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NOTE ON PLANAR FUNCTIONS OVER THE REALS

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Dedicated to the memory of Professor Ferenc Kárteszi

1. INTRODUCTION

The following construction was used in a paper of Kárteszi [7] illustrating the role of Cremona transformations for secondary school students. This is a typical construction in the theory of flat affine planes, see Salzmann [9], Groh [4] and due to Dembowski and Ostrom [3] for the case of finite ground fields. Let R^2 be the classical euclidean affine plane and $\tilde{f}$ be the graph of a real function $f : R \to R$ (R denotes the field of real numbers). Define a new incidence structure $A = A(f)$ on the points of R^2 in which the «new lines» are the vertical lines of R^2 and the translates of $\tilde{f}$. The incidence is the set-theoretical «element of» relation. (For the definition of «incidence structure», «affine plane» etc. we refer to Dembowski [2]).

The following theorem is due to Salzmann (see [9], and can be found in Kárteszi [6] and in [3], [4] for the other cases).

Theorem 0. *If $\tilde{f}$ is a parabola, then $A(f)$ is an affine plane isomorphic to R^2.*

Definition 1. *A function f is called planar if $A(f)$ is an affine plane.*

Salzmann showed that there are a lot of planar functions over the reals, and proved that continuous planar functions are related to convex functions.

Theorem 1. *A continuous function f is planar iff f (or $-f$) is strictly convex and satisfies*

$$\lim_{x\to+\infty} f(x)/x = +\infty; \quad \lim_{x\to-\infty} f(x)/x = -\infty.$$

The purpose of this paper is to characterize the parabolas.

The question about the validity of the Theorem of Desargues in $A(f)$ comes from Kárteszi. The following theorem can be regarded as a partial converse of Theorem 0. It was proved originally by Salzmann (see [9]) using different arguments.

Theorem 2. *Let f be a continuous planar function. If $A(f)$ is desarguesian, then f is a parabola.*

In Part 4 we prove a little more.

Theorem 3. *Let f be a continuous planar function. If the projective closure of $A(f)$ has a translation line, then f is a parabola.*

(For the definition we refer to [7, pp. 237-240] or again to [2, pp. 98-101]).

Some of the results are probably not new, but they have been approached form different viewpoints. (See Groh [4] or Salzmann [9]). But this paper is a self-contained (except of Part 4) elementary solution to an elementary problem of Kárteszi.

2. PRELIMINARIES

The following observation can be found in Dembowski and Ostrom [3].

Proposition 1. *A function f is planar if and only if*

(SA) $f(x+a) - f(x) = b$ has exactly one solution for every $a, b \in R, \quad a \neq 0$.

We use the idea of the proof of Theorem 0 (see [7]). Set

$$f(a+x) = f(x) + g(x,a) + f(a).$$

Thus the equation of the translate $y = f(x+a) + b_1$ of $y = f(x)$ can be written in the form

$$y = f(x) + g(x,a) + (f(a) + b_1).$$

Applying the 1-1 transformation

$$\begin{cases} x' = x \\ y' = y - f(x) \end{cases}$$

the equation of the lines of $A(f)$ will be

(L) $$x' = C \quad or \quad y' = g(x',a) + b.$$

Definition 2. *Define a new multiplication by the rule*

$$a * u = g(a,u).$$

From now on we will suppose that $f(0) = 0$. The following Proposition can be proved without difficulty using the algebraic consequence (SA) of planarity (see Prop. 1).

Proposition 2. *The coordinate structure $(R,+,*)$ (where $+$ is the usual addition) has the following properties:*

(i) $(R,+)$ *is an abelian group with neutral element* 0.

(ii) $(R-\{0\},*)$ *is a commutative quasigroup.*
(iii) $0*x=0$ *for every* $x \in R$.
(iv) *for every* $a \neq b$ *and* c *there exists one and only one* $x \in R$ *such that*

$$a*x-b*x=c.$$

Proof. For example (iv) means that the equation

$$f(x+a)-f(x)-f(a)-f(x+b)+f(x)+f(b)=c$$

has exactly one solution. As this is equivalent to

$$f(x+a)-f(x+b)=c+f(a)+f(b),$$

the assertion is a direct consequence of (SA).

The proof of the other parts is similar.

Remark 1. If an algebraic structure satisfies the conditions (i)-(iv) of Prop. 2, then using (L) an affine plane can be constructed.

