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FINITE C_n GEOMETRIES: A SURVEY

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1. INTRODUCTION

We follow [5], [56] and [45] for all basic notions conceming diagrams, geometries, chamber systems and coverings, except that we use the word «geometry» in a somewhat stricter sense than in [5] or [56], assuming the residual connectedness in any case, as many people do.

The reader is referred to [25] for a survey of results on geometries belonging to Lie diagrams. This paper in a sense completes and updates [25].

People working in diagram geometry often like the idea that so much of information is carried by diagrams that classification theorems are implicit in them. Of course, this cannot be literally true, in generai. In many cases we need to give some substantiai help to the diagrams under consideration, assuming something such as the finiteness or the flag-transitivity or something else. Having done that, it may happen that a classification theorem is then reachable.

However, certain particular diagrams are so rich of information that we can classify all geometries belonging to them without the aid of any additional hypothesis. Sphericai diagrams appear so often in so many different contexts that it is sensible to believe that the most of information is carried by them.

Actually, this is true for A,, D_n ,E_6 and also for E,, E,, H_3 and H_4 (provided that the finiteness is assumed in the last four cases). Indeed all geomeiries of type A,, D_n or E_6 are buildings and all finite thick geometries of type E, or E, are buildings (see Proposition 6 of [56], Lemma 3.3 of [53] and [4]; see also [25]). Thick buildings of irreducible spherical type and rank $n \geq 3$ have been classified by Tits [56]; thin buildings are Coxeter complexes and non thick buildings of sphericai type can be described by means of constructions involving better known buildings (see [43] and [48]; see also [6], [40], [50] and §§7.12 and 10.13 of [56]). Thus, we are done. Infinite or non thick buildings of type E_7 or E_8 are anyway quotient of buildings, by Theorem 1 of [55]: knowing this is already something, albeit quotients of buildings are not so easy to classify, in general. Not so much is known of infinite geometries of type H_3 or H, , but finite geometries of type H_3 or H_4 are thin, by Feit-Higman Theorem [9], and thin geometries of spherical type are not extremely difficult to classify (see [11]).

On the other hand, the cases of C_n and F_4 look a little wilder. The C_3-subdiagram is the source of our iroubles here. If all C_3-residues of a geometry Γ of type C_n or F_4 are 2-covered by buildings, then Γ itself is 2-covered by a building ([55], Theorem 1). If we assume further that Γ is finite and has thick lines, then Γ is a building (see [4]; see also Lemma 6 of [33] for F_4) and we are done: thick buildings of spherical type and rank $n \geq 3$ are classified in [56], as we have recalled above, and non thick buildings of type C_n or F_4

with thick lines are got from thick buildings of type D_n or D_4 in standard ways (§§7.12 and 10.13 of [56]).

However, if we cannot get control over C_3-residues of Γ, then we have to stop at the very beginning of our job. Thus, it would be nice to have a classification of all finite-building C_3-geometries (likely, a classification of all non-building C_3-geometries, infinite ones included, is hopeless). Then, we might hope to be able to study all finite C_n or F_4 geometries considering all possibiiities for their C_3-residues, improving Theorem 1 of [55] in the case of C_n and F_4. Propositions 2 and 3 will give examples of how this strategy can work.

Unfortunately we are still very far from such a classification. However, we will see that what we presently know on finite C_3-geometries is already enough to obtain strong conclusions on C_n-geometries when $n \geq 4$.

We shall not consider F_4 in this paper. Here is the only strong result that we presently have on F_4 : all finite thick flag-transitive F_4-geometries are buildings (see [25] or [30]). The reader is referred to 93 of [33] fora collection of partial results on F_4.

Let us explicitly state two basic results which we have quoted and used in these introductory notes. We shall again use them a number of times in this paper, sometimes implicitly.

Basic Theorem A. *(Tits [55], Theorem i). Let* Γ *be a geometry belonging to a Coxeter diagram. The universal 2-cover of* Γ *is a building iff all residues of* Γ *of type* C_3 *are 2-covered by buildings. In particular,* Γ *is 2-covered by a building f its diagram does not contain any subdiagram of type* C_3 *or* H_3.

Basic Theorem B. *(Tits [55], Prop. 6; Lemma 3.3 of [53], due to Meixner; Brouwer and Cohen [4]). All geometries of type* A_n, D_n *or* E_6 *are buildings. Finite buildings of type* E_7, E_8, C_n *or* F_4 *with thick lines do not admit any proper quotients.*

1.1. Noiation.

Let us recall some standard notation before going on. The n nodes (types) of a C_n-diagram are usually marked by integers, as follows:

Elements of type 0, 1, 2 or 3 are called ***points, lines, pfanesor solids***, respectively. Elements of type $n-2$ are called ***hyperplanes*** (or ***colines***) and those of type $n-1$ are called ***hyperlines*** (§6 of [55]) or *copoints*. Of course, some of these words are synonymous when $n = 3$ or 4. In these cases the words «line», «plane» or «solid» will be preferred for «hyperplane» (or «coline») or «hyperline» (or «copoint»). We shall freely use phrases such as «the point p lies on the plane u» (or «is on u»), «the line r passes through the point p», «the lines r and s meet in the point p», and so on. The collinearity relation and the collinearity graph are defined as usual. We write $a \perp b$ to mean that two distinct points a, b are collinear. Analogously, two distinct hyperlines are said to be *cocollinear* if there is a hyperplane incident with both of them.

