_3) = (A_1, A_4) = 0; \\ (A_1, A_5) = -A_5; (A_1, A_6) = -A_6; (A_1, A_7) = -A_7 \end{gathered}$$

(1) The groups $G_5^1(\lambda, \mu)$ and G_6^1 have been determined by Cirlincione [1].

(2) We use Einstein's convention.

(3.b)

$$(A_2, A_3) = -A_4; (A_2, A_4) = -A_3; (A_2, A_5) = 0; (A_2, A_6) = -A_7; (A_2, A_7) = A_6$$

(3.c) $$(A_3, A_4) = -A_2; (A_3, A_5) = A_7; (A_3, A_6) = 0; (A_3, A_7) = -A_5$$

(3.d) $$(A_4, A_5) = -A_6; (A_4, A_6) = A_5; (A_4, A_7) = 0$$

(3.e) $$(A_5, A_6) = (A_5, A_7) = (A_6, A_7) = 0$$

The change of variables $\alpha'_k = \varphi_k(\alpha_1, \dots, \alpha_7)(k = 1, \dots, 7)$ with $\frac{D(\varphi_1, \dots, \varphi_7)}{D(\alpha_1, \dots, \alpha_7)} \neq 0$ where the functions φ_i satisfy [3]

$$\begin{cases} a_7^k \dfrac{\partial \varphi_i}{\partial \alpha_k} = 0 \quad i = 1, \dots, 6 \\ a_7^k \dfrac{\partial \varphi_7}{\partial \alpha_k} = 1 \end{cases}$$

brings the infinitesimal transformation $A_7 f = a_7^j \frac{\partial f}{\partial \alpha_j}$ in the form $A'_7 f = \frac{\partial f}{\partial \alpha'_7}$ and so we can set $A_7 f = \frac{\partial f}{\partial \alpha_7}$.

By the third equation (3.e) we have that $A_6 f = a_6^j \frac{\partial f}{\partial \alpha_j}$ with $a_6^j = a_6^j(\alpha_1, \dots, \alpha_6)$.

The change of variables

$$\begin{cases} \alpha'_k = \varphi_k(\alpha_1, \dots, \alpha_6) \quad k = 1, \dots, 6 \\ \alpha'_7 = \alpha_7 + \varphi_7(\alpha_1, \dots, \alpha_6) \end{cases}$$

with $\frac{D(\varphi_1, \dots, \varphi_6)}{D(\alpha_1, \dots, \alpha_6)} \neq 0$ and where the funcitons φ_i satisfy

$$\begin{cases} a_6^k \dfrac{\partial \varphi_i}{\partial \alpha_6} = 0 \quad (i = 1, \dots, 5) \\ a_6^k \dfrac{\partial \varphi_6}{\partial \alpha_k} = 1 \\ a_6^k \dfrac{\partial \varphi_7}{\partial \alpha_k} + a_6^7 = 0 \end{cases}$$

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(3) The system is certainly compatible, due to transitivity of the group H_7, the functions $\alpha_7^j(\alpha_1, \dots, \alpha_7)$ are not all equal to zero: likewise for the system below.

leaves $A_7 f$ unchanged and brings the infinitesimal transformation $A_6 f$ in the form $A'_6 f = \dfrac{\partial f}{\partial \alpha'_6}$ and we can set $A_6 f = \dfrac{\partial f}{\partial \alpha_6}$.

By the first two equations (3.e) we have $A_5 f = a_5^j \dfrac{\partial f}{\partial \alpha_j}$ with $a_5^j = a_5^j(\alpha_1, \dots, \alpha_5)$.

The change of variables

$$\begin{cases} \alpha'_k = \varphi_k(\alpha_1, \dots, \alpha_5) \; k = 1, \dots, 5 \\ \alpha'_m = \alpha_m + \varphi_m(\alpha_1, \dots, \alpha_5) \; m = 6, 7 \end{cases}$$

with $\dfrac{D(\varphi_1, \dots, \varphi_5)}{D(\alpha_1, \dots, \alpha_5)} \neq 0$ and where the functions φ_i satisfy

$$\begin{cases} a_5^k \dfrac{\partial \varphi_i}{\partial \alpha_k} = 0 \; (i = 1, \dots, 4) \\ a_5^k \dfrac{\partial \varphi_5}{\partial \alpha_k} = 1 \\ a_5^k \dfrac{\partial \varphi_m}{\partial \alpha_k} + a_5^m = 0 \; m = 6, 7 \end{cases}$$

leaves $A_6 f$ and $A_7 f$ unchanged and brings $A_5 f$ in the form $A'_5 f = \dfrac{\partial f}{\partial \alpha'_5}$ and so we can set $A_5 f = \dfrac{\partial f}{\partial \alpha_5}$. By the last three equations (3.a) we have $A_1 f = \alpha_5 \dfrac{\partial f}{\partial \alpha_5} + \alpha_6 \dfrac{\partial f}{\partial \alpha_6} + \alpha_7 \dfrac{\partial f}{\partial \alpha_7} + a_1^j \dfrac{\partial f}{\partial \alpha_j} (j = 1, \dots, 7)$ with $a_1^j = a_1^j(\alpha_1, \dots \alpha_4)$. The change of variables

$$\begin{cases} \alpha'_k = \varphi_k(\alpha_1, \dots, \alpha_4) \; i = 1, \dots, 4 \\ \alpha'_m = \alpha_m + \varphi_m(\alpha_1, \dots, \alpha_4) \; m = 5, 6, 7 \end{cases}$$

with $\dfrac{D(\varphi_1, \dots, \varphi_4)}{D(\alpha_1, \dots, \alpha_4)} \neq 0$ and where the functions φ_i satisfy

$$\begin{cases} a_1^k \dfrac{\partial \varphi_i}{\partial \alpha_k} = 0 \qquad i = 1, 2, 3 \\ a_1^k \dfrac{\partial \varphi_4}{\partial \alpha_k} = 1 \\ a_1^k \dfrac{\partial \varphi_m}{\partial \alpha_k} + a_1^m - \varphi_m = 0 \quad m = 5, 6, 7 \end{cases}$$

leaves $A_5 f, A_6 f, A_7 f$ unchanged and brings $A_1 f$ in the form $A_1' f = \frac{\partial f}{\partial \alpha_4'} + \alpha_5' \frac{\partial f}{\partial \alpha_5'} + \alpha_6' \frac{\partial f}{\partial \alpha_6'} + \alpha_7' \frac{\partial f}{\partial \alpha_7'}$ and so we can set $A_1 f = \frac{\partial f}{\partial \alpha_4} + \alpha_5 \frac{\partial f}{\partial \alpha_5} + \alpha_6 \frac{\partial f}{\partial \alpha_6} + \alpha_7 \frac{\partial f}{\partial \alpha_7}$.

By the first equation (3a) and by the equations (3b) we have:

$$A_2 f = a_2^k \frac{\partial f}{\partial \alpha_k} + e^{\alpha_4} \left(a_2^h \frac{\partial f}{\partial \alpha_h} \right) - \alpha_7 \frac{\partial f}{\partial \alpha_6} + \alpha_6 \frac{\partial f}{\partial \alpha_7} \quad (k = 1, \ldots, 4; h = 5, 6, 7)$$

with $a_2^k = a_2^k(\alpha_1, \alpha_2, \alpha_3)$. The change of variables

$$\begin{cases} \alpha_k' = \varphi_k(\alpha_1, \alpha_2, \alpha_3) \quad k = 1, 2, 3 \\ \alpha_4' = \alpha_4 + \varphi_4(\alpha_1, \alpha_2, \alpha_3) \\ \alpha_m' = \alpha_m + e^{\alpha_4} \varphi_m(\alpha_1, \alpha_2, \alpha_3) \quad m = 5, 6, 7 \end{cases}$$

with $\frac{D(\varphi_1, \varphi_2, \varphi_3)}{D(\alpha_1, \alpha_2, \alpha_3)} \neq 0$ and where the functions φ_i satisfy

$$\begin{cases} a_2^k \frac{\partial \varphi_i}{\partial \alpha_k} = 0 \quad i = 1, 2 \\ a_2^k \frac{\partial \varphi_3}{\partial \alpha_k} = 1 \\ a_2^k \frac{\partial \varphi_4}{\partial \alpha_k} + a_2^4 = 0 \\ a_2^k \frac{\partial \varphi_5}{\partial \alpha_k} + a_2^4 \varphi_5 + a_2^5 = 0 \\ a_2^k \frac{\partial \varphi_6}{\partial \alpha_k} + a_2^4 \varphi_6 + a_2^6 + \varphi_7 = 0 \\ a_2^k \frac{\partial \varphi_7}{\partial \alpha_k} + a_2^4 \varphi_7 + a_2^7 - \varphi_6 = 0 \end{cases}$$

leaves $A_1 f, A_5 f, A_6 f$ and $A_7 f$ unchanged and brings $A_2 f$ in the form $A_2' f = \frac{\partial f}{\partial \alpha_3'} + \alpha_7' \frac{\partial f}{\partial \alpha_6'} + \alpha_6' \frac{\partial f}{\partial \alpha_7'}$ and so we can set $A_2 f = \frac{\partial f}{\partial \alpha_3} - \alpha_7 \frac{\partial f}{\partial \alpha_6} + \alpha_6 \frac{\partial f}{\partial \alpha_7}$.

The further conditions expressed by the equations (3) give

$$A_3 f = (a_j \cos \alpha_3 + b_j \operatorname{sen} \alpha_3) \frac{\partial f}{\partial \alpha_j} + e^{\alpha_4} (a_5 \cos \alpha_3 + b_5 \operatorname{sen} \alpha_3) \frac{\partial f}{\partial \alpha_5} +$$

$$+e^{\alpha_4}(a_6 \cos 2\alpha_3 + b_6 \operatorname{sen} 2\alpha_3 + a_7)\frac{\partial f}{\partial \alpha_6} + e^{\alpha_4}(a_6 \operatorname{sen} 2\alpha_3 - b_6 \cos 2\alpha_3 + b_7)\frac{\partial f}{\partial \alpha_7}+$$

$$+\alpha_7 \frac{\partial f}{\partial \alpha_5} - \alpha_5 \frac{\partial f}{\partial \alpha_7}$$

$$A_4 f = (a_j \operatorname{sen} \alpha_3 - b_j \cos \alpha_3)\frac{\partial f}{\partial \alpha_j} + e^{\alpha_4}(a_5 \operatorname{sen} \alpha_3 - b_5 \cos \alpha_3)\frac{\partial f}{\partial \alpha_5}+$$

$$+e^{\alpha_4}(a_6 \operatorname{sen} 2\alpha_3 - b_6 \cos 2\alpha_3 - b_7)\frac{\partial f}{\partial \alpha_6}+$$

$$+e^{\alpha_4}(-a_6 \cos 2\alpha_3 - b_6 \operatorname{sen} 2\alpha_3 + a_7)\frac{\partial f}{\partial \alpha_7} - \alpha_6 \frac{\partial f}{\partial \alpha_5} + \alpha_5 \frac{\partial f}{\partial \alpha_6}$$

with $j = 1, \dots, 4$ and where $a_s = a_s(\alpha_1, \alpha_2), b_s = b_s(\alpha_1, \alpha_2); s = 1, \dots, 7$, such that $a_1 b_2 - a_2 b_1 \neq 0$ and $(A_3 f, A_4 f) = A_2 f$.