Remark 2. If for example $f(1)=0$, then put $e=1*1$ and define an other multiplication « $\circ$ » by the rule

$$a \circ b = a_1 * b_1 \quad where \quad a = 1*a_1,\ b=1*b_1.$$

For the case of finite ground fields it was proved by Dembowski and Ostrom [3] that $(K,+,\circ)$ is a commutative cartesian system (see also [10, App.]) and this is the coordinatizing planar ternary ring of $A(f)$. (See also [2, p. 228]).

In the sequel we frequently use the new coordinates x', y' and the equations of lines given in (L).

3. PROOF OF THEOREM 2

In order to study the validity of $\underline{D}_0$ (that version of the Theorem of Desargues, in which the center of the two triangles is ideal and the axis is the ideal line) we build a figure consisting of two centrally perspective triangles $(A_1A_2A_3\triangle, B_1B_2B_3\triangle)$ such that the center is the ideal point of the line $y'=0$, (A_1, B_1 will be on the x'-axis) and A_2A_3 is the vertical line $x'=0$. Set $A_1:(A;0), /; B_1:(B;0)$ and let the slopes of the lines A_1A_3 and A_1A_2 be j and $(-j)$ respectively. Thus the equation of A_1A_2 is $y'=x'*(-j)-(A*(-j))$ and that of A_1A_3 is $y'=x'*j-(A*j)$. As A_2, A_3 are on the y'-axis we have $A_2:(0;-(A*(-j))), A_3:(0;-(A*j))$. The center of these triangles is the ideal point of the x'-axis, hence A_2B_2 has equation $y'=-(A*(-j))$, and A_3B_3 has equation $y'=-(A*j)$.

Since the axis of the configuration is the ideal line, B_1B_2 is parallel to A_1A_2 and B_1B_3 is parallel to A_1A_3. Hence B_2 is the intersection of the lines $y' = x' * (-j) - (B * (-j))$ and $y' = -(A * (-j))$. Thus B_2 must be $(u; -(A * (-j))$, where

$$(1) \qquad u * (-j) - (B * (-j)) = -(A * (-j)).$$

Similarly, B_3 is $(v; -(A * j))$, where

$$(2) \qquad v * j - (B * j) = -(A * j).$$

The theorem $\underline{D}_0$ implies $A_2A_3 || B_2B_3$, i.e. $u = v$. Putting «u» instead of «v» into (2) and using Definition 2 we get

$$(1') \quad f(u-j) - f(u) - f(-j) - f(B-j) + f(B) + f(-j) = -f(A-j) - f(A) - f(-j)$$

$$(2') \quad f(u+j) - f(u) - f(j) - f(B+j) + f(B) + f(j) = -f(A+j) + f(A) + f(j)$$

Adding up these equations (and using $f(0) = 0$) we obtain

$$(3) \quad \begin{aligned} &(f(u+j) + f(u-j) - 2f(u)) + (f(A+j) + f(A-j) - 2f(A)) = \\ &(f(B+j) + f(B-j) - 2f(B)) + (f(j) + f(-j) - 2f(0)). \end{aligned}$$

Define the function d_j for every fixed j by the rule

$$d_j : x \in R \mapsto d_j(x) = f(x+j) + f(x-j) - 2f(x) \in R$$

The strict convexity of f implies that d_j is continuous and $d_j(x) > 0$ for every $x \in R$. Rewriting (3) for the $d_j's$ we have

(3′) for every A, B, j there exists one and only one $u \in R$ such that $d_j(u) + d_j(A) = d_j(B) + d_j(0)$.

We claim that

(4) for every $j \in R$ there is a constant D_j such that $d_j(x) = D_j$ independently of x.

Proof of (4). Suppose indirectly that d_j is not a constant function and let $h_j = \inf\{d_j(x)\} \geq 0$, $H_j = \sup\{d_j(x)\}$.

If $d_j(0) = H_j$ then let $\varepsilon > 0$ be chosen such that $h_j + 3\varepsilon < H_j$ and $d_j(A) < h_j + \varepsilon, d_j(B) > H_j - \varepsilon$. Putting these into (3') we get a contradiction, since

$$d_j(A) + d_j(u) < h_j + \varepsilon + H_j < 2H_j - \varepsilon < d_j(0) + d_j(B).$$

If $d_j(0) < H_j$ then let $\varepsilon > 0$ be chosen such that $d_j(0) + 3\varepsilon < H_j$ and $d_j(B) < h_j + \varepsilon, d_j(A) > d_j(0) + 2\varepsilon$. Substituting these into (3') we get a contradiction again, since

$$h_j + d_j(0) + 2\varepsilon < d_j(u) + d_j(A) = d_j(B) + d_j(0) < h_j + d_j(0) + \varepsilon.$$

Finally, we claim that

(5) If for a continuous function $f, d_j(x) = D_j$ holds for every j independently of x, then f is a parabola.