A line is ***thick*** if it is incident with at least three points. Otherwise, it is *thin.* A C_n-geometry is ***ordinary*** if all its lines are thick. Otherwise, it is ***degenerate.***

The symbols $*$ and τ denote the incidence relation and the type function, as in [55]; σ_i is the i-shadow operator, as in [5]. As for the rest, the same notation is used in [5] and [55] and we follow it.

1.2. Characterizations of C_n-buildings

Buildings of type C_n can be charactenzed by means of elementary properties: a C_n-geometry Γ is a building iff the Intersection Property (IP) holds in Γ ([55], 96). We wam that two kinds of Intersection Properties are considered in [55] and [5] and they are not equivalent in generai: property (Int) of [55] is weaker than property (IP) of [5]. Anyway, they are equivalent if only geometries of spherical type are considered. Moreover, properties rather weaker than (IP) are sufficient to characterize C_n-buildings:

(LL) ([55], 96). ***Any two distincts lines meet in at most one point.***

(0) ([55], §6). ***given any two elements*** a, b ***of type*** $i \leq n-2$, ***we have*** $a = b$ ***if*** $\sigma_0(a) = \sigma_0(b)$.

$(LL)_{res}$ ([25], 52.2). ***Property*** (LL) ***holds in*** Γ ***and in the residue of every flag of*** Γ ***of type*** $\{0, 1, \ldots, i\}$, ***for every*** $i = 0, 1, \ldots, n-4$.

We have:

Proposition 1 ***The following are equivalent on a*** C_n***-geometry*** Γ:

(i) The geometry Γ ***is a building.***

(ii) Both (LL) ***and (0) hold in*** Γ.

(iii) Property $(LL)_{res}$ ***holds in*** Γ.

Proposition 1 is essentially contained in Proposition 9 of [55], but the reader can see §2.2 of [25] for an elementary proof of it.

When $n = 3$, properties (LL) and $(LL)_{res}$ say precisely the same and (LL) implies (0). However, when $n \geq 4$, some C_n-geometries exist which satisfy (LL) and nevertheless are very far from buildings. For insiance, (LL) holds in all flat C_n-geometries when $n \geq 4$ (see §1.4 for the definition of flat geometries). Thus, (0) cannot be dropped in (ii) when $n \geq 4$. Anyway, (LL) fails to hold in each of the known examples of ordinary non-building C_n-geometries (see [32]; we warn that no ordinary flat C_n-geometry is known when $n \geq 4$.). Hence, we might ask if (LL) suffices to make a C_n-geometry Γ a building in the presence of some additional hypotheses quite different from (0): assuming that Γ is ordinary, for instance

[37]; or that it is already covered by a building ([49], §5); or both. What happens if (LL) holds in the 0-shadow space of Γ ? (e.g., see §5.3, Statement 2 and Remark 1; or §5 of [49]).

A number of properties similar to (LL) may be considered. Here are some of them.

$(LL)_i$ (where $i = 1, 2, \ldots,$ or $n-1$). *Given a line r and an element w of type* i, *we have $r * w$ if* $|\sigma_0(r) \cap \sigma_0(w)| \geq 2$.

$(LL)_i^*$ (where $i = 0, 1, \ldots,$ or $n-2$). *Given a hyperplane u and an element w of type* i, *we have $u * w$ if* $|\sigma_{n-1}(u) \cap \sigma_{n-1}(w)| \geq 2$.

Of course, $(LL)_i^*$ is the dual of $(LL)_{n-1-i}$. Property $(LL)_{n-1}$ is the same as (LH) of §6 of [55] and $(LL)_1$ is the same as (LL). It is easily seen that (LL) and $(LL)_i$ are equivalent for every $i = 1, 2, \ldots, n-1$. Property $(LL)_0^*$ implies (LL). Moreover, $(LL)_{res}$ holds in a C_n-geometry Γ (that is, Γ is a building) iff $(LL)_i^*$ holds in Γ for every $i = 0, 1, \ldots, n-1$ (we shall again deal with this equivalence in § 1.6).

Properties $(LL)_0^*, (LL)_1^*, \ldots, (LL)_{n-2}^*$ are not equivalent and $(LL)_{n-2}^*$ (dual of (LL)) is the weakest one: indeed it holds in all C_n-geometries. The reader is referred to §2.2 of [26] for further elementary properties of C_n- buildings holding in arbitrary C_n-geometries as well.

Other ways exist to characterize C_n-buildings. For instance, buildings can be characterized by means of properties of galleries (e.g., property (P_c) of [55], or properties (C_c) or (G_c) of §3 of [49]), but we will not insist on this here.

1.3. A few remarks on degenerate C_n-geometries

Degenerate C_n-geometries (§ 1.1) have been studied by a number of authors (Buekenhout and Sprague [6], Rees [40], [38] and [43], §3 of [34], Scharlau [48], Surowski [50], Hillebrandt [11]), but a complete classification seems to be still far from reach. Unlike the case of ordinary C_n-geometries, classifying degenerate C_3-geometries is not the main problem here. For instance, finite degenerate C_3-geometries have been classified by Rees in [40] (see last lines of [40], in particular), but this does not help us so much in classifying all finite degenerate C_n-geometries.

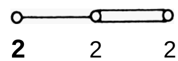
Thus, we will not insist on degenerate C_n-geometries in this paper.