The change of variables

$$\begin{cases} \alpha'_k = \varphi_k(\alpha_1, \alpha_2) \quad k = 1,2 \\ \alpha'_m = \alpha_m + \varphi_m(\alpha_1, \alpha_2) \quad m = 3,4 \\ \alpha'_5 = \alpha_5 + e^{\alpha_4}\varphi_5(\alpha_1, \alpha_2) \\ \alpha'_6 = \alpha_6 + e^{\alpha_4}[\varphi_6(\alpha_1, \alpha_2) \cos \alpha_3 - \varphi_7(\alpha_1, \alpha_2) \operatorname{sen} \alpha_3] \\ \alpha'_7 = \alpha_7 + e^{\alpha_4}[\varphi_6(\alpha_1, \alpha_2) \operatorname{sen} \alpha_3 + \varphi_7(\alpha_1, \alpha_2) \cos \alpha_3] \end{cases}$$

with $\dfrac{D(\varphi_1, \varphi_2)}{D(\alpha_1, \alpha_2)} \neq 0$ and where the functions φ_i satisfy

$$
\begin{cases}
a_1 \dfrac{\partial \varphi_1}{\partial \alpha_1} + a_2 \dfrac{\partial \varphi_1}{\partial \alpha_2} = \sqrt{(1+\varphi_2^2)} \cos \varphi_3 \\
b_1 \dfrac{\partial \varphi_1}{\partial \alpha_1} + b_2 \dfrac{\partial \varphi_1}{\partial \alpha_2} = -\sqrt{(1+\varphi_2^2)} \operatorname{sen} \varphi_3 \\
a_1 \dfrac{\partial \varphi_2}{\partial \alpha_1} + a_2 \dfrac{\partial \varphi_2}{\partial \alpha_2} = (1+\varphi_2^2) \operatorname{sen} \varphi_3 \\
b_1 \dfrac{\partial \varphi_2}{\partial \alpha_1} + b_2 \dfrac{\partial \varphi_2}{\partial \alpha_2} = (1+\varphi_2^2) \cos \varphi_3 \\
a_1 \dfrac{\partial \varphi_3}{\partial \alpha_1} + a_2 \dfrac{\partial \varphi_3}{\partial \alpha_2} = -\varphi_2 \cos \varphi_3 \\
b_1 \dfrac{\partial \varphi_3}{\partial \alpha_1} + b_2 \dfrac{\partial \varphi_3}{\partial \alpha_2} = \varphi_2 \operatorname{sen} \varphi_3 \\
a_1 \dfrac{\partial \varphi_4}{\partial \alpha_1} + a_2 \dfrac{\partial \varphi_4}{\partial \alpha_2} + a_4 = 0 \\
b_1 \dfrac{\partial \varphi_4}{\partial \alpha_1} + b_2 \dfrac{\partial \varphi_4}{\partial \alpha_2} + b_4 = 0 \\
a_1 \dfrac{\partial \varphi_5}{\partial \alpha_1} + a_2 \dfrac{\partial \varphi_5}{\partial \alpha_2} + a_4 \varphi_5 + a_5 = \varphi_7 \\
b_1 \dfrac{\partial \varphi_5}{\partial \alpha_1} + b_2 \dfrac{\partial \varphi_5}{\partial \alpha_2} + b_4 \varphi_5 + b_5 = \varphi_6 \\
a_1 \dfrac{\partial \varphi_6}{\partial \alpha_1} + a_2 \dfrac{\partial \varphi_6}{\partial \alpha_2} - a_3 \varphi_7 + a_4 \varphi_6 + a_6 + a_7 = 0 \\
b_1 \dfrac{\partial \varphi_6}{\partial \alpha_1} + b_2 \dfrac{\partial \varphi_6}{\partial \alpha_2} - b_3 \varphi_7 + b_4 \varphi_6 + b_6 + b_7 + \varphi_5 = 0 \\
a_1 \dfrac{\partial \varphi_7}{\partial \alpha_1} + a_2 \dfrac{\partial \varphi_7}{\partial \alpha_2} + a_3 \varphi_6 + a_4 \varphi_7 - b_6 + b_7 + \varphi_5 = 0 \\
b_1 \dfrac{\partial \varphi_7}{\partial \alpha_1} + b_2 \dfrac{\partial \varphi_7}{\partial \alpha_2} + b_3 \varphi_6 + b_4 \varphi_7 + a_6 - a_7 = 0
\end{cases}
$$

leaves $A_1 f, A_2 f, A_5 f, A_6 f$ and $A_7 f$ unchanged and brings $A_3 f$ and $A_4 f$ in the form

$$
A_3 f = \sqrt{1+\alpha_2^2} \cos \alpha_3 \frac{\partial f}{\partial \alpha_1} + (1+\alpha_2^2) \operatorname{sen} \alpha_3 \frac{\partial f}{\partial \alpha_2} +
$$

$$
-\alpha_2 \cos \alpha_3 \frac{\partial f}{\partial \alpha_3} + \alpha_7 \frac{\partial f}{\partial \alpha_5} - \alpha_5 \frac{\partial f}{\partial \alpha_7}
$$

$$A_4 f = \sqrt{1+\alpha_2^2}\,\mathrm{sen}\,\alpha_3 \frac{\partial f}{\partial \alpha_1} - (1+\alpha_2^2)\cos\alpha_3 \frac{\partial f}{\partial \alpha_2} + \\ -\alpha_2\,\mathrm{sen}\,\alpha_3 \frac{\partial f}{\partial \alpha_3} - \alpha_6 \frac{\partial f}{\partial \alpha_5} + \alpha_5 \frac{\partial f}{\partial \alpha_6}$$

The measurable group $H_7 = [A_1 f, A_2 f, A_3 f, A_4 f, A_5 f, A_6 f, A_7 f]$ we have obtained admits the (elementary) measure

$$e^{-3\alpha_4}(1+\alpha_2^2)^{-3/2} d\alpha_1 \wedge d\alpha_2 \wedge \ldots \wedge d\alpha_7$$

where $e^{-3\alpha_4}(1+\alpha_2^2)^{-3/2}$ is the solution, up to a constant factor, of Deltheil's system for the group H_7. The required family of varieties depending on 7 parameters is obtained by integrating the system of type (2).

$$\begin{cases} \dfrac{\partial F}{\partial z} + \dfrac{\partial F}{\partial \alpha_7} = 0 \\ \dfrac{\partial F}{\partial y} + \dfrac{\partial F}{\partial \alpha_6} = 0 \\ \dfrac{\partial F}{\partial x} + \dfrac{\partial F}{\partial \alpha_5} = 0 \\ x\dfrac{\partial F}{\partial x} + y\dfrac{\partial F}{\partial y} + z\dfrac{\partial F}{\partial z} + \dfrac{\partial F}{\partial \alpha_4} + \alpha_5 \dfrac{\partial F}{\partial \alpha_5} + \alpha_6 \dfrac{\partial F}{\partial \alpha_6} + \alpha_7 \dfrac{\partial F}{\partial \alpha_7} = 0 \\ y\dfrac{\partial F}{\partial z} - z\dfrac{\partial F}{\partial y} + \dfrac{\partial F}{\partial \alpha_3} - \alpha_7 \dfrac{\partial F}{\partial \alpha_6} + \alpha_6 \dfrac{\partial F}{\partial \alpha_7} = 0 \\ -x\dfrac{\partial F}{\partial z} + z\dfrac{\partial F}{\partial x} + \sqrt{1+\alpha_2^2}\cos\alpha_3 \dfrac{\partial F}{\partial \alpha_1} + (1+\alpha_2^2)\,\mathrm{sen}\,\alpha_3 \dfrac{\partial F}{\partial \alpha_2} - \alpha_2 \cos\alpha_3 \dfrac{\partial F}{\partial \alpha_3} + \\ \quad + \alpha_7 \dfrac{\partial F}{\partial \alpha_5} - \alpha_5 \dfrac{\partial F}{\partial \alpha_7} = 0 \\ x\dfrac{\partial F}{\partial y} - y\dfrac{\partial F}{\partial x} + \sqrt{1+\alpha_2^2}\,\mathrm{sen}\,\alpha_3 \dfrac{\partial F}{\partial \alpha_1} - (1+\alpha_2^2)\cos\alpha_3 \dfrac{\partial F}{\partial \alpha_2} - \alpha_2\,\mathrm{sen}\,\alpha_3 \dfrac{\partial F}{\partial \alpha_3} + \\ \quad - \alpha_6 \dfrac{\partial F}{\partial \alpha_5} + \alpha_5 \dfrac{\partial F}{\partial \alpha_6} = 0 \end{cases}$$

So we obtain the family of varieties $\mathscr{F}_7$ [(4)]

[(4)] We shall use the notation $\mathscr{F}_k$ for the measurable families of varieties depending on k parameters and $\overline{\mathscr{F}}_k$ for the non-measurable ones.

$$F(\widetilde{x},\widetilde{y},\widetilde{z}) = 0$$

where

$$\begin{cases} \widetilde{x} = \dfrac{e^{-\alpha_4}}{\sqrt{1+\alpha_2^2}}[\alpha_2(x-\alpha_5) + (y-\alpha_6)\cos\alpha_3 + (z-\alpha_7)\operatorname{sen}\alpha_3] \\ \widetilde{y} = \dfrac{e^{-\alpha_4}}{\sqrt{1+\alpha_2^2}}[(x-\alpha_5)\cos\alpha_1 + (y-\alpha_6)(\sqrt{1+\alpha_2^2}\operatorname{sen}\alpha_1 - \alpha_2\cos\alpha_1)\cos\alpha_3 + \\ \quad -(z-\alpha_7)(\alpha_2\cos\alpha_1 + \sqrt{1+\alpha_2^2}\operatorname{sen}\alpha_1)\operatorname{sen}\alpha_3] \\ \widetilde{z} = \dfrac{e^{-\alpha_4}}{\sqrt{1+\alpha_2^2}}[(x-\alpha_5)\operatorname{sen}\alpha_1 - (y-\alpha_6)(\alpha_2\operatorname{sen}\alpha_1 + \sqrt{1+\alpha_2^2}\cos\alpha_1)\cos\alpha_3 + \\ \quad +(z-\alpha_7)(\sqrt{1+\alpha_2^2}\cos\alpha_1 - \alpha_2\operatorname{sen}\alpha_1)\operatorname{sen}\alpha_3] \end{cases}$$

3. FAMILIES OF VARIETIES DEPENDING ON 6 PARAMETERS

3.1. Families admitting the group G_7 as maximal group

By using the same process as in section 2, one finds the groups:

$$H_6^1(\lambda) = \left[\frac{\partial f}{\partial\alpha_3} + \alpha_4\frac{\partial f}{\partial\alpha_4} + \alpha_5\frac{\partial f}{\partial\alpha_5} + \alpha_6\frac{\partial f}{\partial\alpha_6}; -(1+\alpha_1^2)\operatorname{sen}\alpha_2\frac{\partial f}{\partial\alpha_1} + \right.$$

$$+\alpha_1\cos\alpha_2\frac{\partial f}{\partial\alpha_2} + \lambda\sqrt{1+\alpha_1^2}\cos\alpha_2\frac{\partial f}{\partial\alpha_3} - \alpha_6\frac{\partial f}{\partial\alpha_5} + \alpha_5\frac{\partial f}{\partial\alpha_6};$$

$$(1+\alpha_1^2)\cos\alpha_2\frac{\partial f}{\partial\alpha_1} + \alpha_1\operatorname{sen}\alpha_2\frac{\partial f}{\partial\alpha_2} + \lambda\sqrt{1+\alpha_1^2}\operatorname{sen}\alpha_2\frac{\partial f}{\partial\alpha_3} + \alpha_6\frac{\partial f}{\partial\alpha_4} - \alpha_4\frac{\partial f}{\partial\alpha_6};$$

$$\left.\frac{\partial f}{\partial\alpha_2} - \alpha_5\frac{\partial f}{\partial\alpha_4} + \alpha_4\frac{\partial f}{\partial\alpha_5}, \frac{\partial f}{\partial\alpha_4}, \frac{\partial f}{\partial\alpha_5}, \frac{\partial f}{\partial\alpha_6}\right] \lambda \in \mathbf{R};$$

$$H_6^2 = \left[\frac{\partial f}{\partial\alpha_6}, \frac{\partial f}{\partial\alpha_3} - \alpha_5\frac{\partial f}{\partial\alpha_4} + \alpha_4\frac{\partial f}{\partial\alpha_5}, \sqrt{1+\alpha_2^2}\cos\alpha_3\frac{\partial f}{\partial\alpha_1} + (1+\alpha_2^2)\operatorname{sen}\alpha_3\frac{\partial f}{\partial\alpha_2} + \right.$$

$$-\alpha_2\cos\alpha_3\frac{\partial f}{\partial\alpha_3} - \alpha_4\alpha_5\frac{\partial f}{\partial\alpha_4} - (1+\alpha_5^2)\frac{\partial f}{\partial\alpha_5} + \alpha_5\frac{\partial f}{\partial\alpha_6}; \sqrt{1+\alpha_2^2}\operatorname{sen}\alpha_3\frac{\partial f}{\partial\alpha_1} +$$

$$-(1+\alpha_2^2)\cos\alpha_3\frac{\partial f}{\partial\alpha_2} + \alpha_2\operatorname{sen}\alpha_3\frac{\partial f}{\partial\alpha_3} + (1+\alpha_4^2)\frac{\partial f}{\partial\alpha_4} + \alpha_4\alpha_5\frac{\partial f}{\partial\alpha_5} - \alpha_4\frac{\partial f}{\partial\alpha_6},$$

$$e^{-\alpha_6}\left(-\alpha_4\frac{\partial f}{\partial \alpha_4}-\alpha_5\frac{\partial f}{\partial \alpha_5}+\frac{\partial f}{\partial \alpha_6}\right), e^{-\alpha_6}\frac{\partial f}{\partial \alpha_4}, e^{-\alpha_6}\frac{\partial f}{\partial \alpha_5}\Bigg]$$

