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MINIMUM COVERING AND MAXIMUM MATCHING

D. MARCU

Abstract. *This paper describes simple polynomial-bounded transformations between the problems of minimum edge cover and maximum matching. More exactly, the equivalence between the minimum covering problem and the maximum matching problem is point out.*

INTRODUCTION

For a given undirected graph G, without loops and multiple edges, $V(G)$ and $E(G)$ will denote its vertex set and edge set, respectively. A set $S \subseteq E(G)$ is an *edge covering* of G, if any vertex of G is incident to at least one edge of S, and S is a *matching* of G, if no two edges of S are adjacent. If each edge $v_iv_j \in E(G)$ has associated a cost c_{ij} (a real number), then the cost of S is defined to be the sum of its element costs. The *minimum covering problem* is to find a c-minimum covering of G. The *maximum matching problem* is to find a c-maximum matching of G. Both these problems are well-known (e.g., see [1, 4, 5]). A polynomial algorithm, for the maximum matching problem, was developed by Edmonds, in 1965, and an $0(n^3)$ implementation can be found in [4]. White [8] has used the Edmonds' method and has given a polynomial algorithm, for the minimum covering problem (see, also, [5]), but the development is incomplete. In the special case, when all the costs are equal, the problems are equivalent, according to Gallai [3] and Norman and Rabin [7]. Although Tutte, in 1954, and Edmonds, in 1967, have found transformations of certain degree-constrained subgraph problems, to the maximum matching problem, the obtained problems are often very large (e.g., see [4]) and, therefore, direct algorithms were developed [2].

In [6] is described a primal-dual algorithm, for finding a minimum-weight edge cover. This algorithm is based on the blossom algorithm, for maximum weighted matching problems, and has running time $0(n^2 m)$, where n is the number of vertices and m is the number of edges.

In [9] and [10], the covering problem is tackled, by using the concept of "reducing paths".

In this note, we shall point out the equivalence between the minimum covering problem and the maximum matching problem, in the sense that either problem, in an n-vertex graph, can be transformed, in time $0(n^2)$, into the other problem, in a complete graph with n or $n+1$ vertices.

FROM MINIMUM COVERING TO MAXIMUM MATCHING

One can suppose that G is a graph without isolated vertices and that all its costs are nonnegative. Our transformation consists of the following steps:

Step 1. For each vertex $v_i \in V(G)$, compute $d_i = \min\{c_{ij}.: v_iv_j \in E(G)\}$ and choose an edge e_i, incident to v_i and having $c(e_i) = d_i$.

Step 2. On the set $V(G)$, form a complete graph with costs

$$\widehat{c}_{ij} = \begin{cases} \min\{c_{ij}, d_i + d_j\}, & if\ v_i v_j \in E(G), \\ d_i + d_j, & otherwise. \end{cases}$$

Moreover, if $|V(G)|$ is odd, then add a new vertex w_0 and, for each $v_i \in V(G)$, an edge $v_i w_0$, with cost $\widehat{c}_{io} = d_i$. The obtained complete graph will be denoted by $\widehat{G}$.

Step 3. Choose a number $A > \max\{\widehat{c}_{ij} : v_i v_j \in E(\widehat{G})\}$ and, for every $v_i v_j \in E(\widehat{G})$, define $\overline{c}_{ij} = A - \widehat{c}_{ij}$.

Step 4. Find a $\overline{c}$-maximum matching M of $\widehat{G}$.

Step 5. Construct a set S, by choosing one or two edges of G, for every edge $v_i v_j \in M$, as follows:

(a) if $v_i v_j \in E(G)$ and $c_{ij} \leq d_i + d_j$, then choose the edge $v_i v_j$ (we see that $c_{ij} = \widehat{c}_{ij}$);

(b) if either $v_i v_j \in E(G)$ and $c_{ij} > d_i + d_j$ or $v_i v_j \notin E(G)$ and $v_i, v_j \in V(G)$, then choose the edges e_i and e_j, defined in Step 1 (clearly, $c(e_i) + c(e_j) = d_i + d_j = \widehat{c}_{ij}$);

(c) if one of the vertices v_i, v_j does not belong to $V(G)$, say $v_j = w_0$, then choose the edge e_i (we see that $c(e_i) = d_i = \widehat{c}_{io}$).

Proposition 1. *The set S of edges chosen in Step 5 is a c-minimum covering of G, and $c(S) = \widehat{c}(M) = A\widehat{n}/2 - \overline{c}(M)$, where $\widehat{n} = |V(\widehat{G})|$.*

Proof. Since each $\overline{c}_{ij} > 0$, then M is a $\widehat{c}$-minimum perfect matching of $\widehat{G}$ and, hence, S is a covering of G, with $c(S) \leq \widehat{c}(M)$. On the other hand, for a given c-minimum covering S' of G, one can construct a perfect matching M' of $\widehat{G}$, with $\widehat{c}(M') = c(S')$. Namely, one can suppose that the subgraph of G, formed by S', consists of stars (trees of radius one). Constructing M', we, at first, choose one edge from every star and, then, the remaining vertices are paired arbitrarily. Since $\widehat{c}(M) \leq \widehat{c}(M')$, the proof follows.

FROM MAXIMUM MATCHING TO MINIMUM COVERING

One can suppose that all costs of G are nonnegative. Then, we shall proceed as follows:

Step 1. By adding new edges, form a complete graph $\widehat{G}$, with $V(\widehat{G}) = V(G)$, if $|V(G)|$ is even, and with $V(\widehat{G}) = V(G) \cup \{w_0\}$, if $|V(G)|$ is odd, where w_0 is a new vertex. Define costs

$$\widehat{c}_{ij} = \begin{cases} c_{ij}, & if\ v_i v_j \in E(G), \\ 0, & otherwise. \end{cases}$$

Step 2. Letting $\widehat{c}_{\max} = \max\{\widehat{c}_{ij} : v_i v_j \in E(\widehat{G})\}$, choose a number $B > 2\widehat{c}_{\max}$. Then, define $\overline{c}_{ij} = B - \widehat{c}_{ij}$, for all $v_i v_j \in E(\widehat{G})$.

Step 3. Find a $\bar{c}$-minimum covering Y of $\widehat{G}$ and construct a set $M = Y \cap E(G)$.

Proposition 2. *The set M, obtained in Step 3, is a c-maximum matching of G, and $c(M) = \widehat{c}(Y) = B\widehat{n}/2 - \bar{c}(Y)$, where $\widehat{n} = |V(\widehat{G})|$.*

Proof. Since each $\bar{c}_{ij} > 0$, then every component of the subgraph of $\widehat{G}$, formed by Y, is a star. Moreover, as B is sufficiently large and $\widehat{n}$ is even, every one of these stars has only two vertices (otherwise, a better covering than Y can be made from Y, by deleting two edges and adding one edge). Thus, Y is a $\bar{c}$-minimum perfect matching of $\widehat{G}$ and, therefore, it is a $\widehat{c}$-maximum perfect matching of $\widehat{G}$. Hence, $Y \cap E(G)$ is a c-maximum matching of G.

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D. Marcu
Str. Pasului 3, Sect. 2
70241 - Bucharest, Rumania