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DERIVATION BY COORDINATES

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1. INTRODUCTION

In the '60's, T.G. Ostrom invented the process of *derivation in finite affine planes*; the replacement or renaming of certain Baer subplanes of the plane as *lines* which together with certain *old lines* of the plane determined a new plane called the *derived plane*.

Originally, most if not all affine planes that permitted derivation were seen as such due to a choice of their coordinate structures. It was Albert [1] who recognized that the Desarguesian planes of order q^2 became derivable where the Baer subplanes involved in the construction are those of the substructure of lines whose slopes belong to the square root subfield $GF(q)$ or are infinite. Of course, the field $GF(q^2)$ coordinatizing the Desarguesian plane is a 2-dimensional vector space over $GF(q)$. But, it was soon realized that this was sufficient for derivation in arbitrary finite planes; a coordinate structure Q of order q^2 contains a subfield $F \cong GF(q)$ such that Q is a *right vector space over* F while writing the slopes determined by elements ifn F on the right. In [10], Ostrom gives certain conditions that are slightly weaker than the properties of a right vector space which are also sufficient for derivation but the only known examples satisfied the stronger conditions.

The derivation process was soon extended to include the infinite situation (Johnson [6]) but still there existed coordinate systems of the known infinite derivable affine planes which were right 2-dimensional vector spaces over appropriate fields or skewfields.

The coordinate approach to derivation is continued in the work of Lunardon [8] and Gründhofer [3] who show that for derivable translation planes of finite and infinite order respectively there is always an associated coordinate system which is a right 2-dimensional vector space over some skewfield. Furthermore, this coordinate property has been recently established by Krüger [7] for derivable affine planes which may be coordinatized by cartesian groups.

Now derivation does not essentially depend on an affine plane. That is, the derivation process renames as lines the Baer subplanes of a certain substructure within an affine plane and certainly could be accomplished without this structure being embedded in an affine plane. Furthermore, Ostrom [11] has shown how to determine coordinate structures for arbitrary (affine) nets that have at least three distinct parallel classes.

In [9], Ostrom defines the structure of a *finite derivable net*. This concept is generalized to the infinite case in the author's papers [4] and [5]. In particular, a coordinate system may be chosen for any derivable net which automatically becomes a coordinate system for any net or affine plane which contains the derivable net.

Concerning coordinate structures for derivable nets (derivable affine planes), the following is then the most fundamental question:

Given a derivable net $\mathscr{R}$, is there a coordinate structure Q for $\mathscr{R}$ which contains a subskewfield L such that Q is a right 2-dimensional vector space over L while writing the slopes determined by the elements of L on the right?

The purpose of this note is to answer the above question. Also, part of the rational for writing this paper is to give proofs to some *folklore* results on the connections with the coordinates and the *replacement* procedures inherent within derivation.

2. COFMAN'S IDEAS AND THE EXTENSION TO PROJECTIVE SPACE

In [2], Cofman studies arbitrary derivable affine planes and establishes an affine space associated with the derivable plane. This structure is used to show that the Baer subplanes involved in the derivation process are always Desarguesian thus extending Prohaska [12] who proved the same result for finite derivable affine planes. While this associated affine space does provide some information, it gives essentially no insight as to the structure of the affine plane or to the possible coordinate structures.

In [4] and [5], the author extends the ideas of Cofman in two ways. First it is realized that Cofman's arguments are valid for arbitrary derivable nets whether they may be embedded in an affine plane or not. Second, the associated affine space may be extended to a projective space in such a way so as to obtain complete structural information on the derivable net. Further, the association with the projective space is tight enough so as to determine the full collineation group of any derivable net. It is the intent of this note to show how to answer the question on coordinates with the main results of the author [4], [5].

3. EVERY DERIVABLE NET MAY BE COORDINATIZED BY A RIGHT 2-DIMENSIONAL VECTOR SPACE OVER A SKEWFIELD.

We recall the main result of the author [5]. For the definition of an arbitrary derivable net and the derivation process, the reader is referred to the authors articles [4] and [5].

Theorem 3.1. *(Johnson [5] Lemma (3.6), Theorem (3.8) and Corollary (3.9). Let $\mathscr{R}$ denote a derivable net*

(1) Then there is a vector space V of dimension 4 over a skew field K such that the points of $\mathscr{R}$ may be identified with the vectors of V and the associated translation group of V is a collineation group of $\mathscr{R}$.