The group $H_6^1(\lambda)$ is measurable if and only if $\lambda = 0$ and, in this case, it has measure

$$e^{-3\alpha_3}(1+\alpha_1^2)^{-3/2}\, d\alpha_1 \wedge \ldots \wedge d\alpha_6$$

On the contrary the group H_6^2 is not measurable. In correspondence to the group $H_6^1(\lambda)$ one finds the families of varieties:

$$\overline{\mathcal{F}}_6^1(\lambda) : F\left(\frac{\tilde{y}}{\tilde{x}}\,\text{sen}\,\frac{ln|\tilde{x}|}{\lambda}+\frac{\tilde{z}}{\tilde{x}}\cos\frac{ln|\tilde{x}|}{\lambda}, \frac{\tilde{y}}{\tilde{x}}\cos\frac{ln|\tilde{x}|}{\lambda}-\frac{\tilde{z}}{\tilde{x}}\,\text{sen}\,\frac{ln|\tilde{x}|}{\lambda}\right)=0$$

if $\lambda \neq 0$ and

$$\mathcal{F}_6^1 : F(\tilde{x}, \tilde{y}^2+\tilde{z}^2) = 0 \quad if\ \lambda = 0$$

where

$$\begin{cases} \tilde{x} = \dfrac{e^{-\alpha_3}}{\sqrt{1+\alpha_1^2}}[(x-\alpha_4)\cos\alpha_2+(y-\alpha_5)\,\text{sen}\,\alpha_2-\alpha_1(z-\alpha_6)] \\ \tilde{y} = e^{-\alpha_3}[(y-\alpha_5)\cos\alpha_2-(x-\alpha_4)\,\text{sen}\,\alpha_2] \\ \tilde{z} = \dfrac{e^{-\alpha_3}}{\sqrt{1+\alpha_1^2}}[(z-\alpha_6)+\alpha_1(x-\alpha_4)\cos\alpha_2+\alpha_1(y-\alpha_5)\,\text{sen}\,\alpha_2] \end{cases}$$

In correspondence to the group H_6^2 one finds the family

$$\overline{\mathcal{F}}_6^2 : F\left(\frac{\tilde{x}}{\tilde{y}}, \frac{\tilde{z}}{\tilde{y}}\right) = 0$$

where

$$\begin{cases} \tilde{x} = \dfrac{1}{\sqrt{1+\alpha_2^2}}[xe^{-\alpha_6}-1-\alpha_2(ye^{-\alpha_6}-\alpha_4)\cos\alpha_3-\alpha_2(ze^{-\alpha_6}-\alpha_5)\,\text{sen}\,\alpha_3] \\ \quad \cos\alpha_1-[(ze^{-\alpha_4}-\alpha_5)\cos\alpha_3-(ye^{-\alpha_6}-\alpha_4)\,\text{sen}\,\alpha_3]\,\text{sen}\,\alpha_1 \\ \tilde{y} = \dfrac{1}{\sqrt{1+\alpha_2^2}}[(ye^{-\alpha_6}-\alpha_4)\cos\alpha_3+(ze^{-\alpha_6}-\alpha_5)\,\text{sen}\,\alpha_3+\alpha_2 xe^{-\alpha_6}-\alpha_2] \\ \tilde{z} = \dfrac{\text{sen}\,\alpha_1}{\sqrt{1+\alpha_2^2}}[xe^{-\alpha_6}-1-\alpha_2(ye^{-\alpha_6}-\alpha_4)\cos\alpha_3-\alpha_2(ze^{-\alpha_6}-\alpha_5)\,\text{sen}\,\alpha_3]+ \\ \quad +\cos\alpha_1[(ze^{-\alpha_6}-\alpha_5)\cos\alpha_3-(y-e^{-\alpha_6}-\alpha_4)\,\text{sen}\,\alpha_3] \end{cases}$$

3.2. Families admitting the group G_6 as maximal group

One finds the group

$$H_6^3 = \Big[\sqrt{1+\alpha_2^2}\cos\alpha_3 \frac{\partial f}{\partial \alpha_1} - (1+\alpha_2^2)\,\mathrm{sen}\,\alpha_3 \frac{\partial f}{\partial \alpha_2} + \alpha_2 \cos\alpha_3 \frac{\partial f}{\partial \alpha_3} - \alpha_6 \frac{\partial f}{\partial \alpha_5} + \alpha_5 \frac{\partial f}{\partial \alpha_6};$$

$$\sqrt{1+\alpha_2^2}\,\mathrm{sen}\,\alpha_3 \frac{\partial f}{\partial \alpha_1} + (1+\alpha_2^2)\cos\alpha_3 \frac{\partial f}{\partial \alpha_2} + \alpha_2\,\mathrm{sen}\,\alpha_3 \frac{\partial f}{\partial \alpha_3} + \alpha_6 \frac{\partial f}{\partial \alpha_4} - \alpha_4 \frac{\partial f}{\partial \alpha_6};$$

$$\frac{\partial f}{\partial \alpha_3} - \alpha_5 \frac{\partial f}{\partial \alpha_4} + \alpha_4 \frac{\partial f}{\partial \alpha_5}, \frac{\partial f}{\partial \alpha_4}, \frac{\partial f}{\partial \alpha_5}, \frac{\partial f}{\partial \alpha_6}\Big].$$

This group is measurable and it has measure

$$(1+\alpha_2^2)^{-3/2} d\alpha_1 \wedge \ldots \wedge d\alpha_6$$

It is associated to the family of varieties

$$\mathscr{F}_6^2 : F(\widetilde{x}, \widetilde{y}, \widetilde{z}) = 0$$

where

$$\begin{cases} \widetilde{x} = \dfrac{1}{\sqrt{1+\alpha_2^2}}[(x-\alpha_4)\cos\alpha_3 + (y-\alpha_5)\,\mathrm{sen}\,\alpha_3 - \alpha_2(z-\alpha_6)] \\ \widetilde{y} = [(y-\alpha_5)\cos\alpha_3 - (x-\alpha_4)\,\mathrm{sen}\,\alpha_3]\cos\alpha_1 + \\ \quad + \dfrac{\mathrm{sen}\,\alpha_1}{\sqrt{1+\alpha_2^2}}[(z-\alpha_6) + \alpha_2(x-\alpha_4)\cos\alpha_3 + \alpha_2(y-\alpha_5)\,\mathrm{sen}\,\alpha_3] \\ \widetilde{z} = \dfrac{\cos\alpha_1}{\sqrt{1+\alpha_2^2}}[(z-\alpha_6) + \alpha_2(x-\alpha_4)\cos\alpha_3 + \alpha_2(y-\alpha_5)\,\mathrm{sen}\,\alpha_3] + \\ \quad - [(y-\alpha_5)\cos\alpha_3 - (x-\alpha_4)\,\mathrm{sen}\,\alpha_3]\,\mathrm{sen}\,\alpha_1 \end{cases}$$

4. FAMILIES OF VARIETIES DEPENDING ON 5 PARAMETERS

4.1. Families admitting the group G_7 as maximal group

One finds the groups

$$H_5^1 = \Big[\frac{\partial f}{\partial \alpha_5}, (1+\alpha_1^2)\cos\alpha_2 \frac{\partial f}{\partial \alpha_1} + \alpha_1\,\mathrm{sen}\,\alpha_2 \frac{\partial f}{\partial \alpha_2} - \alpha_4 \frac{\partial f}{\partial \alpha_3} + \alpha_3 \frac{\partial f}{\partial \alpha_4};$$

$$(1+\alpha_1^2)\,\text{sen}\,\alpha_2 \frac{\partial f}{\partial \alpha_1} - \alpha_1 \cos\alpha_2 \frac{\partial f}{\partial \alpha_2} - \alpha_3\alpha_4 \frac{\partial f}{\partial \alpha_3} - (1+\alpha_4^2)\frac{\partial f}{\partial \alpha_4} + \alpha_4 \frac{\partial f}{\partial \alpha_5};$$

$$\frac{\partial f}{\partial \alpha_2} + (1+\alpha_3^2) + \alpha_3\alpha_4 \frac{\partial f}{\partial \alpha_4} - \alpha_3 \frac{\partial f}{\partial \alpha_5} : e^{-\alpha_5}\left(-\alpha_3 \frac{\partial f}{\partial \alpha_3} - \alpha_4 \frac{\partial f}{\partial \alpha_4} + \frac{\partial f}{\partial \alpha_5}\right);$$

$$\left. e^{-\alpha_5}\frac{\partial f}{\partial \alpha_3}, e^{-\alpha_5}\frac{\partial f}{\partial \alpha_4}\right]$$

and

$$H_5^2 = \left[\frac{\partial f}{\partial \alpha_5}, \alpha_2 \frac{\partial f}{\partial \alpha_1} - \alpha_1 \frac{\partial f}{\partial \alpha_2} - \alpha_4 \frac{\partial f}{\partial \alpha_3} + \alpha_3 \frac{\partial f}{\partial \alpha_4};\ (1+\alpha_1^2)\frac{\partial f}{\partial \alpha_1} + \alpha_1\alpha_2 \frac{\partial f}{\partial \alpha_2} + \right.$$

$$+\alpha_2\alpha_4 \frac{\partial f}{\partial \alpha_3} + \alpha_1\alpha_4 \frac{\partial f}{\partial \alpha_4}; -\alpha_1\alpha_2 \frac{\partial f}{\partial \alpha_1} - (1+\alpha_1^2)\frac{\partial f}{\partial \alpha_2} - \alpha_2\alpha_3 \frac{\partial f}{\partial \alpha_3} - \alpha_1\alpha_3 \frac{\partial f}{\partial \alpha_4};$$

$$\left. e^{-\alpha_5}\alpha_2 \frac{\partial f}{\partial \alpha_3} + \alpha_1 \frac{\partial f}{\partial \alpha_4}; e^{-\alpha_5}\frac{\partial f}{\partial \alpha_3}; E^{-\alpha_5}\frac{\partial f}{\partial \alpha_4}\right].$$

The group H_5^1 is not measurable and it corresponds to the family of varieties

$$\overline{\mathscr{F}}_5^1 : F\left(\frac{\tilde{x}^2 + \tilde{z}^2}{\tilde{y}^2}\right) = 0$$

where

$$\begin{cases} \tilde{x} = (xe^{-\alpha_5} - 1)\cos\alpha_2 + (ye^{-\alpha_5} - \alpha_3)\,\text{sen}\,\alpha_2 \\ \tilde{y} = \dfrac{1}{\sqrt{1+\alpha_1^2}}[(ye^{-\alpha_5} - \alpha_3)\cos\alpha_2 - (xe^{-\alpha_5} - 1)\,\text{sen}\,\alpha_2 + \alpha_1(ze^{-\alpha_5} - \alpha_4)] \\ \tilde{z} = \dfrac{1}{\sqrt{1+\alpha_1^2}}[(ze^{-\alpha_5} - \alpha_4) - \alpha_1(ye^{-\alpha_5} - \alpha_3)\cos\alpha_2 + \alpha_1(xe^{-\alpha_5} - 1)\,\text{sen}\,\alpha_2] \end{cases}$$

On the contrary the group H_5^2 is measurable and has measure

$$(1+\alpha_1^2+\alpha_2^2)^{-2}\, d\alpha_1 \wedge \ldots \wedge d/\alpha_5$$

and corresponds to the family of varieties:

$$\mathscr{F}_5^1 : F\left(\frac{(1+\alpha_1^2+\alpha_2^2)(\tilde{y}+\alpha_2\tilde{x})^2 + [(1+\alpha_2^2)\tilde{z} + \alpha_1\tilde{x} - \alpha_1\alpha_2\tilde{y}]^2}{(1+\alpha_2^2)(1+\alpha_1^2+\alpha_2^2)}\right) = 0$$

where

$$\begin{cases} \widetilde{x} = xe^{-\alpha_5} \\ \widetilde{y} = ye^{-\alpha_5} - \alpha_3 \\ \widetilde{z} = ze^{-\alpha_5} - \alpha_4 \end{cases}$$

4.2. Families admitting the group G_6 as maximal group

One finds the measurable group

$$H_5^3 = \Big[-(1+\alpha_1^2)\,\mathrm{sen}\,\alpha_2 \frac{\partial f}{\partial \alpha_1} + \alpha_1 \cos\alpha_2 \frac{\partial f}{\partial \alpha_2} - \alpha_5 \frac{\partial f}{\partial \alpha_4} + \alpha_4 \frac{\partial f}{\partial \alpha_5};$$

$$(1+\alpha_1^2)\cos\alpha_2 \frac{\partial f}{\partial \alpha_1} + \alpha_1\,\mathrm{sen}\,\alpha_2 \frac{\partial f}{\partial \alpha_2} + \alpha_5 \frac{\partial f}{\partial \alpha_3} - \alpha_3 \frac{\partial f}{\partial \alpha_5};$$

$$\frac{\partial f}{\partial \alpha_2} - \alpha_4 \frac{\partial f}{\partial \alpha_3} + \alpha_3 \frac{\partial f}{\partial \alpha_4}; \frac{\partial f}{\partial \alpha_3}; \frac{\partial f}{\partial \alpha_4}; \frac{\partial f}{\partial \alpha_5}\Big].$$

