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A FRAMEWORK INCLUDING THE THEORIES OF CONTINUOUS FUNCTIONS AND CERTAIN NON-CONTINUOUS FUNCTIONS

J.K. KOHLI

Abstract. *A framework unifying the theories of continuous functions and certain non-continuous functions including weakly continuous functions, almost continuous functions, c-continuous functions, c^*-continuous functions, ℓ-continuous functions, s-continuous functions, H-continuous functions, ε-continuous functions and others, is presented. This unifies and frequently improves known results.*

1. INTRODUCTION

Several types of non-continuous functions occur in literature. The theories of certain of these non-continuous functions run, either in part or in whole, parallel to the theory of continuous functions. In [10] we proposed and developed a unified theory which encompasses continuous functions and certain non-continuous functions having properties analogous to the properties of continuous functions; that theory was further elaborated in [11]. However, the framework of ([10], [11]) does not seem to include, among others, the theory of weakly continuous functions. The purpose of the present paper is to enlarge the framework of ([10] [11]) to include the theory of weakly continous functions along with several other such types of functions.

2. A UNIFIED FRAMEWORK

Let P denote a property, not necessarily topological, possessed by certain subsets of a topological space.

The properties P that we consider in this paper are listed in the accompanying table. It may be observed tht some of these properties are not necessarily preserved under topological embeddings, while others are only continuous invariants; note that the property of having a complement of diameter $\geq \varepsilon$ is not even a topological property.

A π*-set* [28] in a space is the intersection of finitely many regularly closed setss. In the sequel, the complement of a π*-set* will be referred to as π-open. The interior of a set A in a topological space B will be denoted by int A or $\text{int}_B A$ and closure by $\overline{A}$.

Definition 2.1. *Let X be a topological space, and let $A \subset X$. Then we say that*

(i) A is a P-set if A possesses property P; and

(ii) A has P-complement if $X - A$ possesses property P.

Definition 2.2. *Let $f : X \to Y$ be a function from a topological space X into a topological space Y. Then f is said to be*

(i) P-continuous [10] if for each $x \in X$ and each open set V containing $f(x)$ and having P-complement, there is an open set U containing x such that $f(U) \subset V$; and

(ii) P^*-continuous if for each $x \in X$ and set B with $f(x) \in \operatorname{int} B$, whenever B has P-complement, there is an open set U containing x such that $f(U) \subset B$.

The following table lists the type of P-continuous function and P^*-continuous function induced by a property P, and gives explicitly the weak forms of continuity to which they correspond.

Table

	Property P	Type of P^*-continuous function	Type of P-continuous function
1.	closed set	continuous	continuous
2.(a)	open set		
(b)	regularly open set	weakly continuous [14]	mildly continuous[26]
(c)	δ-open set		
(d)	π-open set		
3.	zero set	z-continuous [25]	z-continuous
4.(a)	regularly closed set	almost continuous [24]	almost continuous
(b)	δ-closed set [27]		
5.	closed compact set	c-continuous [2]	c-continuous
6.	regularly closed compact set	almost c-continuous[5]	almost c-continuous
7.	closed countably compact set	c^*-continuous [21]	c^*-continuous
8.	closed connected set	s-continuous [8]	s-continuous
9.	closed Lindelöf set	ℓ-continuous [12]	ℓ-continuous
10.	H-closed	H-continuous [16]	H-continuous
11.	closed set having complement of diameter $\geq \varepsilon$	ε-continuous ([6] [7])	ε-continuous
12.	hard set ([22] [23])	h-continuous [13]	h-continuous
13.	closed G_δ-set.	D-continuous [13]	D-continuous
14.	strongly closed G_δ-set ([3] [4])	weakly D-continuous [13]	weakly D-continuous
15.	the boundary of an open set	w^*-continuous [14]	w^*-continuous
16.	clopen set	mildly continuous [26]	mildly continuous

3. BASIC PROPERTIES OF P^*-CONTINUOUS FUNCTIONS

The class of P^*-continuous functions constitutes a subclass of the class of P-continuous functions, and the two coincide if the property P implies the property of being a closed set.

Theorem 3.1. *Let $f : X \to Y$ be a function from a topological space X into a topological space Y. Consider the following statements:*

(a) f is P^*-continuous,

(b) for every subset A of Y having P-complement

$$f^{-1}(\text{int } A) \subset \text{int } f^{-1}(A)$$

(c) for every open subset V of Y having P-complement $f^{-1}(V)$ is open.