1.4. Flat geometries

A C_n-geometry Γ, where $n \geq 3$, is *flat* if all elements of Γ of type less than $n-2$ are incident with all hyperlines of Γ. We warn that our definition of flat geometries is much more restrictive than that of [49] when $n \geq 4$; in [49] Shult requires only that all points are incident with all hyperlines.

We are not going to list all elementary properties of flat geometries here. A lot of information on this matter can be found in chp. 5 of [42].

Degenerate flat geometries can be produced easily (see [40], §3 of [34] and [50]). On the contrary, ordinary flat C_n-geometries are not so frequent. Finite examples cannot exist if $n \geq 4$ (see [26] or [34]). But some non finite examples of rank 3 exist associated with ordered fields (92.2 (ii) of [39]). However, just one finite ordinary example is presently known. namely the A, -geometry. This is the only flat C_3-geometry with uniform parameter 2



([42], Lemma 5.4). Perhaps, it is the only ordinary finite flat C_3-geometry.

Several different ways exist to produce the A, -geometry: see [39], [21], [25] (Example 4 of §2.3) or [16]. The construction given by Rees in [39] by means of maximal exterior sets looked most interesting, although it is not the simplest one (we give the details of it in §5.3). Indeed that construction seemed to be general enough to produce several flat geometries other than the A_7-geometry. Actually, it does so if we are satisfied of non finite examples ([39], §2.2.(ii)). However the A, -geometry is the only finite example that can be got in that way (this follows from Thas [60]).

The A, -geometry is the only finite ordinary non-building C_n-geometry presently known. As it is flat, we might consider the flatness to be the most important pathology that can occur in C_3-geometries. This point of view is implicit in the following propositions.

Proposition 2. *([34]). Let Γ be an ordinary C_n-geometry and let us assume that every C_3-residue of Γ is either a building or flat. Then one of the following holds:*

(i) The univeral 2-cover of Γ is a building.

(ii) The geometry Γ is flat.

Here is a sketch of the proof. Residues of points are either quotients of buildings or flat, by the inductive hypothesis. If Γ is not flat, then, given any point a having flat residue Γ_a, we find a point b non collinear with a and such that the residue Γ_b of b is a quotient of a building. Next, we can construct a 2-covering from Γ_b to Γ_a exploiting the flatness of Γ_a and the fact that $b \not\perp a$ (the reader is referred to [34] for details). Thus, all residues of points of Γ are 2-covered by buildings. The conclusion follows from Basic Theorem **A.**

A number of non finite examples exist satisfying the hypotheses of Proposition 2 which are proper quotients of buildings (see [32]). But, if we assume the finiteness, then (i) and (ii) can be substituted with a much stronger conclusion. Indeed, by Basic Theorem B we obtain:

Proposition 3. *([26]). Let Γ be as in Proposition 2 and let us assume that $n \geq 4$ and that Γ is finite. Then Γ is a building.*

Thus, it would be nice to succeed in proving at least the following:

Conjecture 1. *All finite ordinary non-building C_3-geometries are flat.*

If this were true, then finite ordinary C_n-geometries would **be** buildings when $n \geq 4$, by Proposition 3. We should still classify finite ordinary flat C_3-geometries and the finite non degenerate case would **be** done.

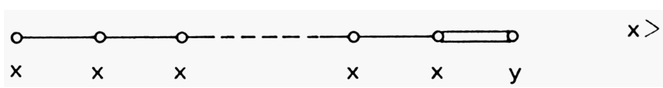
We shail see later that a statement on C_3-geometries weaker ihan conjecture 1 would suffice to obtain the same conclusion in the case of $n \geq 4$ (see conjecture 2 of §3.2 and proposition 3.bis of 93.4).

Of course, we might conjecture even that the A, -geometry is the only non-building finite ordinary C_3-geometry. Theorems **5** and 6 will give some evidence of this.

Anyway, conjecture **1** gives us a motivation for the following definition: a C_3-geometry is *anomalous* if it is neither a building nor flat. Several examples of anomalous C_3-geometries are known ([40] and [32]), but each of them is either degenerate of infinite, of course.

1.5. Parameters

The notion of parameters of a geometry (i-orders in [5]) is a well known one. Anyway, the reader can find a definition of it in [30] or [26]. Ordinary C_n-geometries admit parameters x, y:



The letters x, y will aiways denote parameters, as above. A finite ordinary C_n-geometry Γ is said to have parameters of ***known** type* if one of the following holds on x and y:

(1) $x = y$ (uniform parameter)

(2) $y = 1$ (non thick case)

(3) $y = x^2$

(4) $x = y^2$

(5) $y^2 = x^3$

(6) $x^2 = y^3$

(7) $x = y - 2$ and $x \geq 3$

(8) $y = x - 2$ and $y \geq 3$

(6)–(8): We will see later that cases (6), (7) and (8) are impossible when $n \geq 3$.

These are actually all relations occurring between parameters of known examples of finite generaiized quadrangles with thick lines (see [36] and [35]).

The geometry Γ has parameters of *classical type* if one of (1)-(6) holds ($n = 2$ in case (6)) and x is a prime power.

The geometry Γ is *locally classical* if **all** projective planes and generaiized quadrangles occurring as rank 2 residues of Γ **are** classical.

Needless to say that all projective planes occurring as rank 2 residues in a finite ordinary C_n-geometry are classical if $n \geq 4$. Thus, the parameter x is a prime power in that case. A finite ordinary C_n-geometry Γ is said to have parameters of *semi-classical* type if both x and y are powers of the same prime number (of course, $y = 1 = p^0$ is allowed).