This group has measure

$$(1+\alpha_1^2)^{-3/2} d\alpha_1 \wedge \ldots \wedge d\alpha_5$$

and it is associated to the family

$$\mathscr{F}_5^2 : F(\widetilde{x}, \widetilde{y}^2 + \widetilde{z}^2) = 0$$

where

$$\begin{cases} \widetilde{x} = \dfrac{1}{\sqrt{1+\alpha_1^2}}[(x-\alpha_3)\cos\alpha_2 + (y-\alpha_4)\,\mathrm{sen}\,\alpha_2 - \alpha_1(z-\alpha_5)] \\ \widetilde{y} = (y-\alpha_4)\cos\alpha_2 - (x-\alpha_3)\,\mathrm{sen}\,\alpha_2 \\ \widetilde{z} = \dfrac{1}{\sqrt{1+\alpha_1^2}}[\alpha_1(x-\alpha_3)\cos\alpha_2 + \alpha_1(y-\alpha_4)\,\mathrm{sen}\,\alpha_2 + (z-\alpha_5)] \end{cases}$$

4.3. Families admitting the group $G_5(\lambda,\mu)$ as maximal group

For fixed $\lambda, \mu \in \mathbf{R}$ one finds the group

$$H_5^4 = \Big[\frac{\partial f}{\partial \alpha_1} + \alpha_3 \frac{\partial f}{\partial \alpha_3} + \alpha_4 \frac{\partial f}{\partial \alpha_4} + \alpha_5 \frac{\partial f}{\partial \alpha_5}; \frac{\partial f}{\partial \alpha_2} + (\lambda\alpha_5 - \mu\alpha_4)\frac{\partial f}{\partial \alpha_3} +$$

$$+(\mu\alpha_3 - \alpha_5)\frac{\partial f}{\partial \alpha_4} + (\alpha_4 - \lambda\alpha_3)\frac{\partial f}{\partial \alpha_5}; \frac{\partial f}{\partial \alpha_3}; \frac{\partial f}{\partial \alpha_4}; \frac{\partial f}{\partial \alpha_5}\Big]$$

which is measurable with measure

$$e^{3\alpha_1} d\alpha_1 \wedge \ldots \wedge d\alpha_5 .$$

In correspondence one obtains the family

$$\mathscr{F}_5^3 : F(\widetilde{x}, \widetilde{y}, \widetilde{z}) = 0$$

where

$$\begin{cases} \widetilde{x} = \dfrac{e^{-\alpha_1}}{1+\lambda^2+\mu^2}[(x-\alpha_3)+\lambda(y-\alpha_4)+\mu(z-\alpha_5)] \\ \widetilde{y} = \dfrac{e^{-\alpha_1}}{\sqrt{1+\lambda^2+\mu^2}}\left[-(x-\alpha_3)\left(\dfrac{\mu \operatorname{sen}\sqrt{1+\lambda^2+\mu^2}\,\alpha_2}{\sqrt{1+\lambda^2+\mu^2}}+\right.\right. \\ \qquad \left. +\dfrac{\lambda(1+\lambda^2+\mu^2)\cos\sqrt{1+\lambda^2+\mu^2}\,\alpha_2}{1+\lambda^2}\right)+ \\ \qquad +(y-\alpha_4)\left(\dfrac{(1+\lambda^2+\mu^2)\cos\sqrt{1+\lambda^2+\mu^2}\,\alpha_2}{1+\lambda^2}-\dfrac{\mu\lambda \operatorname{sen}\sqrt{1+\lambda^2+\mu^2}\,\alpha_2}{\sqrt{1+\lambda^2+\mu^2}}\right)+ \\ \qquad \left. +(z-\alpha_5)\dfrac{(1+\lambda^2)\operatorname{sen}\sqrt{1+\lambda^2+\mu^2}\,\alpha_2}{\sqrt{1+\lambda^2+\mu^2}}\right] \\ \widetilde{z} = \dfrac{e^{-\alpha_1}}{\sqrt{1+\lambda^2+\mu^2}}\left[(x-\alpha_3)\left(\dfrac{\cos\sqrt{1+\lambda^2+\mu^2}\,\alpha_2}{\sqrt{1+\lambda^2+\mu^2}}+\right.\right. \\ \qquad \left. -\dfrac{\lambda(1+\lambda^2+\mu^2)\operatorname{sen}\sqrt{1+\lambda^2+\mu^2}\,\alpha_2}{1+\lambda^2}\right)+ \\ \qquad +(y-\alpha_4)\left(\dfrac{\lambda\cos\sqrt{1+\lambda^2+\mu^2}\,\alpha_2}{\sqrt{1+\lambda^2+\mu^2}}+\dfrac{(1+\lambda^2+\mu^2)\operatorname{sen}\sqrt{1+\lambda^2+\mu^2}\,\alpha_2}{1+\lambda^2}\right)+ \\ \qquad \left. +(z-\alpha_5)\dfrac{(1+\lambda^2)\cos\sqrt{1+\lambda^2+\mu^2}\,\alpha_2}{\sqrt{1+\lambda^2+\mu^2}}\right] \end{cases}$$

5. FAMILIES OF VARIETIES DEPENDING ON 4 PARAMETERS

5.1. Families admitting the group G_7 as maximal group

One obtains the groups:

$$H_4^1 = \left[\frac{\partial f}{\partial \alpha_1} + \alpha_2 \frac{\partial f}{\partial \alpha_2} + \alpha_3 \frac{\partial f}{\partial \alpha_3} + \alpha_4 \frac{\partial f}{\partial \alpha_4}; -\alpha_4 \frac{\partial f}{\partial \alpha_3} + \alpha_3 \frac{\partial f}{\partial \alpha_4};\right.$$

$$\left.\alpha_4 \frac{\partial f}{\partial \alpha_2} - \alpha_2 \frac{\partial f}{\partial \alpha_4}; -\alpha_3 \frac{\partial f}{\partial \alpha_2} + \alpha_2 \frac{\partial f}{\partial \alpha_3}; \frac{\partial f}{\partial \alpha_2}; \alpha_3; \frac{\partial f}{\partial \alpha_4}\right];$$

$$H_4^2 = \left[\frac{\partial f}{\partial \alpha_4}; -\alpha_1 \alpha_2 \frac{\partial f}{\partial \alpha_1} - (1+\alpha_2^2) \frac{\partial f}{\partial \alpha_2} - \alpha_2 \alpha_3 \frac{\partial f}{\partial \alpha_3}; (1+\alpha_1^2) \frac{\partial f}{\partial \alpha_1} + \alpha_1 \alpha_2 \frac{\partial f}{\partial \alpha_2} +\right.$$

$$\left.+\alpha_1 \alpha_3 \frac{\partial f}{\partial \alpha_3}; -\alpha_2 \frac{\partial f}{\partial \alpha_1} + \alpha_1 \frac{\partial f}{\partial \alpha_2}; e^{-\alpha_4} \alpha_1 \frac{\partial f}{\partial \alpha_3}; e^{-\alpha_4} \alpha_2 \frac{\partial f}{\partial \alpha_3}; e^{-\alpha_4} \frac{\partial f}{\partial \alpha_3}\right];$$

$$H_4^3 = \left[\frac{\partial f}{\partial \alpha_4}; \alpha_2 \frac{\partial f}{\partial \alpha_1} - \alpha_1 \frac{\partial f}{\partial \alpha_2} + (1+\alpha_3^2) \frac{\partial f}{\partial \alpha_3} - \alpha_3 \frac{\partial f}{\partial \alpha_4}; (1+\alpha_1^2) \frac{\partial f}{\partial \alpha_1} + \alpha_1 \alpha_2 \frac{\partial f}{\partial \alpha_2} +\right.$$

$$+(\alpha_1 \alpha_3 - \alpha_2 \alpha_3^2) \frac{\partial f}{\partial \alpha_3} + \alpha_2 \alpha_3 \frac{\partial f}{\partial \alpha_4}; -\alpha_1 \alpha_2 \frac{\partial f}{\partial \alpha_1} - (1+\alpha_2^2) \frac{\partial f}{\partial \alpha_2} +$$

$$+(\alpha_2 \alpha_3 - \alpha_1) \frac{\partial f}{\partial \alpha_3} - \alpha_2 \frac{\partial f}{\partial \alpha_4}; e^{-\alpha_4} (\alpha_1 - \alpha_2 \alpha_3) \frac{\partial f}{\partial \alpha_3} +$$

$$\left.+\alpha_2 \frac{\partial f}{\partial \alpha_4}; e^{-\alpha_4} \left(-\alpha_3 \frac{\partial f}{\partial \alpha_3} + \frac{\partial f}{\partial \alpha_4}\right); e^{-\alpha_4} \frac{\partial f}{\partial \alpha_3}\right]$$

The group H_4^1 is measurable and it has measure

$$e^{-3\alpha_1} d\alpha_1 \wedge \ldots \wedge d\alpha_4$$

the corresponding family of varieties is:

$$\mathscr{F}_4^1 : F(e^{-2\alpha_1}[(x-\alpha_2)^2 + (y-\alpha_3)^2 + (z-\alpha_4)^2]) = 0$$

which includes as a particular case, the spheres with centre $(\alpha_2, \alpha_3, \alpha_4)$ and radius e^{α_1} : $e^{-2\alpha_1}[(x-\alpha_2)^2 + (y-\alpha_3)^2 + (z-\alpha_4)^2] - 1 = 0$ [2].

The group H_4^2 is measurable and it has measure

$$(1+\alpha_1^2+\alpha_2^2)^{-2} d\alpha_1 \wedge \ldots \wedge d\alpha_4$$

The corresponding family of varieties is:

$$\mathscr{F}_4^2 : F\left(\frac{ze^{-\alpha_4} + \alpha_2 ye^{-\alpha_4} + \alpha_1 xe^{-\alpha_4} - \alpha_3}{\sqrt{1+\alpha_1^2+\alpha_2^2}}\right) = 0$$

The group H_4^3 is not measurable; the corresponding family of varieties is:

$$\overline{\mathscr{F}}_4^1 : F\left(\frac{(1+\alpha_2^2)(ze^{-\alpha_4} - \alpha_3) + \alpha_1 xe^{-\alpha_4} - \alpha_1\alpha_2 ye^{-\alpha_1} + \alpha_1\alpha_2}{\sqrt{1+\alpha_1^2+\alpha_2^2}(ye^{-\alpha_4} + \alpha_2 xe^{-\alpha_4} - 1)}\right) = 0$$

5.2. Families admitting the group G_6 as maximal group

One only finds the group

$$H_4^4 = \left[-\alpha_2\frac{\partial f}{\partial\alpha_1} + \alpha_1\frac{\partial f}{\partial\alpha_2} - \alpha_4\frac{\partial f}{\partial\alpha_3} + \alpha_3\frac{\partial f}{\partial\alpha_4}; \alpha_1\alpha_2\frac{\partial f}{\partial\alpha_1} + (1+\alpha_2^2)\frac{\partial f}{\partial\alpha_2} + \right.$$

$$+\alpha_1\alpha_4\frac{\partial f}{\partial\alpha_3} + \alpha_2\alpha_4\frac{\partial f}{\partial\alpha_4}; -(1+\alpha_1^2)\frac{\partial f}{\partial\alpha_1} - \alpha_1\alpha_2\frac{\partial f}{\partial\alpha_2} - \alpha_1\alpha_3\frac{\partial f}{\partial\alpha_3} - \alpha_2\alpha_3\frac{\partial f}{\partial\alpha_4};$$

$$\left.\alpha_1\frac{\partial f}{\partial\alpha_3} + \alpha_2\frac{\partial f}{\partial\alpha_4}; \frac{\partial f}{\partial\alpha_3}; \frac{\partial f}{\partial\alpha_4}\right].$$

This group has measure

$$(1+\alpha_1^2+\alpha_2^2)^{-2} d\alpha_1 \wedge \ldots \wedge d\alpha_4$$

and it is associated to the family of varieties

$$\mathscr{F}_4^3 : F\left(\frac{[\alpha_1 x + (1+\alpha_2^2)(y-\alpha_3) - \alpha_1\alpha_2(z-\alpha_4)]^2}{(1+\alpha_1^2+\alpha_2^2)(1+\alpha_2^2)} + \frac{(z+\alpha_2 x - \alpha_4)^2}{1+\alpha_2^2}\right) = 0$$

which includes, as particular case, the family of straight lines [2].