(2) Let $Z(K)$ denote the center of the skewfield K. Let $V = \{x_1, x_2, y_1, y_2)|x_i, y_i \in K, i = 1, 2\}$. The lines of the derivable net are translates of the following sets: $\{(x_1, x_2, \delta x_1, \delta x_2)|x_i \in K, i - 1, 2\}$ for δ fixed in K and $\{(0, 0, y_1, y_2)|y_i \in K\ i = 1, 2\}$. Furthermore, each line incident with $(0,0,0,0)$ is a $2[K : Z(K)]$-dimensional subspace over the center $Z(K)$ with the scalar action of $Z(K)$ on V defined by $\alpha(x_1, x_2, y_1, y_2) = (\alpha x_1, \alpha x_2, \alpha y_1, \alpha y_2)$.

We now choose $Q = K \oplus K, \overline{0} = (0,0), \overline{1} = (1,0), (x = \overline{0}) = \{(0,0,y_1,y_2) | y_i \in K, i = 1,2\}, (y = \overline{0}) = \{(x_1,x_2,0,0) | x_i \in K, i = 1,2\}$ and $(y = x) = \{(x_1,x_2,x_1,x_2) | x_i \in K, i = 1,2\}$. We form the Hall coordinate system for the net based on $x = \overline{0}, y = \overline{0}, y = x, (\overline{0},\overline{0})$ and $(\overline{1},\overline{1})$.

Recall, to obtain $a + b$ where $a, b \in Q$: Form the line parallel to $y = x$ thru $(\overline{0}, b)$ on $x = \overline{0}$. Form the line thru the point (a,a) parallel to $x = \overline{0}$. Form the intersection point of these latter two lines to obtain the point $(a, a+b)$.

We know that the lines of the net $\mathscr{R}$ are translates of the lines thru the origin since there is an associated translation group. Thus, for $b = (b_1,b_2)$ for $b_i \in K, i = 1,2$, the line $\mathscr{L}_b$ defined by $\{(x_1,x_2,x_1,x_2) | x_i \in K, i = 1,2\} + (0,0,b_1,b_2)$ contains $(\overline{0},b) = (0,0,b_1,b_2)$. Thus, $\mathscr{L}_b$ must be the line $y = x + b$. For $x = (x_1,x_2), x_i \in K, i = 1,2$, and $y = (y_1,y_2)$ for $y_i \in K, i = 1,2$ we obtain, $(y_1,y_2) = (x_1,x_2) + (b_1,b_2) = (x_1 + b_1, x_2 + b_2)$ since $(x, x+b)$ is on $\mathscr{L}_b$.

Thus, we have shown that

$(Q,+)$ *in the coordinate system is vector addition on* $K \oplus K$.

We now wish to form $a \cdot \alpha$ for $a \in Q$ and for $\alpha \in K \oplus \{0\}$.

To form $a \cdot \alpha$: First determine $(\overline{0}, \alpha)$ on $x = \overline{0}$ and form the line $y = \alpha$ parallel to $y = \overline{0}$ thru $(\overline{0},\alpha)$. Then determine $(\overline{1},\overline{0})$ on $y = \overline{0}$ and form the line $x = \overline{1}$ parallel to $x = \overline{0}$ thru $(\overline{1},\overline{0})$. Now intersect these latter lines to obtain the point $(\overline{1},\alpha)$. Finally, form the line $y = x \cdot \alpha$ as the join of $(\overline{0},\overline{0})$ and $(\overline{1},\alpha)$.

Now $\{(x_1,x_2,\alpha x_1,\alpha x_2) | x_i \in K, i = 1,2\}$ contains $(0,0,0,0) = (\overline{0},\overline{0})$ and $(1,0,\alpha,0) = (\overline{1},\alpha)$ (note the two uses of α). Hence, $y = x \cdot \alpha = \{(x_1,x_2,\alpha x_1,\alpha x_2) | x_i \in K, i = 1,2\}$. Now to obtain the point $(a, a \cdot \alpha)$: Find (a,a) on $y = x$ and form $x = a$ parallel to $x = \overline{0}$ thru (a,a). Then intersect $y = x \cdot \alpha$ and $x = a$ to obtain $(a, a \cdot \alpha) = \{(x_1,x_2,\alpha x_1,\alpha x_2) | x_i \in K, i = 1,2\} \cap \{(a_1,a_2,y_1,y_2) | y_i \in K, i = 1,2\}$ where $a = (a_1,a_2)$. Hence, this forces $x_i = a_i$ for $i = 1,2$ so that

$$a \cdot \alpha = (a_1,a_2) \cdot \alpha = (\alpha a_1, \alpha a_2).$$

Now let $\overline{K}$ denote the dual skewfield to K: Let $a \circ b = ba$ where juxtaposition denotes multiplication in K and $\circ$ denotes multiplication in $\overline{K}$.