1.6. The Ott-Liebler number

The Ott-Liebler number α of a C_n-geometry Γ is defined inductively as follows [59]. If $n = 2$, then $\alpha = 0$, by definition. Let $n \geq 3$ and let Γ be a C_n-geometry. Given a point-hyperline flag (a,u) of Γ, let $\bar{\alpha}(a,u)$ be the number of hyperlines v different from u, cocollinear with u, incident with a and such that the hyperplane w incident with both u and v (uniquely determined by $(LL)^*_{n-2}$ of §1.2) is not incident with a. Let $\alpha(a)$ be the Ott-Liebler number of the residue Γ_a of a (already defined by the inductive hypothesis). Then $(\bar{\alpha}(a,u)+1)(\alpha(a)+1)$ does not depend on the choice of the flag (a,u). In particular, $\bar{\alpha}(a,u)$ does not depend on the choice of the hyperline u in Γ_a. The reader may find a proof of this claim in [59]. We write $\alpha+1$ instead of $(\bar{\alpha}(a,u)+1)(\alpha(a)+1)$ and $\bar{\alpha}(a)$ instead of $\bar{\alpha}(a,u)$, for short. The constant α is the *Ott-Liebler number* of Γ. The numbers $\alpha(a)$ and $\bar{\alpha}(a)$ are respectively the *inner* and *outer local Ott-Liebler numbers of* Γ *at* a. Of course, we have $\bar{\alpha}(a) = \alpha$ and $\alpha(a) = 0$ if $n = 3$.

Proposition 4. *We have* $\alpha = 0$ *iff* Γ *is a building.*

Indeed we have $\alpha = 0$ iff $(LL)^*_i$ of §1.2 holds in Γ for every $i = 0,1,\dots,n-3$. Property $(LL)^*_i$ holds for every $i = 0,1,\dots,n-3$ iff $(LL)_{res}$ holds. Then $\alpha = 0$ iff Γ is a building, by Proposition 1.

Some interesting relations exist between Ott-Liebler numbers and orders of groups of deck transformations. Let $\varphi : \Gamma_1 \to \Gamma_2$ be a 2-covering and let $A \leq Aut(\Gamma_1)$ be the group of deck transformations of φ (see [55]), assuming that $\Gamma_2 \cong \Gamma_1/A$. Let α_1, α_2 be the Ott-Liebler numbers of Γ_1 and Γ_2, respectively. Then we have [59]:

$$\alpha_2 + 1 = (\alpha_1 + 1) \cdot |A|.$$

Moreover, given a point b of Γ_1, let $\alpha_1(b), \bar{\alpha}_1(b)$ be the inner and outer local Ott-Liebler numbers of Γ_1 at b and let $\alpha_2(a), \bar{\alpha}_2(a)$ have a similar meaning in Γ_2 with respect to $a = \varphi(b)$. Then we have: $\alpha_2(a)+1 = |A_b|\cdot(\alpha_1(b)+1)$ and $\bar{\alpha}_2(a)+1 = (\bar{\alpha}_1(b)+1)\cdot[A : A_b]$ where A_b is the stabilizer of b in A. In particular, if Γ_1 is a building (so that $\alpha_1 = 0$), the previous relations become as follows:

$$\alpha_2 + 1 = |A|, \quad \alpha_2(a) + 1 = |A_b| \quad \text{and} \quad \bar{\alpha}_2(a) + 1 = [A : A_b].$$

From this it is clear that $\alpha(a)$ may depend on the choice of a when $n \geq 4$, so that the same happens to $E(a)$. The reader is referred to [32] and [59] for some examples were this phenomenon occurs. Of course, they are non finite examples.

All previous claims are easy consequences of the following statement [59]:

Let (a,v) be a non incident point-hyperline pair in Γ. Then there are exactly $\overline{\alpha}(a) + 1$ hyperlines w incident with a and cocollinear with v.

Let us briefly explain how that statement implies the previous claims on $|A|, |A_b|$ and $[A : A_b]$. Let $\varphi : \Gamma_1 \rightarrow \Gamma_2, \alpha_1, \alpha_2$ and A be as above. Let (a,u) be a non incident point-hyperline pair in Γ_2 and let $(\overline{a}, \overline{u})$ E $\varphi^{-1}(a,u)$. Let X be the orbit of $\overline{a}$ under the action of A. Then each b E $X - \{E\}$ contributes $\overline{\alpha}_1(b) + 1$ configurations to the computation of $\overline{\alpha}_2(a,u) = \overline{\alpha}_2(a)$.

Of course, we have $\overline{\alpha}_1(b) = \overline{\alpha}_1(E)$ for every b E X. Hence

$$\overline{\alpha}_1(\overline{a}) + ([A : A_a] - 1) \cdot (\overline{\alpha}_1(\overline{a}) + 1) = \overline{\alpha}_1(a),$$

that is:

$$[A : A_a] \cdot (\overline{\alpha}_1(\overline{a}) + 1) = \overline{\alpha}_2(a) + 1.$$

The rest easily follows from this.

We note that, when $n = 3$, the Ott-Liebler number a of Γ equals the number of closed galleries of type 012012012 based at a given chamber of Γ. We don't know any nice way to generalize this to the case of $n \geq 4$. Yet, the constant a first appeared in [24] precisely in this way, as the number of closed galleries as above, while the definition that we have given has been inspired by Liebler [16]. However, the constant a arises in both [24] and [16] in the context of a representation-theoretic approach to finite C_3-geometries. Thus, neither Ott nor Liebler could fully realize how general this concept was and their proofs of the constancy of α heavily depended on techniques from representation theory, so that they were valid only for finite C_3-geometries admitting parameters.