5.3. Families admitting the group $G_5(\lambda, \mu)$ as maximal group

One obtaines the groups

$$H_4^5(a,\epsilon) = \left[\frac{\partial f}{\partial\alpha_4}; (\lambda\alpha_3 - \mu\alpha_2 - a\alpha_1 + \epsilon)\frac{\partial f}{\partial\alpha_1} + (\mu\alpha_1 - a\alpha_2 - \alpha_3)\frac{\partial f}{\partial\alpha_2} + \right.$$

$$+(\alpha_2 - a\alpha_3 - \lambda\alpha_1)\frac{\partial f}{\partial \alpha_3} + a\frac{\partial f}{\partial \alpha_4}; e^{-\alpha_4}\frac{\partial f}{\partial \alpha_1}; e^{-\alpha_4}\frac{\partial f}{\partial \alpha_2}; e^{-\alpha_4}\frac{\partial f}{\partial \alpha_3}\right]$$

with $a \in \mathbf{R}; \epsilon \in \{-1, 0, 1\}; a \cdot \epsilon = 0$ and

$$H_4^6 = \left[\frac{\partial f}{\partial \alpha_4}, \frac{\partial f}{\partial \alpha_1} + (\mu\alpha_2^2 + \mu - \alpha_3 - \lambda\alpha_2\alpha_3)\frac{\partial f}{\partial \alpha_2} + (\alpha_2 + \mu\alpha_2\alpha_3 - \lambda - \lambda\alpha_3^2)\frac{\partial f}{\partial \alpha_3} + \right.$$

$$\left. +(\lambda\alpha_3 - \mu\alpha_2)\frac{\partial f}{\partial \alpha_4}; e^{-\alpha_4}\left(-\alpha_2\frac{\partial f}{\partial \alpha_2} - \alpha_3\frac{\partial f}{\partial \alpha_3} + \frac{\partial f}{\partial \alpha_4}\right); e^{-\alpha_4}\frac{\partial f}{\partial \alpha_2}; e^{-\alpha_4}\frac{\partial f}{\partial \alpha_3}\right]$$

The group $H_4^5(a, \epsilon)$ is measurable if and only if $a = 0$ and in this case one finds the family of varieties

$$\mathscr{F}_4^4 : F(\tilde{y}, \tilde{z}) = 0$$

where

$$\begin{cases} \tilde{y} = [(x - \alpha_1)^2 + (y - \alpha_2)^2 + (z - \alpha_3)^2]e^{-2\alpha_4} \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad if \epsilon = 0 \\ \tilde{z} = [(x - \alpha_1) + \lambda(y - \alpha_2) + \mu(z - \alpha_3)]e^{-\alpha_4} \end{cases}$$

and

$$\begin{cases} \tilde{y} = \left[k(y - \alpha_2)\cos kx' - \epsilon\lambda\mu(y - \alpha_2)k \operatorname{sen} kx' - \epsilon(1 + \mu^2)(z - \alpha_3)\operatorname{sen} kx' + \right. \\ \qquad \left. -e\frac{\alpha_4\lambda x' - \epsilon\mu}{k}\cos kx' - e^{\alpha_1}\frac{\epsilon\mu b^2 x' + \lambda}{k^2}\operatorname{sen} kx'\right] e^{-\alpha_4} \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad if \epsilon \neq 0 \\ \tilde{z} = \left[\epsilon k(y - \alpha_2)\operatorname{sen} kx' + \lambda\mu k(y - \alpha_2)\cos kx' + (1 + \mu^2)(z - \alpha_2)\cos kx' + \right. \\ \qquad \left. +e^{\alpha_4}\frac{\mu k^2 x' + \lambda}{k^2}\cos kx' - e^{\alpha_4}\epsilon\frac{\lambda x - \epsilon\mu}{k}\operatorname{sen} kx'\right] e^{-\alpha_4} \end{cases}$$

where $k = \sqrt{1 + \lambda^2 + \mu^2}$ and $x' = [(x - \alpha_1) + \lambda(y - \alpha_2) + \mu(z - \alpha_3)]e^{-\alpha_4}$.

If $a \neq 0$ one finds the family

$$\overrightarrow{\mathscr{F}}_4^3 : F(\tilde{y}, \tilde{z}) = 0$$

where

$$\begin{cases} \tilde{y} = \left[\frac{(y - \alpha_2)}{ax'}k\cos k\, ln|x'| - \mu\lambda\frac{(y - \alpha_2)}{ax'}k \operatorname{sen} k\, ln|x'| - (1 + \mu^2)\frac{(z - \alpha_3)}{ax'}\right. \\ \qquad \left.\operatorname{sen} k\, ln|x'| + \mu e^{\alpha_4}\operatorname{sen} k\, ln|x'| - \frac{\lambda}{k}\cos k\, ln|x'|\right] e^{-\alpha_4} \\ \tilde{z} = \left[\frac{(y - \alpha_2)}{ax'}k\cos k\, ln|x'| - \mu\lambda\frac{\mu\lambda k(y - \alpha_2)}{ax'}\cos\, ln|x'| + \frac{(y - \alpha_2)}{ax'}\operatorname{sen}\, ln|x'| + \right. \\ \qquad \left.(1 + \mu^2)\frac{(z - \alpha_3)}{ax'}\cos k\, ln|x'| - \mu e^{\alpha_4}\cos k\, ln|x'| - \frac{\lambda}{k}\operatorname{sen} k\, ln|x'|\right] e^{\alpha_4} \end{cases}$$

where $k = \sqrt{1+\lambda^2+\mu^2}$ and $x' = e^{\alpha_4}[(x-\alpha_1)+\lambda(y-\alpha_2)+\mu(z-\alpha_3)]$.

The group H_4^6 is not measurable and it is associated to the family of varieties

$$\mathscr{F}_4^4 : F\left(\frac{\tilde{y}}{\tilde{x}}, \frac{\tilde{z}}{\tilde{x}}\right) = 0$$

where

$$\begin{cases} \tilde{x} = xe^{-\alpha_4} + \lambda(ye^{-\alpha_4} - \alpha_2) + \mu(ze^{-\alpha_4} - \alpha_3) - \dfrac{1}{1+\lambda^2} \\ \tilde{y} = \dfrac{1}{1+\lambda^2+\mu^2}[-xe^{-\alpha_4}(\mu \operatorname{sen} \sqrt{1+\lambda^2+\mu^2}\alpha_1 + \\ \quad + \lambda\sqrt{1+\lambda^2+\mu^2} \cos\sqrt{1+\lambda^2+\mu^2}\alpha_1) + (ye^{-\alpha_4} - \alpha_2)(\sqrt{1+\lambda^2+\mu^2} \\ \quad \cos\sqrt{1+\lambda^2+\mu^2}\alpha_1 - \lambda\mu \operatorname{sen}\sqrt{1+\lambda^2+\mu^2}\alpha_1) + (1+\lambda^2)(ze^{-\alpha_4} - \alpha_3) \\ \quad \operatorname{sen}\sqrt{1+\lambda^2+\mu^2}\alpha_1 + \lambda\cos\sqrt{1+\lambda^2+\mu^2}\alpha_1] \\ \tilde{z} = \dfrac{1}{1+\lambda^2+\mu^2}[e^{-\alpha_4}x(\lambda\sqrt{1+\lambda^2+\mu^2}\operatorname{sen}\sqrt{1+\lambda^2+\mu^2}\alpha_1 - \mu \\ \quad \cos\sqrt{1+\lambda^2+\mu^2}\alpha_1) + (e^{-\alpha_4}y - \alpha_2)(\lambda\mu\cos\sqrt{1+\lambda^2+\mu^2}\alpha_1 + \\ \quad + \sqrt{1+\lambda^2+\mu^2}\operatorname{sen}\sqrt{1+\lambda^2+\mu^2}\alpha_1) + (1+\lambda^2)(ze^{-\alpha_4} - \alpha_3) \\ \quad \cos\sqrt{1+\lambda^2+\mu^2}\alpha_1 - \lambda\operatorname{sen}\sqrt{1+\lambda^2+\mu^2}\alpha_1] \end{cases}$$

5.4. Families admitting the groups G_4^i as maximal group

One obtains the groups below ($H_4^{k+6} \simeq G_4^k$)

$$H_4^7 = \left[\frac{\partial f}{\partial \alpha_4}; \sqrt{1+\alpha_2^2}\cos\alpha_3\frac{\partial f}{\partial \alpha_1} - (1+\alpha_2^2)\operatorname{sen}\alpha_3\frac{\partial f}{\partial \alpha_2} + \right.$$

$$+\alpha_2\cos\alpha_3\frac{\partial f}{\partial \alpha_3}; \sqrt{1+\alpha_2^2}\operatorname{sen}\alpha_3\frac{\partial f}{\partial \alpha_1} +$$

$$\left. +(1+\alpha_2^2)\cos\alpha_3\frac{\partial f}{\partial \alpha_2} + \alpha_2\operatorname{sen}\alpha_3\frac{\partial f}{\partial \alpha_3}; \frac{\partial f}{\partial \alpha_3}\right];$$

$$H_4^8 = \left[\frac{\partial f}{\partial \alpha_1} + \alpha_3\frac{\partial f}{\partial \alpha_3} + \alpha_4\frac{\partial f}{\partial \alpha_4}; \frac{\partial f}{\partial \alpha_2} - \alpha_4\frac{\partial f}{\partial \alpha_3} + \alpha_3\frac{\partial f}{\partial \alpha_4}; \frac{\partial f}{\partial \alpha_3}; \frac{\partial f}{\partial \alpha_4}\right];$$

$$H_4^9 = \left[\frac{\partial f}{\partial \alpha_1} + \alpha_2 \frac{\partial f}{\partial \alpha_2} + \alpha_3 \frac{\partial f}{\partial \alpha_3} + \alpha_4 ; \frac{\partial f}{\partial \alpha_2}; \frac{\partial f}{\partial \alpha_3}; \frac{\partial f}{\partial \alpha_4}\right];$$

$$H_4^{10} = \left[\frac{\partial f}{\partial \alpha_1} - \alpha_3 \frac{\partial f}{\partial \alpha_2} + \alpha_2 \frac{\partial f}{\partial \alpha_3}; \frac{\partial f}{\partial \alpha_2}; \frac{\partial f}{\partial \alpha_3}; \frac{\partial f}{\partial \alpha_4}\right].$$

Each of these groups is associated to a measurable family of varieties ($H_4^k \longleftrightarrow \mathscr{F}_4^{k-2}$)

$\mathscr{F}_4^5$: $F\left(\frac{e^{-\alpha_4}}{\sqrt{1+\alpha_2^2}}(x \cos \alpha_3 + y \operatorname{sen} \alpha_3 - \alpha_2 z), e^{-\alpha_4}(y \cos \alpha_3 - x \operatorname{sen} \alpha_3), \frac{e^{-\alpha_4}}{\sqrt{1+\alpha_2^2}}\right.$
$\left.(z + \alpha_2 x \cos \alpha_3 + \alpha_2 y \operatorname{sen} \alpha_3)\right) = 0$
with measure $d\mathscr{F}_4^5 = (1+\alpha_2^2)^{-3/2} d\alpha_1 \wedge \ldots \wedge d\alpha_4$;

$\mathscr{F}_4^6$: $F(e^{-\alpha_1} x, e^{-\alpha_1}[(y - \alpha_3) \cos \alpha_2 + (z - \alpha_4) \operatorname{sen} \alpha_2], e^{-\alpha_1}[(z - \alpha_4) \cos \alpha_2 - (y +$
$\alpha_3) \operatorname{sen} \alpha_2]) = 0$
with measure $d\mathscr{F}_4^6 = e^{-2\alpha_1} d\alpha_1 \wedge \ldots \wedge d\alpha_4$;

$\mathscr{F}_4^7$: $F(e^{-\alpha_1}(x - \alpha_2), e^{-\alpha_1}(y - \alpha_3), e^{-\alpha_1}(z - \alpha_4)) = 0$
with measure $d\mathscr{F}_4^7 = e^{-3\alpha_1} d\alpha_1 \wedge \ldots \wedge d\alpha_4$;

$\mathscr{F}_4^8$: $F((x-\alpha_2) \cos \alpha_1 + (y-\alpha_3) \operatorname{sen} \alpha_1, (y-\alpha_3) \cos \alpha_1 - (x-\alpha_2) \operatorname{sen} \alpha_1, z-\alpha_4) = 0$
with measure $d\mathscr{F}_4^8 = d\alpha_1 \wedge \ldots \wedge d\alpha_4$.