$(Q,+,\cdot)$ *is a right 2-dimensional vector space over the dual of* $K, \overline{K}$.

Proof. Recall, $a \cdot \alpha = (a_1,a_1) \cdot \alpha = (\alpha a_1, \alpha a_2) = (a_1 \circ \alpha, a_2 \circ \alpha)$.

Then $(a \cdot \alpha) \cdot \beta$ for $\alpha, \beta \in \overline{K}, a \in Q = (\alpha a_1, \alpha a_2) \cdot \beta = ((\beta(\alpha a_1, \beta(\alpha a_2)) = ((\beta\alpha)a_1, (\beta\alpha)a_2) = a \cdot (\beta\alpha) = a \cdot (\alpha \circ \beta)$. Note that $\alpha \cdot \beta = \alpha \circ \beta$ since $\alpha = (\alpha, 0)$ and $\alpha \cdot \beta = (\alpha,0) \cdot \beta = (\beta\alpha, 0) = (1,0) \cdot (\beta\alpha) = \overline{1} \cdot (\beta\alpha) = (\beta\alpha) = \alpha \circ \beta$.

$(a+b)\cdot\alpha = ((a_1,a_2)+(b_1,b_2))\cdot\alpha = (a_1+b_1,a_2+b_2)\cdot\alpha = (\alpha(a_1+b_1),\alpha(a_2+b_2)) = (\alpha a_1,\alpha a_2) + (\alpha b_1,\alpha b_2) = a\cdot\alpha + b\cdot\alpha$.

Similarly, $a\cdot(\alpha+\beta) = a\cdot\alpha + a\cdot\beta$ for all $a \in Q$ and $\alpha \in \overline{K}$.

This proves the assertion.

4. THE ALBERT «SWITCH»

In this section, we show how to choose coordinates for the derived net $\overline{\mathscr{R}}$ based on coordinates in the original derivable net $\mathscr{R}$. This is slightly different then the original version due to Albert which we call the «Albert Switch» where a point with coordinates (x_1,x_2,x_3,x_4) in $\mathscr{R}$ is represented by (x_1,x_3,x_2,x_4) in $\overline{\mathscr{R}}$ (see Ostrom [10]).

Theorem 4.1. *Let $\mathscr{R}$ denote a derivable net with corresponding representation over a skew field K as $\{(x_1,x_2,x_3,x_4)|x_i \in K, i=1,2,3,4\}$ as a left $Z(K)$-vector space of dimension $4\cdot[K:Z(K)]$. Let $\overline{\mathscr{R}}$ denote the derived net. Then coordinates for a point may be taken by (x_1,x_2,x_3,x_4) in $\mathscr{R} \Leftrightarrow (x_1,x_3,x_2,x_4)$ in $\overline{\mathscr{R}}$ where $\overline{\mathscr{R}}$ is viewed as represented as a left $Z(\overline{K})$-vector space of dimension $4\cdot[\overline{K}:Z(\overline{K})]$ where $\overline{K}$ is the dual skew field to K.*

Proof. Let $\overline{K} = (K,+,\cdot)$ where $\alpha\cdot\beta = \beta\alpha$ and juxtaposition denotes multiplication in K. Then a line of $\mathscr{R}$ incident with $O=(0,0,0,0)$ has the form

$$\{(0,0,x_3,x_4)|x_3,x_4 \in K\} \equiv (x=O)$$

or

$$\{(x_1,x_2,\delta x_1,\delta x_2)|x_1,x_2 \in K, \delta \text{ fixed in } K\}.$$

The Baer subplanes of $\mathscr{R}$ incident with O have the form

$$\pi_{d_1,d_2} = \{(\alpha d_1,\alpha d_2,\beta d_1,\beta d_2)|\alpha,\beta \in K, d_1,d_2 \text{ fixed in } K, d_1,d_2 \text{ not both zero }\}.$$