For the rest of this paragraph we assume that Γ is a finite C_3-geometry admitting parameters x, y. Then we have $a \leq x^2 y$ and:

Proposition 5. *We have $a = x^2 y$ iff Γ is flat.*

The reader may see [27] for the (easy) proof. We will see later that the upper bound $x^2 y \geq \alpha$ can be improved as follows: we have $a \geq m^2 y$ where $m = \min(x, y)$ (Proposition 9 of §3). Hence, Γ cannot be flat if $x > y$ (but this statement can be proved easily in an elementary way; see Lemma 5.10 of [42]).

Let n_0, n_1, n_2 be the number of points, lines and planes of Γ, respectively. By easy computations we get:

(1) $(a+1)n_0 = (x^2 y + 1)(x^2 + x + 1)$

(2) $(\alpha+1)n_1 = (x^2y+1)(xy+1)(x^2+x+1)$

(3) $(\alpha+1)n_2 = (x^2y+1)(xy+1)(y+1)$

(see [27]). The following proposition is an easy consequence of (1) and (3):

Proposition 6. *The number* $\alpha+1$ *divides* $(1+x^2y)d$ *where*

$$d = g.c.d.(x^2+x+1, (xy+1)(y+1))$$

The Ott-Liebler number will be exploited in the context of representation theory (see §3). But it also has nice elementary applications. We give one of them here.

Given two distinct collinear points a, b of a finite C_3-geometry Γ, let $n(a,b)$ be the number of lines of Γ through a and b. A point a of Γ is homogeneous if $n(a,b) = n(a,c)$ for every choice of the points b, c collinear with a and distinct from a.

Proposition 7. *([28], Theorem 2). A finite ordinary C_3-geometry is either a building or flat if it admits some homogeneous point.*

The proof consists of a series of computations involving the Ott-Liebler number. The reader is referred to [28] for details. Proposition 7 will play a relevant role in the inquiry into flag-transitive finite C_3-geometries (§5).

1.7. Statements of the theorems

We denote the parameters and the Ott-Liebler number of Γ by x, y and α, respectively, as in §§1.5 and 1.6. In Theorem 1 we consider ordinary non thick C_n-geometries ($y = 1 < x$). We examine them separately because they can be classified fairly easily, exploiting the assumption that $y = 1$ and the correspondence between non thick polar spaces of rank n and D_n buildings ([56], chp. 7). Thick C_n-geometries are much harder to study. The remaining theorems deal with them.

Theorem 1. *(Rees [41]). Let Γ be an ordinary non thick C_n-geometry. Then one of the following holds.*

(i) The geometry Γ is a building.

(ii) The geometry Γ is infinite and it is the quotient of a building $\overline{\Gamma}$ over an involutory automorphism of $\overline{\Gamma}$ induced by a diagram automorphism of the D_n-building associated with $\overline{\Gamma}$. We have $\alpha = 1$.

A sketch of the proof of this theorem will be given in §2.

Theorem 2. *Let Γ be a finite ordinary C_n-geometry admitting parameters x, y of known type. Then either Γ is a building or we have $n = 3$ and one of the following holds:*

(i) We have $x = y$ and Γ is flat.

(ii) We have $x^3 = y^2$, the geometry Γ is flat and the C_2-residues of Γ cannot be isomorphic with any of the known generalized quadrangles.

(iii) We have $x = y^2$ and $\alpha = y^3$. The geometry Γ is anomalous.

Theorem 2 is contained in [27], [26] and [31]. We will give a sketch of the proof in 993.3 and 3.4. Here are two consequences of Theorem 2.

Theorem 3. *Let Γ be a locally classical finite ordinary C_n-geometry. Then either Γ is a building or $n = 3$ and one of the following holds:*

(i) We have $x = y$, the geometry Γ is flat and $\Gamma_a \cong Q_4(x)$ for every point a of Γ (the points of $Q_4(x)$ are lines of Γ through a).

(ii) We have $x = y^2$, $\alpha = y^3$ (hence, Γ is anomalous) and $\Gamma_a \cong H_3(y^2)$ for every point a of Γ.

Theorem 4. *A finite ordinary C_n-geometry or rank $n \geq 4$ is a building if it admits parameters of semi-classical type.*

Theorem 3 is a tnvial corollary of Theorem 2 (see 93.3). On the contrary, the proof of Theorem 4 is not so tnvial. We will give it in 94. Here we make some comments. If Γ is a finite C_3-geometry admitting parameters x, y of semi-classical type, then it is easily seen that xy divides α (see 94). However we cannot say so much more when $n = 3$. Things were different in Theorem 2. Indeed a relation between x and y was assumed in Theorem 2, so that we had only 2 unknowns there, namely α and one of x or y. On the contrary, in the semi-classicai case we really have 3 unknowns. However, things become easier when $n \geq 4$, as we will see in §4.

The foliowing theorem is contained in [29], [30] and [20]. We will sketch its proof in §§5.3, 5.4 and 5.5.

Theorem 5. *Let Γ be a finite ordinary C_n-geometry and let $Aut(\Gamma)$ be flag-transitive. Then Γ is a building or it is the A_7-geometry or we have $n = 3$ and Γ is anomalous.*

If Γ is anomalous, then all the following properties hold:

(a) Residues of planes of Γ are non desarguesian flag-transitive projective planes.