Then (a) and (b) are equivalent, and either implies (c).

Proof. (a) $\Rightarrow$ (b). Let A be a subset of Y having P-complement, and let $x \in f^{-1}(\text{int } A)$. Then $f(x) \in \text{int } A \subset A$ and so by P^*-continuity of f, there is an open set U containing x such that $f(U) \subset A$. Thus $U \subset f^{-1}(A)$ and hence $x \in \text{int } f^{-1}(A)$.

(b) $\Rightarrow$ (a). Let $x \in X$ and let V be a neighbourhood of $f(x)$ having P-complement. Then $f(x) \in \text{int } V$ and hence $x \in f^{-1}(\text{int } V) \subset \text{int } f^{-1}(V)$. Consequently, $U = \text{int } f^{-1}(V)$ is an open set containing x and $f(U) \subset V$ and, so f is P^*-continuous.

(b) $\Rightarrow$ (c). Obvious.

Without restrictions on the type of property P, a P-continuous function may not be P^*-continuous. For example, if P denotes the property of being an open set, then the concept of P-continuous function coincides with that of mildly continuous function, and P^*-continuous functions are precisely weakly continuous functions. Examples of mildly continuous functions which are not weakly continuous abound. For instance, every discontinuous function into a connected regular space is mildly continuous, but fails to be weakly continuous.

Corollary 3.2. *For a function f from a topological space X into a topological space Y the following statements are equivalent:*

(a) f is weakly continuous,

(b) for every closed subset A of Y $f^{-1}(\text{int } A) \subset \text{int } f^{-1}(A)$,

(c) for every regularly closed subset B of Y $f^{-1}(\text{int } B) \subset \text{int } f^{-1}(B)$,

(d) for every open subset V of Y $f^{-1}(V) \subset \text{int } f^{-1}(\overline{V})$,

(e) for every regularly open subset W of Y $f^{-1}(W) \subset \text{int } f^{-1}(\overline{W})$.

Proof. (a) $\Rightarrow$ (b) follows from Theorem 3.1 with P = open set. (b) $\Rightarrow$ (c) is obvious. (c) $\Rightarrow$ (d). Let V be an open subset of Y. Then $B = \overline{V}$ is a regularly closed set and $V \subset \text{int } \overline{V}$. So $f^{-1}(V) \subset f^{-1}(\text{int } \overline{V}) \subset \text{int } f^{-1}(\overline{V})$. (d) $\Rightarrow$ (e) is obvious. (e) $\Rightarrow$ (a). Let $x \in X$ and let V be an open set containing $f(x)$. Then $W = \text{int } \overline{V}$ is a regularly open set containing V and $\overline{W} = \overline{V}$. Thus $f^{-1}(V) \subset f^{-1}(\text{int } \overline{V}) \subset \text{int } f^{-1}(\overline{V})$. Let $U = \text{int } f^{-1}(\overline{V})$. Then $x \in U$ and $f(U) \subset \overline{V}$, and so f is weakly continuous.

Remark 3.3. Equivalence of (a) and (d) in Corollary 3.2 is due to Levine [14]. Again, using 3.2 (b) it is easily shown that for a weakly continuous function f and for any open set $V, \overline{f^{-1}(V)} \subset f^{-1}(\overline{V})$, a result due to Noiri [19].

Theorem 3.4. *Let $f : X \to Y$ be any function. The following statements are true:*

(a) If f is P^*-continuous and $A \subset X$, then $f|A : A \to Y$ is P^*-continuous.

(b) If $\{U_\alpha : \alpha \in \Lambda\}$ is an open cover of X and for each $\alpha \in \Lambda$, $f_\alpha = f|U_\alpha$ is P^*-continuous, then f is P^*-continuous.

Proof. (a) Easy.

(b) Let A be any subset of Y having P complement. Since each f_α is P^*-continuous, by Theorem 3.1 $f_\alpha^{-1}(\text{int } A) \subset \text{int}_{U_\alpha} f_\alpha^{-1}(A)$ and hence

$$f^{-1}(\text{int } A) \cap U_\alpha \subset \text{int}_{U_\alpha}(f^{-1}(A) \cap U_\alpha) = \text{int}_X f^{-1}(A) \cap U_\alpha.$$

Again, since $\{U_\alpha : \alpha \in \Lambda\}$ is a cover of X, by taking union over α we get

$$f^{-1}(\text{int } A) \subset \text{int}_X f^{-1}(A)$$

and so in view of Theorem 3.1 f is P^*-continuous.