6. FAMILIES OF VARIETIES DEPENDING ON 3 PARAMETERS

6.1. Families admitting the group G_7 as maximal group

One obtains the groups

$$H_3^1 = \left[\frac{\partial f}{\partial \alpha_3}, -\alpha_2 \frac{\partial f}{\partial \alpha_1} + \alpha_1 \frac{\partial f}{\partial \alpha_2}; \alpha_1 \alpha_2 \frac{\partial f}{\partial \alpha_1} + (1+\alpha_2^2) \frac{\partial f}{\partial \alpha_2} - \alpha_2 \frac{\partial f}{\partial \alpha_3};\right.$$

$$-(1+\alpha_1^2) \frac{\partial f}{\partial \alpha_1} - \alpha_1 \alpha_2 \frac{\partial f}{\partial \alpha_2} + \alpha_1 \frac{\partial f}{\partial \alpha_3}; e^{-\alpha_3}\left(\alpha_1 \frac{\partial f}{\partial \alpha_1} + \alpha_2 \frac{\partial f}{\partial \alpha_2} - \frac{\partial f}{\partial \alpha_3}\right);$$

$$\left. e^{-\alpha_3} \frac{\partial f}{\partial \alpha_1}; e^{-\alpha_3} \frac{\partial f}{\partial \alpha_2}\right]$$

$$H_3^2 = \left[\frac{\partial f}{\partial \alpha_3}; -\alpha_1 \alpha_2 \frac{\partial f}{\partial \alpha_1} - (1+\alpha_2^2) \frac{\partial f}{\partial \alpha_2} - \alpha_2 \frac{\partial f}{\partial \alpha_3}; (1+\alpha_1^2) \frac{\partial f}{\partial \alpha_1} + \alpha_1 \alpha_2 \frac{\partial f}{\partial \alpha_2} + \alpha_1 \frac{\partial f}{\partial \alpha_3};\right.$$

$$-\alpha_2 \frac{\partial f}{\partial \alpha_1} + \alpha_1 \frac{\partial f}{\partial \alpha_2}; \alpha_1 e^{-\alpha_3} \frac{\partial f}{\partial \alpha_3}; \alpha_2 e^{-\alpha_3} \frac{\partial f}{\partial \alpha_3}; e^{-\alpha_3} \frac{\partial f}{\partial \alpha_3}\Big]$$

The group H_3^1 is not measurable and it is associated to the family of varieties

$$\overline{\mathscr{F}}_3^1 : (ye^{-\alpha_3} - \alpha_1)^2 = k[(xe^{-\alpha_3} + 1)^2 + (ze^{-\alpha_3} - \alpha_2)^2] \qquad (k = \text{constant})$$

that is a family of cones or cylinders and, if $k = -1$, the family consisting of single points. The group H_3^2 is not measurable and it is associated to the family of varieties

$$\overline{\mathscr{F}}_3^2 : \frac{z + \alpha_1 x + \alpha_2 y - e^{-\alpha_3}}{\sqrt{1 + \alpha_1^2 + \alpha_2^2}} = \text{constant}$$

This family is obviously the family of the planes.

6.2. Families admitting the group G_6 as maximal group

One obtains the groups

$$H_3^3 = \left[-\alpha_3 \frac{\partial f}{\partial \alpha_2} + \alpha_2 \frac{\partial f}{\partial \alpha_3}; \alpha_3 \frac{\partial f}{\partial \alpha_1} - \alpha_1 \frac{\partial f}{\partial \alpha_3}; -\alpha_2 \frac{\partial f}{\partial \alpha_1} + \alpha_1 \frac{\partial f}{\partial \alpha_2}; \frac{\partial f}{\partial \alpha_1}; \frac{\partial f}{\partial \alpha_2}; \frac{\partial f}{\partial \alpha_3}\right]$$

$$H_3^4 = \Big[-\alpha_1 \alpha_2 \frac{\partial f}{\partial \alpha_1} - (1 + \alpha_2^2) \frac{\partial f}{\partial \alpha_2} - \alpha_2 \alpha_3 \frac{\partial f}{\partial \alpha_3}; (1 + \alpha_1^2) \frac{\partial f}{\partial \alpha_1} + \alpha_1 \alpha_2 \frac{\partial f}{\partial \alpha_2} +$$

$$+ \alpha_1 \alpha_3 \frac{\partial f}{\partial \alpha_3}; -\alpha_2 \frac{\partial f}{\partial \alpha_1} + \alpha_1 \frac{\partial f}{\partial \alpha_2}; \alpha_1 \frac{\partial f}{\partial \alpha_3}; \alpha_2 \frac{\partial f}{\partial \alpha_3}; \frac{\partial f}{\partial \alpha_3}\Big]$$

This groups are both measurable: the first has measure

$$d\alpha_1 \wedge d\alpha_2 \wedge d\alpha_3$$

and it is associated to the family of varieties

$$\mathscr{F}_3^1 : F((x - \alpha_1)^2 + (y - \alpha_2)^2 + (z - \alpha_3)^2) = 0$$

which includes, as a particular case $(x - \alpha_1)^2 + (y - \alpha_2)^2 + (z - \alpha_3)^2 = R^2$ i.e. the family of the spheres of fixed radius and, if $R = 0$, the family consisting of single points [2].

The group H_4^3 has measure

$$(1 + \alpha_1^2 + \alpha_2^2)^{-2} d\alpha_1 \wedge d\alpha_2 \wedge d\alpha_3$$

and it is associated to the family

$$\mathscr{F}_3^2 : F\left(\frac{z + \alpha_1 x + \alpha_2 y - \alpha_3}{\sqrt{1 + \alpha_1^2 + \alpha_2^2}}\right) = 0$$

which has, as particular case, the family of the planes [2].

6.3. Families admitting the group $G_5(\lambda,\mu)$ as maximal group

One obtains the group

$$H_3^5 = \left[\frac{\partial f}{\partial \alpha_3}; (\lambda\alpha_1\alpha_2 + \mu\alpha_2 - \alpha_1^2 - 1)\frac{\partial f}{\partial \alpha_1} + (\lambda\alpha_2^2 - \alpha_1\alpha_2 - \mu\alpha_1 + \lambda)\frac{\partial f}{\partial \alpha_2} + \right.$$

$$\left. +(\alpha_1 - \lambda\alpha_2)\frac{\partial f}{\partial \alpha_3}; e^{-\alpha_3}\frac{\partial f}{\partial \alpha_2}; e^{-\alpha_3}\frac{\partial f}{\partial \alpha_1}; e^{-\alpha_3}\left(-\alpha_1\frac{\partial f}{\partial \alpha_1} - \alpha_2\frac{\partial f}{\partial \alpha_2} + \frac{\partial f}{\partial \alpha_3}\right)\right]$$

which is not measurable and it is associated to the family of varieties

$$\overline{\mathscr{F}}_3^3 : F\left(\frac{e^{-\alpha_3}x - \alpha_2 + \lambda(e^{-\alpha_3}y - \alpha_1) + \mu(e^{-\alpha_3}z - 1)}{\sqrt{(e^{-\alpha_3}x - \alpha_2)^2 + (e^{-\alpha_3}y - \alpha_1)^2 + (e^{-\alpha_3}z - 1)^2}}\right) = 0$$

6.4. Families admitting one of the groups G_4^i as maximal group

One finds the family of groups $H_3^6(\lambda)$ each of one isomorphic to

$$G_4^1 : H_3^6(\lambda) = \left[\frac{\partial f}{\partial \alpha_3}; \frac{\partial f}{\partial \alpha_2}; -(1+\alpha_1^2)\ \mathrm{sen}\ \alpha_2\frac{\partial f}{\partial \alpha_1} + \alpha_1 \cos\alpha_2\frac{\partial f}{\partial \alpha_2} + \right.$$

$$+\lambda\sqrt{1+\alpha_1^2}\cos\alpha_2\frac{\partial f}{\partial \alpha_3};$$

$$\left.(1+\alpha_1^2)\cos\alpha_2\frac{\partial f}{\partial \alpha_1} + \alpha_2\ \mathrm{sen}\ \alpha_2\frac{\partial f}{\partial \alpha_2} + \lambda\sqrt{1+\alpha_1^2}\ \mathrm{sen}\ \alpha_2\frac{\partial f}{\partial \alpha_3}\right]$$

with $\lambda \in \mathbf{R}$.

For each value of λ, $H_3^6(\lambda)$ is a measurable group and it has measure

$$(1+\alpha_1^2)^{-3/2}d\alpha_1 \wedge d\alpha_2 \wedge d\alpha_3$$

it is associated to the families of varieties

$$\mathscr{F}_3^3(\lambda) : \begin{cases} F\left(\frac{\overline{z}}{\overline{y}}\cos\frac{ln|\overline{y}|}{\lambda} - \frac{\overline{x}}{y}\ \mathrm{sen}\ \frac{ln|\overline{y}|}{\lambda}, \frac{\overline{z}}{\overline{y}}\ \mathrm{sen}\ \frac{ln|\overline{y}|}{\lambda} + \frac{\overline{x}}{\overline{y}}\cos\frac{ln|\overline{y}|}{\lambda}\right) = 0 & \text{if } \lambda \neq 0 \\ F(\overline{x}^2 + \overline{z}^2, \overline{y}) = 0 & \text{if } \lambda = 0 \end{cases}$$

where

$$\begin{cases} \widetilde{x} = \dfrac{e^{-\alpha_3}}{\sqrt{1+\alpha_1^2}}\,(x + \alpha_1 y \cos\alpha_2 + \alpha_1 z \,\mathrm{sen}\,\alpha_2) \\ \widetilde{y} = \dfrac{e^{-\alpha_3}}{\sqrt{1+\alpha_1^2}}\,(y\cos\alpha_2 + z\,\mathrm{sen}\,\alpha_2 - \alpha_1 x) \\ \widetilde{z} = e^{-\alpha_3}(z\cos\alpha_2 - y\,\mathrm{sen}\,\alpha_2) \end{cases}$$

There exists a family of groups and two groups isomorphic to G_4^2 :

$$H_3^7(\lambda) = \left[\frac{\partial f}{\partial\alpha_1} + \alpha_2\frac{\partial f}{\partial\alpha_2} + \alpha_3\frac{\partial f}{\partial\alpha_3}, \lambda\frac{\partial f}{\partial\alpha_1} - \alpha_3\frac{\partial f}{\partial\alpha_2} + \alpha_2\frac{\partial f}{\partial\alpha_3}, \frac{\partial f}{\partial\alpha_2}, \frac{\partial f}{\partial\alpha_3}\right],$$

with $\lambda \in \mathbf{R}$

$$H_3^8 = \left[\alpha_2\frac{\partial f}{\partial\alpha_2} + \alpha_3\frac{\partial f}{\partial\alpha_3}, \frac{\partial f}{\partial\alpha_1} - \alpha_3\frac{\partial f}{\partial\alpha_2} + \alpha_2\frac{\partial f}{\partial\alpha_3}, \frac{\partial f}{\partial\alpha_2}, \frac{\partial f}{\partial\alpha_3}\right]$$

$$H_3^9 = \left[\frac{\partial f}{\partial\alpha_1} + \alpha_3\frac{\partial f}{\partial\alpha_3}; -(1+\alpha_2^2)\frac{\partial f}{\partial\alpha_2} - \alpha_2\alpha_3\frac{\partial f}{\partial\alpha_3}; \alpha_2\frac{\partial f}{\partial\alpha_3}; \frac{\partial f}{\partial\alpha_3}\right]$$

The group $H_3^7(\lambda)$ is measurable if and only if $\lambda = 0$ and in this case has measure

$$e^{-2\alpha_1} d\alpha_1 \wedge d\alpha_2 \wedge d\alpha_3$$

To this group are associated the families of varieties

$$\mathscr{F}_3^4(\lambda) : F\left(\frac{\widetilde{y}}{\widetilde{x}}\cos\frac{ln|\widetilde{x}|}{\lambda} - \frac{\widetilde{z}}{\widetilde{x}}\,\mathrm{sen}\frac{ln|\widetilde{x}|}{\lambda}; \frac{\widetilde{y}}{\widetilde{x}}\,\mathrm{sen}\frac{ln|\widetilde{x}|}{\lambda} + \frac{\widetilde{z}}{\widetilde{x}}\cos\frac{ln|\widetilde{x}|}{\lambda}\right) = 0 \quad if\ \lambda \neq 0$$

$$\mathscr{F}_3^4 : F(\widetilde{x}, \widetilde{y}^2 + \widetilde{z}^2) = 0 \quad if\ \lambda = 0$$

where, in both cases,

$$\begin{cases} \widetilde{x} = e^{-\alpha_1} x \\ \widetilde{y} = e^{-\alpha_1}(y - \alpha_2) \\ \widetilde{z} = e^{-\alpha_1}(z - \alpha_3) \end{cases}$$