(b) The number x is even, $1 + x + x^2$ is prime, $x \equiv 2 \pmod{3}$ and $x > 10^3$.

(c) Let $d = g.c.d.(x^2, y)$. We have $x > d^2$ and $x^2 - x > y \geq (x-1)d^2 + d$. Moreover, xd divides α, $(1+\alpha)xd$ divides $x^2y - \alpha$, $(1+\alpha)(x+y)x$ divides $(1+xy)(x^2y-\alpha)$ and $(1+\alpha)(x^2+y)x$ divides $(1+x^2y)(x^4y-\alpha)$.

(d) If $Aut(\Gamma)$ acts primitively on the set of points of Γ, then y is odd.

Needless to say that conditions (a)-(d) look quite strange. We remark also that something more can be said in (b): x cannot be a prime power (i.e., a power of 2) if $x \leq 3006$ (see [8], page 209, footnote 2). Trivially, conditions (b) and (c) are impossible to satisfy if x, y are of known type or of semi-classical type. Therefore:

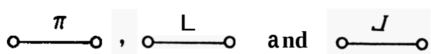
Theorem 6. *(Main Theorem). Let Γ be a finite ordinary C_n-geometry admitting parameters of known type or of semi-classical type and let $Aut(\Gamma)$ be flag-transitive. Then Γ is either a building or the A_7-geometry.*

The celebrated theorem of Aschbacher [1] is included in Theorem 6.

2. NON THICK ORDINARY C_n-GEOMETRIES

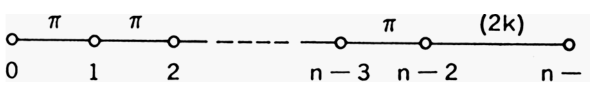
We give a sketch of the proof of Theorem 1. This proof (due to Rees [41]) needs some lemmas, which are actually fairly stronger than we might believe from [41], as it has been pointed out by Rinauro [45]. We follow the more general exposition of [45], generalizing it a bit further.

We recall that the next pictures



denote the class of partial planes, the class of linear spaces and the class of dual linear spaces, respectively [5]. A gallery $\gamma = (C_0, C_1, \ldots, C_m)$ of a chamber system $\mathscr{C}$ is *non stammering* if $C_{i-1} \neq C_i$ for every $i = 1, \ldots, m$. Given a non stammering gallery $\gamma = (C_0, C_1, \ldots, C_m)$ of a chamber system $\mathscr{C}$, the type of γ is the mapping τ_γ from $\{1, \ldots, m\}$ to the set of types of $\mathscr{C}$ defined as follows: $\tau_\gamma(i) = j$ iff C_{i-1} and C_i are j-adjacent $(i = 1, \ldots, m)$. Given a type j of the chamber system $\mathscr{C}$, the *j-section* of γ is the inverse image $\tau_\gamma^{-1}(j)$ of j under τ_γ and $|\tau_\gamma^{-1}(j)|$ is the *j-length* of γ.

Lemma 1. *Let Γ be a 2-simply connected geometry belonging to the following diagram:*



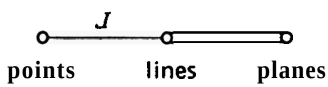
where $0, 1, \ldots, n-1$ are types and $k \geq 2$. Let us assume that every element of Γ of type $n-2$ is incident with exactly two elements of type $n-1$. Then every non stammering closed gallery of the chamber system $\mathscr{C}(\Gamma)$ of Γ has even $(n-1)$-length.

This lemma has been proved by Rees [41] in the particular case of C_3, but the argument by Rees can be easily generalized so that to obtain the previous lemma.

Let Γ be as in the hypotheses of Lemma 1. That lemma says that we can share the elements of Γ of type $n-1$ in two disjoint classes so that, if u, v are distinct elements of type $n-1$ in the same class, then we have $\sigma_{n-2}(u) \cap \sigma_{n-2}(v) = \phi$. This suggest that Γ might be obtained as 0-linearization ([25], page 317) from a geometry belonging to a diagram as in the following picture:

However, to prove this, we should be able to prove that, given elements u, v, w, a of Γ where w has type $n-2$, a has type less than $n-1$ and $\{u, v\} = \sigma_{n-1}(w)$, we have $a * w$ iff a is incident with both u and v. Of course, if we know that the Intersection Property (IP) holds in Γ, then we are done. Let us consider (LL), first.

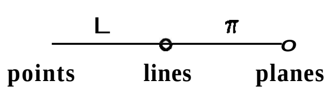
Lemma 2. ***Let*** Γ ***be a simply connected geometry belonging to the following diagram:***



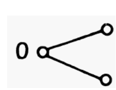
and let us assume that every line of Γ ***is incident with precisely 2 planes. Then any two distinct lines of*** Γ ***meet in at most one point.***

Indeed, otherwise we can construct a closed gallery in Γ involving three planes, and we contradict Lemma 1.

It is easily seen that (IP) holds in a geometry Γ belonging to the following diagram



if (LL) holds in Γ. Thus, (IP) holds in every simply connected ordinary non thick C_3-geometry, by Lemma 2. Then every such geometry is obtained as 0-linearization from an A, -geometry



by the previous remarks. However A, -geometries are projective geometries (Basic Theorem B). Hence, we have:

Lemma 3. ***Simply connected ordinary non thick*** C_3***-geometries are Klein quadrics.***

Of course, we could also get Lemma 3 directly from Lemma 2 and Proposition 1. But we have preferred the way above in order to show how atypicai the non thick case is.