Let $f : X \to Y$ be any function. Then the function $g : X \to X \times Y$ defined by $g(x) = (x, f(x))$ is called the *graph function* with respect to f.

Theorem 3.5. *Suppose P is a finitely productive property and let $f : X \to Y$ be a function such that graph function is P^*-continuous. If X possesses property P, then f is P^*-continuous.*

Proof. Let $x \in X$ and let V be a neighbourhood of $f(x)$ having P-complement. Since X possesses property P and since P is finitely productive, $X \times V$ is a neighbourhood of $g(x) = (x, f(x))$ having P-complement. By P^*-continuity of g, there is an open set U containing x such that $g(U) \subset X \times V$. Consequently, $f(U) \subset V$, and so f is P^*-continuous.

Remark 3.6. Using the table Theorem 3.5 includes several results in the literature. For example, with P = compactness it yields Theorem 9 of Long and Hendrix [17] pertaining to c-continuous function, and for P = connectedness it gives [8, Theorem 2.7] pertaining to s-continuous functions.

Theorem 3.7. *If $f : X \to Y$ is continuous and $g : Y \to Z$ is P^*-continuous, then $gof : X \to Z$ is P^*-continuous.*

Proof. Let $x \in X$ and let W be a neighbourhood in Z of $g(f(x))$ having P-complement. By P^*-continuity of g, there is an open set V containing $f(x)$ such that $g(V) \subset W$. Since f is continuous, there is an open set U containing x such that $f(U) \subset V$, and hence $gof(U) = g(f(U)) \subset W$. Thus gof is P^*-continuous.

Remark 3.8. The substitution P =closed set in Theorem 3.7 gives the well known result that the composition of continuous functions in continuous. The above theorem yields [2, Theorem 3], [8, Theorem 2.9], and [16, Theorem 5], respectively, on substituting P = closed compact set, P = closed connected set and P = quasi H-closed set, respectively. Moreover, if P = regularly closed set, then Theorem 3.7 yields corresponding version pertaining to almost continuous functions.

Remark 3.9. In the general the roles of f and g cannot be interchanged in Theorem 3.7 (See [8, Remark 2.2]). However, in particular, it is true that if f is weakly continuous and g is continuous, the gof is weakly continuous.

Theorem 3.10. *Let P denote a property such that every clopen set is a P-set. Then every P^*-continuous image of a connected space is connected.*

Proof. Let $f : X \to Y$ be a P^*-continuous function from a connected space X onto a space Y. Suppose Y is not connected and let $Y = A \cup B$ be a partition of Y. Then A and B are nonempty disjoint clopen sets and hence P-sets whose union is Y. Since f is P^*-continuous, by Theorem 3.1 $f^{-1}(A) \subset \text{int } f^{-1}(A)$ and $f^{-1}(B) \subset \text{int } f^{-1}(B)$. Thus, the nonempty disjoint open setss $f^{-1}(A)$ and $f^{-1}(B)$ constitute a partition of X. This contradiction to the fact that X is connected shows that Y is connected.

Remark 3.11. Reading from the table Theorem 3.10 contains several results in the literature. For example, with P = open set it gives that weakly continuous image of a connected space is connected, a result due to Noiri [19]. Thus, θ-continuous [1] image of a connected space is connected. Similarly, with P = regularly closed set it yields a result due to Long and Carnahan [15].

Corollary 3.12. *If P denotes a property such that every clopen set is a P-set, then every P^*-continuous function is a connected function. In particular, every weakly continuous function is a connected function.*

Proof. This is immediate in view of Theorems 3.4 and 3.10.

Theorem 3.13. *Let $f : X \to \pi X_\alpha$ be a function into a product space such that each X_α possesses property P, and suppose that P is productive. If f is P^*-continuous, then each $P_\alpha of$ is P^*-continuous.*

Proof. Let $x \in X$ and let V_β be a neighbourhood in X_β of $p_\beta(f(x))$ having P-complement. Then $p_\beta^{-1}(V_\beta) = V_\beta \times \left(\underset{\alpha \neq \beta}{\pi} X_\alpha\right)$ is a neighbourhood of $f(x)$ in πX_α. Since each X_α possesses property P, and since P is productive, $p_\beta^{-1}(V_\beta)$ has P-complement. So by P^*-continuity of f, there is an open set U containing x such that $f(U) \subset p_\beta^{-1}(V_\beta)$. Thus $p_\beta o f(U) \subset V_\beta$ and hence $p_\beta o f$ is P^*-continuous.