The group H_3^8 is not measurable, it is associated to the family of varieties:

$$\mathscr{F}_3^5 : F\left(\frac{y-\alpha_2}{x}\cos\alpha_1 + \frac{z-\alpha_3}{x}\,\mathrm{sen}\,\alpha_1; \frac{z-\alpha_3}{x}\cos\alpha_1 - \frac{y-\alpha_2}{x}\,\mathrm{sen}\,\alpha_1\right) = 0$$

The group H_3^9 is measurable and it has measure

$$e^{-\alpha_1}(1+\alpha_2^2)^{-3/2}d\alpha_1\wedge d\alpha_2\wedge d\alpha_3$$

It is associated to the family of varieties:

$$\mathscr{F}_3^5 : F\left(\frac{e^{-\alpha_1}(z-\alpha_3+\alpha_2 y)}{\sqrt{1+\alpha_2^2}}, e^{-\alpha_1}x\right)=0$$

One finds only one group isomorphic to G_4^3:

$$H_3^{10}=\left[\frac{\partial f}{\partial\alpha_1}, e^{-\alpha_1}\frac{\partial f}{\partial\alpha_2}, e^{-\alpha_1}\frac{\partial f}{\partial\alpha_3}, e^{-\alpha_1}\left(\frac{\partial f}{\partial\alpha_1}-\alpha_2\frac{\partial f}{\partial\alpha_2}-\alpha_3\frac{\partial f}{\partial\alpha_3}\right)\right]$$

which is not measurable; it is associated to the family of varieties:

$$\overline{\mathscr{F}}_3^6 : F\left(\frac{e^{-\alpha_1}z-1}{e^{-\alpha_1}x-\alpha_2}, \frac{e^{-\alpha_1}y-\alpha_3}{e^{-\alpha_1}x-\alpha_2}\right)=0$$

One finds two families of groups and one group isomorphic to G_4^4 :

$$H_3^{11}(\lambda)=\left[\lambda\frac{\partial f}{\partial\alpha_1}-\alpha_3\frac{\partial f}{\partial\alpha_2}+\alpha_2\frac{\partial f}{\partial\alpha_3}; \frac{\partial f}{\partial\alpha_2}; \frac{\partial f}{\partial\alpha_3}; \frac{\partial f}{\partial\alpha_1}\right], \qquad \lambda\in\mathbf{R};$$

$$H_3^{12}(\lambda)=\left[\frac{\partial f}{\partial\alpha_1}-\alpha_3\frac{\partial f}{\partial\alpha_2}+\alpha_2\frac{\partial f}{\partial\alpha_3}; \frac{\partial f}{\partial\alpha_2}; \frac{\partial f}{\partial\alpha_3}; \lambda\cos\alpha_1\frac{\partial f}{\partial\alpha_2}+\lambda\,\mathrm{sen}\,\alpha_1\frac{\partial f}{\partial\alpha_3}\right], \lambda\in\mathbf{R};$$

$$H_3^{13}=\left[-(1+\alpha_2^2)\frac{\partial f}{\partial\alpha_2}-\alpha_2\alpha_3\frac{\partial f}{\partial\alpha_3}; \alpha_2\frac{\partial f}{\partial\alpha_3}; \frac{\partial f}{\partial\alpha_3}; \frac{\partial f}{\partial\alpha_1}\right]$$

The group $H_3^{11}(\lambda)$ is measurable and its has measure

$$d\alpha_1\wedge d\alpha_2\wedge d\alpha_3$$

It is associated to the family

$$\mathscr{F}_3^6(\lambda) : \begin{cases} F\Bigg((x-\alpha_2)\cos\dfrac{z-\alpha_1}{\lambda}+(y-\alpha_3)\,\mathrm{sen}\,\dfrac{z-\alpha_1}{\lambda}, \\ \quad (x-\alpha_2)\,\mathrm{sen}\,\dfrac{z-\alpha_1}{\lambda}-(y-\alpha_3)\cos\dfrac{z-\alpha_1}{\lambda}\Bigg)=0 & \text{if } \lambda\neq 0 \\ F((x-\alpha_2)^2+(y-\alpha_3)^2, z-\alpha_1)=0 & \text{if } \lambda=0 \end{cases}$$

The group $H_3^{12}(\lambda)$ is measurable and it has measure

$$d\alpha_1 \wedge d\alpha_2 \wedge d\alpha_3$$

It is associated to the family of varieties

$$\mathscr{F}_3^7(\lambda) : F((x-\alpha_2)\cos\alpha_1 + (y-\alpha_3)\operatorname{sen}\alpha_1 + \lambda z, (y-\alpha_3)\cos\alpha_1 - (x-\alpha_2)\operatorname{sen}\alpha_1) = 0$$

The group H_3^{13} is measurable and it has measure

$$(1+\alpha_2^2)^{-3/2}\, d\alpha_1 \wedge d\alpha_2 \wedge d\alpha_3$$

It is associated to the family of varieties

$$\mathscr{F}_3^8 : F\left(\frac{y-\alpha_3+\alpha_2 x}{\sqrt{1+\alpha_2^2}}; z-\alpha_1\right) = 0$$

6.5. Families admitting one of the groups G_3^i as maximal group

One finds the measurable groups H_3^k where $H_3^k \simeq G_3^{k-13}$

$H_3^{14} = \left[\frac{\partial f}{\partial \alpha_1} + \alpha_3 \frac{\partial f}{\partial \alpha_3}; \frac{\partial f}{\partial \alpha_2}; \frac{\partial f}{\partial \alpha_3}\right]$ which is associated to the family;

$\mathscr{F}_3^9 : F(e^{-\alpha_1}(x\cos\alpha_2 + y\operatorname{sen}\alpha_2), e^{-\alpha_1}(y\cos\alpha_2 - x\operatorname{sen}\alpha_2), e^{-\alpha_1}(z-\alpha_3)) = 0$

with measure $d\mathscr{F}_3^1 = e^{-\alpha_1} d\alpha_1 \wedge d\alpha_2 \wedge d\alpha_3$;

$H_3^{15} = \left[\frac{\partial f}{\partial \alpha_1} + (\alpha_2 - \lambda\alpha_3)\frac{\partial f}{\partial \alpha_2} + (\alpha_3 + \lambda\alpha_2)\frac{\partial f}{\partial \alpha_3}; \frac{\partial f}{\partial \alpha_2}; \frac{\partial f}{\partial \alpha_3}\right]$ which is associated to the family:

$\mathscr{F}_3^{10}$: $F(e^{-\alpha_1}[(x-\alpha_2)\cos\lambda\alpha_1 + (y-\alpha_3)\operatorname{sen}\lambda\alpha_1], e^{-\alpha_1}[(y-\alpha_3)\cos\lambda\alpha_1 - (x+\alpha_2)\operatorname{sen}\lambda\alpha_1], e^{-\alpha_1}z) = 0$

with measure $d\mathscr{F}_3^{10} = e^{-2\alpha_1} d\alpha_1 \wedge d\alpha_2 \wedge d\alpha_3$

$H_3^{16} = \left[\frac{\partial f}{\partial \alpha_1} - \alpha_3 \frac{\partial f}{\partial \alpha_2} + \alpha_2 \frac{\partial f}{\partial \alpha_3}; \frac{\partial f}{\partial \alpha_2}; \frac{\partial f}{\partial \alpha_3}\right]$ which is associated to the family:

$\mathscr{F}_3^{11} : F((x-\alpha_2)\cos\alpha_1 + (y-\alpha_3)\operatorname{sen}\alpha_1, (y-\alpha_3)\cos\alpha_1 - (x-\alpha_2)\operatorname{sen}\alpha_1, z-\varepsilon\alpha_1) = 0$

with measure $d\mathscr{F}_3^{11} = d\alpha_1 \wedge d\alpha_2 \wedge d\alpha_3$;

$$H_3^{17} = \left[\sqrt{1+\alpha_2^2}\cos\alpha_3\frac{\partial f}{\partial\alpha_1} - (1+\alpha_2^2)\text{ sen }\alpha_3\frac{\partial f}{\partial\alpha_2} + \alpha_2\cos\alpha_3\frac{\partial f}{\partial\alpha_3}; \sqrt{1+\alpha_2^2}\text{ sen }\alpha_3\right.$$
$$\left.\frac{\partial f}{\partial\alpha_1} + (1+\alpha_2^2)\cos\alpha_3\frac{\partial f}{\partial\alpha_2} + \alpha_2\text{ sen }\alpha_3\frac{\partial f}{\partial\alpha_3}; \frac{\partial f}{\partial\alpha_3}\right]$$ which is associated to the family:

$$\mathscr{F}_3^{12} : F\left(\frac{x\cos\alpha_3 + y\text{ sen }\alpha_3 - \alpha_2 z}{\sqrt{1+\alpha_2^2}}, (y\cos\alpha_3 - x\text{ sen }\alpha_3)\cos\alpha_1 + \right.$$
$$+\frac{\alpha_2(x\cos\alpha_3 + y\text{ sen }\alpha_3)+z}{\sqrt{1+\alpha_2^2}}\text{ sen }\alpha_1, \frac{\alpha_2(x\cos\alpha_3 + y\text{ sen }\alpha_3)+z}{\sqrt{1+\alpha_2^2}} +$$
$$\left. -(y\cos\alpha_3 - x\text{ sen }\alpha_3)\text{ sen }\alpha_1\right) = 0$$

with measure $d\mathscr{F}_3^{12} = (1+\alpha_2^2)^{-3/2}d\alpha_1\wedge d\alpha_2\wedge d\alpha_3$;

$H_3^{18} = \left[\frac{\partial f}{\partial\alpha_1}, \frac{\partial f}{\partial\alpha_2}, \frac{\partial f}{\partial\alpha_3}\right]$ which is associated to the family:

$\mathscr{F}_3^{13} : F(x-\alpha_1, y-\alpha_2, z-\alpha_3) = 0$

with measure $d\mathscr{F}_3^{13} = d\alpha_1\wedge d\alpha_2\wedge d\alpha_3$.

7. FAMILIES OF VARIETIES DEPENDING ON 2 PARAMETERS

7.1. Families admitting one of the groups $G_5(\lambda,\mu), G_6, G_7$ as maximal group

There are not groups isomorphic to G_6 or G_7; there is the group H_2^1 isomorphic to $G_5(\lambda,\mu)$:

$$H_2^1 = \left[\frac{\partial f}{\partial\alpha_2}; \frac{\partial f}{\partial\alpha_1}; e^{-\alpha_2}\frac{\mu\sqrt{1+\lambda^2+\mu^2}\cos\sqrt{1+\lambda^2+\mu^2}\,\alpha_1 - \lambda\text{ sen }\sqrt{1+\lambda^2+\mu^2}\,\alpha_1}{1+\mu^2}\right.$$

$$\frac{\partial f}{\partial\alpha_2}; e^{-\alpha_2}\text{ sen }\sqrt{1+\lambda^2+\mu^2}\,\alpha_1\frac{\partial f}{\partial\alpha_2};$$

$$\left. e^{-\alpha_2}\frac{\mu\lambda\text{ sen }\sqrt{1+\lambda^2+\mu^2}\,\alpha_1 - \sqrt{1+\lambda^2+\mu^2}\cos\sqrt{1+\lambda^2+\mu^2}\,\alpha_1}{1+\mu^2}\frac{\partial f}{\partial\alpha_2}\right]$$

This group is not measurable; it is associated to the family of varieties:

$$\overline{\mathscr{F}}_2^1 : \begin{cases} e^{-\alpha_2}(x + \lambda y + \mu z) = k_1 \\ e^{-\alpha_2}[-(\lambda \cos\sqrt{1+\lambda^2+\mu^2}\alpha_1 + \mu\sqrt{1+\lambda^2+\mu^2}\,\mathrm{sen}\sqrt{1+\lambda^2+\mu^2}\alpha_1)x + \\ \quad t\cos\sqrt{1+\lambda^2+\mu^2}\alpha_1 + (\sqrt{1+\lambda^2+\mu^2}\,\mathrm{sen}\sqrt{1+\lambda^2+\mu^2} + \\ \quad -\mu\lambda\cos\sqrt{1+\lambda^2+\mu^2}\alpha_1)z] = k_2 \\ e^{-\alpha_2}[(\lambda\,\mathrm{sen}\sqrt{1+\lambda^2+\mu^2}\alpha_1 - \mu\sqrt{1+\lambda^2+\mu^2}\cos\sqrt{1+\lambda^2+\mu^2}\alpha_1)x + \\ \quad -y\,\mathrm{sen}\sqrt{1+\lambda^2+\mu^2}\alpha_1 + (\sqrt{1+\lambda^2+\mu^2}\cos\sqrt{1+\lambda^2+\mu^2}\alpha_1 + \\ \quad \lambda\mu\,\mathrm{sen}\sqrt{1+\lambda^2+\mu^2}\alpha_1)z] = k_3 \ \textit{where}\ k_i = \textit{constants} \end{cases}$$