We can prove Theorem 1 now. By Lemma 3 and Basic Theorem A, every ordinary non thick C_n-geometry Γ is 2-covered by a building. Let $\varphi : \overline{\Gamma} \to \Gamma$ be the universai 2-covering of Γ and let A be the group of deck transformations of Γ. As D_n-buildings do not admit proper quotients (Basic Theorem B), every non identical element of A acts as an involutory diagram automorphism on the D_n-building associated with $\overline{\Gamma}$ ([56], 57.10). Then A has order 2. Theorem 1 follows from this and from Basic Theorem B.

3. PARAMETERS OF KNOWN TYPE

Most of what we will say in this section will depend on representation theory. Classical matrix theoretic techniques need some regularity assumptions on the adjacency graph of Γ in order to work, or on some other graph related to Γ (see [9], [4] or §§1.2.2, 1.4, 1.9 or 1.10 of [36]). Representation theory is rather more generai.

Two different approaches exists to the algebraic representation of C_n-geometries: the one of [24], which is an application of the very generai theory by Ott [22] and [23]; and the one by Liebler [19], which developes previous work by Hoefsmit [13] on representations of groups with BN-pair of classical Lie type.

Hoefsmit used ideas and results taken from papers by Carter, Curtis, Iwahori, Kilmoyer, Steinberg, Tits and other ones. He developed those ideas to a very far reaching point and gave effective procedures to explicitly find all irreducible components of the induced representation l_B^G of a finite Chevalley group G admitting a BN-pair (B,N) of type A_n, D_n or C_n. The algebra affording l_B^G is the *Hecke* ***algebra*** $H(G,E?)$ **of** G with respect to the Borel subgroup B of G. The algebra $H(G,B)$ is presented by a nice set of relations (see (1) of 53.1) and Hoefsmit fully exploited also this fact, of course. Finally, he computed the multiplicities of the irreducible representations of $H(G,B)$ in almost all cases. An inquiry into F_4 in this style has been done by Surowski [51] shortly afterwards.

Hoefsmit focused onto groups rather than onto geometries. That is, his work immediately fits for any building of classical Lie type but not for any possible geometry of that type. The job to adapt that work for geometries of Lie type (in particular, for C_n-geometries) has been done by Liebler in [19]. Unfortunately, a gap occurred in one part of [19] and the author decided not to publish anything of [19], though it still remained a good and useful paper. We will use a number of things from [19].

In [22] and [23] a different approach to this matter is developed, which could in principle be applied to any chamber system, even far from buildings of Lie type. However, because of its very generaiity, this approach cannot immediately give us effective procedures to compute everything in every case. Ott applied that machinery to finite C_3-geometries with uniform parameter in [24]. Rees and Scharlau [44] continued that work, considering finite

C_3-geometries with parameters of classical type and could settle the cases listed as (3), (5) and (6) in §1.5. Unfortunately, they did not publish their work. They could not get any satisfactory result in case (4) of §1.5 and, perhaps, they thought that this was a fault in their work (actually, it was not really such: see §1.7, Theorem 2.(iii) and Theorem 3.(ii)).

We shall pragmatically mix the previous two approaches together so that to get the most of profit from each of them.

3.1. Hecke algebras of geometries

Let Γ be a (residually connected) geometry over a finite set of types $I = \{0, 1, \ldots, n-1\}$, admitting finite parameters $x_0, x_1, \ldots, x_{n-1}$ and let $\mathscr{C}(\Gamma)$ be the set of chambers of Γ. We can define a vector space V_Γ over the complex field taking $\mathscr{C}(\Gamma)$ as a basis of V_Γ. Let $\mathscr{L}(V_\Gamma)$ be the algebra of all linear mappings of V_Γ. For every type $i = 0, 1, \ldots, n-1$, let σ_i be the linear mapping of V_Γ acting on $\mathscr{C}(\Gamma)$ as follows:

$$\sigma_i(C) = \sum_{X \underset{i}{\sim} C} X$$

(where $\underset{i}{\sim}$ means i-adjacency of chambers).

Let $H(\Gamma) = \langle \sigma_i | i = 0, 1, \ldots, n-1 \rangle$ be the subalgebra of $\mathscr{L}(V_\Gamma)$ generated by $\{\sigma_i | i = 0, 1, \ldots, n-1\}$. The algebra $H(\Gamma)$ is the *Hecke algebra* of Γ ([22] and [23]). It is semisimple if Γ is finite (see [23]).

Denoted by I the identity mapping, we define $\pi^n(\sigma_i, \sigma_j)$ inductively as follows:

$$\begin{cases} \pi^0(\sigma_i, \sigma_j) = I \\ \pi^{n+1}(\sigma_i, \sigma_j) = \sigma_i \cdot \pi^n(\sigma_j, \sigma_i) \end{cases}$$

Let us assume further that Γ belongs to a Coxeter diagram $\mathscr{C} = (m_{ij} | i, j = 0, 1, \ldots, n-1)$. Then the generators σ_i of $H(\Gamma)$ satisfy the following relations:

$$(1) \qquad \begin{cases} \sigma_i^2 = (x_i - 1)\sigma_i + x_i I & (i = 0, 1, \ldots, n-1) \\ \pi^{m_{ij}}(\sigma_i, \sigma_j) = \pi^{m_{ij}}(\sigma_j, \sigma_i) & (i, j = 0, 1, \ldots, n-1; i \neq j) \end{cases}$$

Let us write $\mathscr{X} = (x_0, x_1, \ldots, x_{n-1})$, for short, and let $H_{\mathscr{C}\mathscr{X}}$ be the algebra over the complex field presented by the set of relations (1). If Γ is a finite building defined by a BN-pair (B, N) of a group G, then $H_{\mathscr{C}\mathscr{X}} \cong H(\Gamma)$. Indeed, in this case $H(\Gamma)$ is the

Hecke algebra $H(G,B)$ of G with respcct to B and the relation $H(G,B) \cong H_{\mathscr{C},\mathscr{X}}$ is well known.