Remark 3.14. According to the table, Theorem 3.13 represents the unification of various results in the literature. For example, if P = open set, we get a result pertaining to weakly continuous functions; and if P = regularly closed set, the same yields Theorem 3.17 of Singal and Singal [24] on almost continuous functions. Similarly, for P equal to H-closed, the outcome is Theorem 4.7 of Noiri [20] pertaining to H-continuous functions; and for P = closed connected set this yields [9, Theorem 2.3] on s-continuous functions. For P = closed set, this gives the classical result pertaining to continuous functions.

Theorem 3.15. *For each $\alpha \in I$, let $f_\alpha : X_\alpha \to Y_\alpha$ be a function; and let $f : \pi X_\alpha \to \pi Y_\alpha$ be defined by $f((x_\alpha)) = (f_\alpha(x_\alpha))$ for each (x_α) in πX_α. If f is P^*-continuous and if each Y_α possesses property P and P is productive, then each f_α is P^*-continuous.*

Proof. For each $\alpha \in I$, let $p_\alpha : \pi X_\alpha \to Y_\alpha$ be projections. Then by definition of f, we have $q_\alpha o f = f_\alpha o p_\alpha$ for each $\alpha \in I$. Let $x_\alpha \in X_\alpha$ and let V_α be a neighbourhood in Y_α of $f_\alpha(x_\alpha)$ having P-complement. Let $x \in p_\alpha^{-1}(x_\alpha)$ be any point. Since each Y possesses property P and since P is productive, $q_\alpha^{-1}(V_\alpha) = \left(\underset{\beta \neq \alpha}{\pi} Y_\beta\right) \times V_\alpha$ is a neighbourhood in πY_α of P-complement. By P^*- continuity of f, there is an open set U containing x such that $f(U) \subset q_\alpha^{-1}(V_\alpha)$ and so $q_\alpha(f(U)) \subset V_\alpha$. Now, since projections are open maps, $p_\alpha(U)$ is an open set in X_α containing x_α and $f_\alpha(p_\alpha(U)) = f_\alpha o p_\alpha(U) = q_\alpha o f(U) \subset V_\alpha$. Thus f_α is P^*-continuous.

Remark 3.16. Using the table Theorem 3.15 unifies several known results. For example, if P = regularly closed set, then it yields Theorem 1 of Long and Herrington [18] on almost continuous functions, and for P = H-closed, the same gives Theorem 4.3 of Noiri [20] pertaining to H-continuous functions. Similarly, the substitution P = closed connected set yield [9, Theorem 2.2] on s-continuous functions; and if P denotes the property of being a zero set, we get a similar result pertaining to z-continuous functions. The classical result pertaining to continuous functions follows by substituting closed set for P.

Theorem 3.17. *If $f : X \to Y$ is P^*-continuous and if Y possesses a base of neighbourhood having complements which are finite unions of P-sets, then f is continuous.*

Proof. Let $x \in X$ and let V be an open set in Y containing $f(x)$. By hypothesis on Y there is a neighbourhood W of $f(x)$ such that $W \subset V$ and $Y - W = \bigcup_{k=1}^{n} S_k$, where each S_k is a P-set in Y. For $k = 1, \ldots, n$, $Y - S_k$ is a neighbourhood of $f(x)$, and so P^*-continuity of f there is an open set U_k containing x such that $f(U_k) \subset Y - S_k$.

Let $U = \bigcap_{k=1}^{n} U_k$. Then U is an open set containing x and $f(U) \subset \bigcap_{k=1}^{n} f(U_k) \subset \bigcap_{k=1}^{n} (Y - S_k) = W$. Thus f is continuous.

Remark 3.18. Reading from the table, Theorem 3.17 contains several known results in the literature. For example, with P = regularly closed set, it gives a result of Singal and Singal [24] pertaining to almost continuous functions; and for P = closed connected set, it yields [8, Theorem 2.12] corresponding to s-continuous functions. For P = open set we get that a weakly continuous function into a regular space is continuous.

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J.K. Kohli
Department of Mathematics
Hindu College, University of Delhi
Delhi - 110007, India