Then $\{(x_1,x_2,\delta x_1,\delta x_2)\}$ in $\mathscr{R}$ is represented by $\{(x_1,\delta x_1,x_2,\delta x_2)\}$ in $\overline{\mathscr{R}} = \{(x_1\cdot 1, x_1\cdot\delta, x_2\cdot 1, x_2\cdot\delta)\}$ and for $x_1=\alpha, x_2=\beta, d_1=1, d_2=\delta$ has the form

$$\{(\alpha\cdot d_1,\alpha\cdot d_2,\beta\cdot d_1,\beta\cdot d_2)|\alpha,\beta \in \overline{K}\} = \pi_{d_1,d_2} = \pi_{1,\delta}$$

in $\overline{\mathscr{R}}$. Similarly $\{(\alpha d_1,\alpha d_2,\beta d_1,\beta d_2)|\alpha,\beta \in K, d_1,d_2$ fixed in K and not both zero$\}$ in $\mathscr{R}$ is $\{(\alpha d_1,\beta d_1,\alpha d_2,\beta d_2)\}$ in $\overline{\mathscr{R}}, = \{(d_1\cdot\alpha, d_1\cdot\beta, d_2\cdot\alpha, d_2\cdot\beta)\}$. Let $x_1 = d_1\cdot\alpha, x_2 = d_1\cdot\beta$. Then $d_2\cdot\alpha = d_2\cdot(d_1^{-1}\cdot x_1)$ and $d_2\cdot\beta = d_2\cdot(d_1^{-1}\cdot x_2)$. Letting $d_2\cdot d_1^{-1} = \delta$ this set has the form

$$\{(x_1,x_2,\delta\cdot x_1,\delta\cdot x_2)|x_1,x_2 \in \overline{K}, \delta \in \overline{K}\}.$$

This completes the proof of (4.1).

Corollary 4.2. *If a derivable net $\mathscr{R}$ is represented by the partial spread $x = O, y = \delta \cdot x$ for $\delta \in K$, K a skewfield then the associated derivable net $\overline{R}$ may be represented in the form $x = O, y = \delta \cdot x$ for $\delta \in \overline{K}$ where $\overline{K}$ is the dual skewfield of K.*

5. DERIVATION IN TRANSLATION PLANES

In this section, we show how to relate the matrix spread sets of a derivable translation plane and its derived plane.

Throughout this section, let π denote a derivable translation plane with derivable net $\mathscr{R}$. Note that we assume that the Baer subplanes of $\mathscr{R}$ are also Baer when considered as subplanes of π. Let K be a skew field such that we represent the set of points of π and $\mathscr{R}$ as $\{(x_1, x_2, x_3, x_4) | x_i \in K, i = 1,2,3,4\}$. Let F denote the prime subfield of the center $Z(K)$ of K so that π is a vector space over F of dimension $4 \cdot [K : Z(K)] \cdot [Z(K) : F]$. Let $x = (x_1, x_2), y = (y_1, y_2); x_i, y_i \in K, i = 1,2$ and represent the lines of $\mathscr{R}$ incident with the zero vector $O = (0,0,0,0)$ by $y = \delta x, x = O$ for $\delta x = (\delta x_1, \delta x_2)$ for each $\delta \in K$. Represent the Baer subplanes of $\mathscr{R}$ incident with O by

$$\pi_{d_1,d_2} = \{(\alpha d_1, \alpha d_2, \beta d_1, \beta d_2) | \alpha, \beta \in K, d_1, d_2 \text{ fixed in } K \text{ and not both zero}.$$

The lines $y = \delta x, x = O$ and subplanes π_{d_1,d_2} are subspaces over $Z(K)$ and hence subspaces over F.