7.2. Families admitting one of the groups G_4^i as maximal group

There are not groups depending on two parameters isomorphic to G_4^1 and to G_4^3; one finds one group isomorphic to G_4^2: the group

$$H_2^2 = \left[\alpha_1\frac{\partial f}{\partial\alpha_1} + \alpha_2\frac{\partial f}{\partial\alpha_2}; -\alpha_2\frac{\partial f}{\partial\alpha_1} + \alpha_1\frac{\partial f}{\partial\alpha_2}; \frac{\partial f}{\partial\alpha_1}; \frac{\partial f}{\partial\alpha_2}\right]$$

isomorphic to G_4^2. This group is not measurable and it is associated to the family of varieties

$$\overline{\mathscr{F}}_2^2 : F(xe^{-2[(y-\alpha_1)^2+(z-\alpha_2)^2]}) = 0$$

Besides one finds the group $H_2^3 = \left[\frac{\partial f}{\partial\alpha_1}; \cos\alpha_1\frac{\partial f}{\partial\alpha_2}; \mathrm{sen}\,\alpha_1\frac{\partial f}{\partial\alpha_2}; \frac{\partial f}{\partial\alpha_2}\right]$ isomorphic to G_4^4. This group is measurable with measure

$$d\alpha_1 \wedge d\alpha^2$$

It is associated to the family

$$\mathscr{F}_2^1 : F(z - \alpha_2 + x\cos\alpha_1 + y\,\mathrm{sen}\,\alpha_1) = 0$$

7.3. Families admitting one of the groups G_3^i as maximal group

There are no two parameter groups isomorphic to G_3^5.

One obtains two groups isomorphic to G_3^1

$$H_2^4 = \left[\alpha_2 \frac{\partial f}{\partial \alpha_2}, \frac{\partial f}{\partial \alpha_1}, \frac{\partial f}{\partial \alpha_2}\right] \text{ and } H_2^5 = \left[\frac{\partial f}{\partial \alpha_1} + \alpha_2 \frac{\partial f}{\partial \alpha_2}, e^{\alpha_1} \frac{\partial f}{\partial \alpha_2}, \frac{\partial f}{\partial \alpha_2}\right].$$

The group H_2^4 is not measurable and it is associated to the family:

$$\overline{\mathscr{F}}_2^3 : F\left(\frac{x \cos \alpha_1 + y \operatorname{sen} \alpha_1}{z - \alpha_2}, \frac{y \cos \alpha_1 - x \operatorname{sen} \alpha_1}{z - \alpha_2}\right) = 0$$

The group H_2^5 is measurable with measure

$$d\alpha_1 \wedge d\alpha_2$$

It is associated to the family:

$$\mathscr{F}_2^2 : F(e^{-\alpha_1} x \operatorname{sen}[e^{-\alpha_1}(z - \alpha_2)] + e^{-\alpha_1} y \cos[e^{-\alpha_1}(z - \alpha_2)],$$

$$e^{-\alpha_1} x \cos[e^{-\alpha_1}(z - \alpha_2)] - e^{-\alpha_1} y \operatorname{sen}[e^{-\alpha_1}(z - \alpha_2)]) = 0$$

One obtains two groups isomorphic to $G_3^2(\lambda)$:

$$H_2^6 = \left[(\alpha_1 - \lambda\alpha_2) \frac{\partial f}{\partial \alpha_1} + (\lambda\alpha_1 + \alpha_2) \frac{\partial f}{\partial \alpha_2}; \frac{\partial f}{\partial \alpha_1}; \frac{\partial f}{\partial \alpha_2}\right]$$

and, if $\lambda \neq 0$,

$$H_2^7 = \left[(1 + \lambda^2 \alpha_1^2) \frac{\partial f}{\partial \alpha_1} + (\lambda^2 \alpha_1 \alpha_2 + \alpha_2) \frac{\partial f}{\partial \alpha_2}, -\lambda\alpha_1 \frac{\partial f}{\partial \alpha_2}, \frac{\partial f}{\partial \alpha_2}\right]$$

The group H_2^6 is not measurable; it is associated to the family of varieties:

$$\overline{\mathscr{F}}_2^4 : F\left(\frac{x - \alpha_1}{z} \cos(\lambda\, ln|z|) + \frac{y - \alpha_2}{z} \operatorname{sen}(\lambda\, ln|z|);\right.$$

$$\left.\frac{y - \alpha_2}{z} \cos(\lambda\, ln|z|) - \frac{x - \alpha_1}{z} \operatorname{sen}(\lambda\, ln|z|)\right) = 0$$

the group H_2^7 is measurable with measure

$$(1+\lambda^2\alpha_1^2)^{-3/2} e^{-\frac{\operatorname{arctg}\lambda\alpha_1}{\lambda}} d\alpha_1 \wedge d\alpha_2$$

It is associated to the family of varieties:

$$\mathscr{F}_2^3 : F((y-\lambda\alpha_1 x-\alpha_2)\sqrt{1+\lambda^2\alpha_1^2}\, e^{-\frac{\operatorname{arctg}\lambda\alpha_1}{\lambda}}; z e^{-\frac{\operatorname{arctg}\lambda\alpha_1}{\lambda}}) = 0$$

One obtains two groups isomorphic to $G_3^3(\varepsilon)$

$$H_2^8 = \left[-\alpha_2\frac{\partial f}{\partial\alpha_1}+\alpha_1\frac{\partial f}{\partial\alpha_2}; \frac{\partial f}{\partial\alpha_1}; \frac{\partial f}{\partial\alpha_2}\right]$$

and

$$H_2^9 = \left[(1+\alpha_1^2)\frac{\partial f}{\partial\alpha_1}+\alpha_1\alpha_2\frac{\partial f}{\partial\alpha_2}; -\alpha_1\frac{\partial f}{\partial\alpha_2}; \frac{\partial f}{\partial\alpha_2}\right].$$

the group H_2^8 is measurable with measure

$$d\alpha_1 \wedge d\alpha_2$$

It is associated to the family of varieties:

$$\mathscr{F}_2^4 : \begin{cases} F((x-\alpha_1)\cos z+(y-\alpha_2)\operatorname{sen} z; (x-\alpha_1)\operatorname{sen} z-(y-\alpha_2)\cos z) = 0 & if \varepsilon = 1 \\ F((x-\alpha_1)^2+(y-\alpha_2)^2, z) = 0 & if \varepsilon = 0 \end{cases}$$

The group H_2^9 is measurable with measure

$$(1+\alpha_1^2)^{-3/2} d\alpha_1 \wedge d\alpha_2$$

It is associated to the family of varieties:

$$\mathscr{F}_2^5 : F(\sqrt{1+\alpha_1^2}(y-\alpha_1 x-\alpha_2), z-\varepsilon\alpha_2 \operatorname{arctg}\alpha_1) = 0$$

One finds only one group isomorphic to G_3^4

$$H_2^9 = \left[\frac{\partial f}{\partial\alpha_2}; (1+\alpha_1^2)\cos\alpha_2\frac{\partial f}{\partial\alpha_1}; \alpha_1\operatorname{sen}\alpha_2\frac{\partial f}{\partial\alpha_1}-\alpha_1\cos\alpha_2\frac{\partial f}{\partial\alpha_2}\right].$$

This groups is measurable and it has measure

$$(1+\alpha_1^2)^{-3/2}\,d\alpha_1 \wedge d\alpha_2$$

It is associated to the family of varieties

$$\mathscr{F}_2^6 : F(\widetilde{x}^2+\widetilde{y}^2,\widetilde{z}) = 0$$

where

$$\begin{cases} \widetilde{x} = \sqrt{1+\alpha_1^2}\,(x + y \operatorname{sen}\alpha_2 - \alpha_1 z \cos\alpha_2) \\ \widetilde{y} = y\cos\alpha_2 + z \operatorname{sen}\alpha_2 \\ \widetilde{z} = \sqrt{1+\alpha_1^2}\,(\alpha_1 x - y \operatorname{sen}\alpha_2 + z\cos\alpha_2) \end{cases}$$

7.4. Families admitting one of the groups G_2^i as maximal group

One finds the group: $H_2^{10} = \left[\dfrac{\partial f}{\partial\alpha_1}; \dfrac{\partial f}{\partial\alpha_2}\right]$ isomorphic to the groups G_2^1, G_2^3 and G_2^4.

This group is measurable with measure

$$d\alpha_1 \wedge d\alpha_2$$

and one obtains respectively the families of varieties associated

$$\mathscr{F}_2^7 : F(e^{-\alpha_1}(x\cos\alpha_2 + y\operatorname{sen}\alpha_2), e^{-\alpha_1}(y\cos\alpha_2 - x\operatorname{sen}\alpha_2), e^{-\alpha_1}z) = 0$$

$$\mathscr{F}_2^8 : F(x\cos\alpha_1 + y\operatorname{sen}\alpha_1, y\cos\alpha_1 - x\operatorname{sen}\alpha_1; z-\alpha_2) = 0$$

$$\mathscr{F}_2^9 : F(x, y-\alpha_1, z-\alpha_2) = 0$$

Then there is the group $H_2^{11} = \left[\dfrac{\partial f}{\partial\alpha_1} + \alpha_2\dfrac{\partial f}{\partial\alpha_2}, \dfrac{\partial f}{\partial\alpha_2}\right]$ isomorphic to $G_2^2(\lambda)$, measurable with measure

$$e^{-\alpha_1}d\alpha_1 \wedge d\alpha_2$$

It is associated to the family of varieties

$$\mathscr{F}_2^{10} : F(e^{-\alpha_1}(x\cos\lambda\alpha_1 + y\operatorname{sen}\lambda\alpha_1), e^{-\alpha_1}(y\cos\lambda\alpha_1 - x\operatorname{sen}\lambda\alpha_1), e^{-\alpha_1}(z-\alpha_2)) = 0$$

8. FAMILIES OF VARIETIES DEPENDING ONE PARAMETER

8.1. Families admitting one of the groups G_k with $k \geq 2$ as maximal group

There are neither one parameter groups isomorphic to a group G_k with $k \geq 3$ nor isomorphic to G_2^1, G_2^3, G_2^4.

There exists a group $H_1^1 = \left[\frac{\partial f}{\partial \alpha_1}, e^{-\alpha_1}\frac{\partial f}{\partial \alpha_1}\right]$ isomorphic to $G_2^2(\lambda)$.

The group H_1^1 is not measurable and it is associated to the family of varieties:

$$\overline{\mathscr{F}}_1^1 : F\left(\frac{x\cos(\lambda\, ln|z - e^{-\alpha_1}|) + y\,\mathrm{sen}(\lambda\, ln|z - e^{-\alpha_1}|)}{z - e^{-\alpha_1}},\right.$$

$$\left.\frac{y\cos(\lambda\, ln|z - e^{-\alpha_1}|) - x\,\mathrm{sen}(\lambda\, ln|z - e^{-\alpha_1}|)}{z - e^{-\alpha_1}}\right) = 0$$

8.2. Families admitting one of the groups G_1^i as maximal group

There is only one group $H_1^2 = \left[\frac{\partial f}{\partial \alpha_1}\right]$ isomorphic to any group G_1^i.

The group H_1^2 is measurable with measure

$$d\alpha_1$$

It is associated (resp. in the case of isomorphism with $G_1^1(\lambda), G_1^2(\varepsilon), G_1^3$) to the family of varieties:

$$\mathscr{F}_1^1 : F(e^{-\alpha_1}(x\cos\lambda\alpha_1 + y\,\mathrm{sen}\,\lambda\alpha_1), e^{-\alpha_1}(y\cos\lambda\alpha_1 - x\,\mathrm{sen}\,\lambda\alpha_1), e^{-\alpha_1}z) = 0$$

$$\mathscr{F}_1^2 : F(x\cos\alpha_1 + y\,\mathrm{sen}\,\alpha_1, y\cos\alpha_1 - x\,\mathrm{sen}\,\alpha_1, z - \varepsilon\alpha_1) = 0$$

$$\mathscr{F}_1^3 : F(x, y, z - \alpha_1) = 0\,.$$

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