Proof of (5). Fix an arbitrary $j_0 \neq 0$ and let $f(j_0) = C_0$. As $f(0) = 0$ may be supposed, the relation

$$f(2j_0) = 2C_0 + D_j, \ldots, f(kj_0) = 2f((k-1)j_0) - f((k-2)j_0 + D_j) \quad (\textit{for every } k \in Z) \tag{6}$$

follows immediately from the definition of $d_j(x)$. Thus it is easy to see (for example by induction) that the points $(kj_0, f(kj_0))$ lie on a suitably chosen parabola with equation $f(x) = a_0x^2 + b_0x$. Choosing $j_0 = 1, j_1 = \frac{1}{2}, j_2 = \frac{1}{3}, \ldots, j_k = \frac{1}{k+1}$ we get a sequence of parabolas with equations $y = a_0x^2 + b_0x, \ldots, y = a_kx^2 + b_kx, \ldots$. As the points of the graph of f with integer x-coordinate are common points of these parabolas, all these parabolas coincide, i.e. there exist a, b R such that $f(x) = ax^2 + bx$ for every rational x. As f is continuous, (5) is proved.

4. FURTHER PROPERTIES OF $A(f)$

In this section the projective closure of $A(f)$ will be denoted by $\overline{A}$.

Proposition 3. *The plane $\overline{A}$ admits a collinetion group of rank 3.*

Proof. The orbits of the group $T = \{\tau_{a,b} : (x, y) \mapsto (x + a, y + b) | a, b \in R\}$ (where x, y are the old coordinates) are $O_1 = \{$the ideal point of $x' = 0\}, O_2 = \{$the other ideal points$\}$ and $O_3 = \{$the points of $A(f)\}$.

Remark 3. The existence of abelian rank 3 collineation groups was the starting point of Dembowski and Ostrom [3].

Proposition 4. *The plane $\overline{A}$ is self-dual.*

Proof. The mapping

$$\pi : (u, v) \leftrightarrow y' = u * x' - v.$$
$$(m) \leftrightarrow x' = m$$
$$(\infty) \leftrightarrow \textit{ideal line} = l_\infty$$

(where (m) is the ideal point of $y' = m * x', (\infty)$ is the ideal point of $x' = 0$, as usual) will be a polarity. (cf. [10, App. 7. pp. 396-398], or [3]).

Proof of Theorem 3. We prove that the plane $\overline{A}$ has no translation line if $\widetilde{f}$ is not a parabola. We have proved in Thm. 1 that l_∞ is not a translation line. (A line r of $\overline{A}$ is a «translation line» if the affine plane $\overline{A} \setminus r$ is a translation plane, cf. Hall [5, p. 336]). If a vertical line r is a translation line, then $r^{\tau(a,0)}$ is also a translation line. But Thm. 20.5.1. of Hall [5] shows that every line through $r \cap r^{\tau(a,0)}$, in particular l_∞, is a translation line, which is a contradiction. If a non-vertical line $\widetilde{f}_1$ is a translation line, then $\widetilde{f}_1^{\tau(0,b)}$ is also a translation line parallel to $\widetilde{f}_1$. As $\widetilde{f}_1 \cap \widetilde{f}_1^{\tau(0,b)}$ is an ideal point, the previous argument (using [5, Thm. 20.5.1]) yields the same contradiction.

Remark 4. Considering the so-called Lenz-Barlotti classification of projective planes (see [1], [8] or Thm. 3.1.20 of [2, pp. 123-126]) the existence of $T_0 = \{\tau_{0,b} | b \in R\}$ implies that $\overline{A}$ is at least of type II.1. (i.e. all vertical translation do exist). The non-existence results listed in Table 1 of [2, p. 126]) together with our Thm. 3 yield that if $\widetilde{f}$ is not a parabola then $\overline{A}$ is either of type II.1. or of type II.2. The existence of T (see Prop. 3) excludes that $\overline{A}$ is of type II.2. Thus if $\widetilde{f}$ is not a parabola then $\overline{A}$ is a projective plane of Lenz-Barlotti type II.1.

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