Lemma 5.1. *The subspaces of π not in $\mathscr{R}$ corresponding to the spread of π may be represented in the form $y = xM$ where $M = \begin{bmatrix} m_1, & m_2 \\ m_3, & m_4 \end{bmatrix}$ and m_i for $i = 1,2,3,4$ is a linear transfomation of K over F where K is considered a F-module of dimension $[K : Z(K)][Z(K) : F]$. Furthermore, the linear transformations m_2 and m_3 are nonsingular.*

Proof. Let $\pi = W \oplus W$ for some vector space W over F. If a component is represented in the form $y = xM$ then M is a nonsingular F-linear transformation of W. In particular, M is a nonsingular F-linear transformation of $K \oplus K$ so clearly $M = \begin{bmatrix} m_1, & m_2 \\ m_3, & m_4 \end{bmatrix}$ where m_i is a F-linear transformation of K (m_1, m_2 act on $K \oplus 0, m_3, m_4$ on $0 \oplus K$).

Assume m_3 is singular and $\overline{x}_2 m_3 = O$ for $\overline{x}_2 \neq O$. Then $\pi_{0,1} = \{(0,\alpha,0,\beta)|\alpha,\beta \in K\}$ is a Baer subplane incident with $O = (0,0,0,0)$ which is disjoint form $y = xM$. But,

$(0,\overline{x}_2,0,\overline{x}_2 m_4) \in \pi_{0,1}$ and since $(0,\overline{x}_2)\begin{bmatrix} m_1, & m_2 \\ m_3, & m_4 \end{bmatrix} = (\overline{x}_2 m_3,\overline{x}_2 m_4) =$ $(0,\overline{x}_2 m_4)$, this element is also in $y = xM$. This proves m_3 is nonsingular. The same argument using $\pi_{1,0}$ shows m_2 is nonsingular.

Lemma 5.2. *Assume the conditions of (5.1). If* $y = xM$ *is a subspace of the spread of* π *which is not in* $\mathscr{R}$ *and* $\overline{K}$ *is the dual skewfield of* K *then, for* $M = \begin{bmatrix} m_1, & m_2 \\ m_3, & m_4 \end{bmatrix}$, *the* m_i *may be considered as* F*-linear transformations over* $\overline{K}$ *or* K.

Proof. Since $(\pi,+) = (K,+)$ and $F \subseteq Z(\overline{K})$, (5.2) follows.

Theorem 5.3. *Let* π *be a derivable translation plane whose spread is represented by the equations* $x = O, y = \delta \cdot x, \delta \in K, K$ *a skewfield and* $y = x\begin{bmatrix} m_1, & m_2 \\ m_3, & m_4 \end{bmatrix}$ *for* m_i F*-linear transformations over* K *where* F *is the prime field of the center of* K. *Then the spread for the derived translation plane* $\overline{\pi}$ *may be represented by the equations* $x = O, y = \delta \cdot x$ *for* $x \in \overline{K}$ *where* $\overline{K}$ *is the dual skewfield to* K, *and* $y = x\begin{bmatrix} -m_1 m_3^{-1}, & m_2 - m_1 m_3^{-1} m_4 \\ m_3^{-1}, & m_3^{-1} m_4 \end{bmatrix}$.

Proof. On $\pi - \mathscr{R}$, the Albert switch amounts to the basis change $\begin{bmatrix} I & O & O & O \\ O & O & I & O \\ O & I & O & O \\ O & O & O & I \end{bmatrix}$ thinking of π as an F-module where I, O refer to appropriate identity and zero mappings.

$$(y = xM) = \{(x_1,x_2,x_1 m_1 + x_2 m_3, x_1 m_2 + x_2 m_4) \mid x_1,x_2 \in K\}$$

in $\pi - \mathscr{R}$ is

$$\{(x_1,x_1 m_1 + x_2 m_3, x_2, x_1 m_2 + x_2 m_4) \mid x_1,x_2 \in K\}$$

in $\overline{\pi} - \overline{\mathscr{R}}$. Let $\overline{x}_1 = x_1, \overline{x}_2 = x_1 m_1 + x_2 m_3$. Since m_3 is nonsingular, $x_2 = (\overline{x}_2 - \overline{x}_1 m_1) m_3^{-1}$ and

$$x_1 m_2 + x_2 m_4 = \overline{x}_1 m_2 + (\overline{x}_2 - \overline{x}_1 m_1) m_3^{-1} m_4.$$

Hence, in $\bar{\pi} - \overline{\mathscr{R}}$, we obtain:

$$y = x \begin{bmatrix} -m_1 m_3^{-1}, & m_2 - m_1 m_3^{-1} m_4 \\ m_3^{-1}, & m_3^{-1} m_4 \end{bmatrix}.$$

Hence, we obtain the proof of (5.3).

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