In gcncral, $H(\Gamma)$ is a homomorphic image of $H_{\mathscr{C},\mathscr{X}}$ and ali irreducible representations of $H(\Gamma)$ of finite dcgree appear among the irreducible represenıations of $H_{\mathscr{C},\mathscr{X}}$ of finite dcgree. We can also study relations between the multiplicities that such a representation has whcn it is viewed as a possible component of two distinct Hecke algebras $H(\Gamma_1)$ and $H(\Gamma_2)$ of two finite geometries Γ_1, Γ_2 relative to the same pair $(\mathscr{C}, \mathscr{X})$ ([19], §2). Translating these things into gcncral and effective computing procedures is not easy at ail.

However, other tricks can be found to compute multiplicities of irreducible representations of $H(\Gamma)$ in special cases and C_3 is one of those lucky cases.

Remark. Since this paper is a continuation of [25], we must warm the reader that the second relation of (1) is stated in a wrong way in [25], as $(\sigma_i, \sigma_j)^{m_{ij}} = I$, which holds in Coxeter groups, not in Hecke algebras.

3.2. Hecke algebras of C_3-geometries

Hocfsmit [13] gives us methods to compute all irreducible representations of $H_{\mathscr{C},\mathscr{X}}$ when $\mathscr{C} = C_n$, A, or D_n. We already know that all geometries of *type* A, or D_n are buildings (Basic Theorem B). As for $C_n (n \geq 4)$, if we get control over C_3-residues, then we are done. Thus, we shall consider only the case of C_3 : henceforth Γ will be a finite C_3-geometry admitting parameters x, y. Hence, we have $\mathscr{X} = (x, x, y)$ and

$$\mathscr{C} = \begin{bmatrix} 1 & 3 & 2 \\ 3 & 1 & 4 \\ 2 & 4 & 1 \end{bmatrix}$$

Hocfsmit [12] has provcd that there are exactly 10 pairwise inequivalent irreducible representations of $H_{\mathscr{C},\mathscr{X}}$. Each of them is associated to a double partition of the set of types $\{0, 1, 2\}$ and here are all double partitions to be considcrcd:

$$((0,1,2);\phi),\quad ((0,1),(2);\phi),\quad ((0),(1),(2);\phi),\quad ((0,1);(2)),$$

$$((0),(1);(2)),\quad ((0);(1),(2)),\quad ((0);(1,2)),\quad (\phi;(0),(1),(2)),$$

$$(\phi;(0,1),(2)),\quad (\phi;(0,1,2))$$

We dcnote thcm by the following shortened symbols: $3/0$, $2 \cdot 1/0$, $1^3/0$, $2/1$, $1^2/1$, $1/1^2$, $1/2$, $0/1^3$, $0/2 \cdot 1$ and $0/3$, respectively.

Hocfsmit attaches a representation to each of these double partitions constructing representative matrices for σ_0, σ_1 and σ_2 with the aid of certain sequences of Young diagrams related to the double partition that he is considering (the reader is referred to chps. 1 and 2 of [13] for details) and proves that those are actually all irreducible representations of $H_{\mathscr{C}\mathfrak{X}}$. We now list all of them. Henceforth $A_k(t)$ will denote the following matrix:

$$\frac{1}{x^k t + 1} \cdot \begin{bmatrix} x-1 & x^{k+1}t+1 \\ x^k t + x & x^k t(x-1) \end{bmatrix}$$

Here are the representations.

1) (Index representation). $3/0$. Degree 1.

$$\sigma_0 \to x, \sigma_1 \to x, \sigma_2 \to y.$$

2) $2 \cdot 1/0$. Degree 2.

$$\sigma_0 \to A_2(-1), \sigma_1 \to \begin{bmatrix} x & 0 \\ 0 & -1 \end{bmatrix}, \sigma_2 \to \begin{bmatrix} y & 0 \\ 0 & y \end{bmatrix}$$

3) $1^3/0$. Degree 1.

$$\sigma_0 \to -1, \sigma_1 \to -1, \sigma_2 \to y.$$

4) (Reflection representation). $2/1$. Degree 3. This representation appears also in [24] in a seemingly different form (but we have equivalent representations, of course).

$$\sigma_0 \to \begin{bmatrix} A_2(y) & 0_{2,1} \\ 0_{1,2} & x \end{bmatrix}, \sigma_1 \to \begin{bmatrix} x & 0_{1,2} \\ 0_{2,1} & A_1(y) \end{bmatrix}, \sigma_2 \to \begin{bmatrix} y & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & -1 \end{bmatrix},$$

where $0_{r,s}$ is the null r-by-s matrix $(r, s = 1, 2)$.

5) $1^2/1$. Degree 3.

$$\sigma_0 \to \begin{bmatrix} A_1(y) & 0_{2,1} \\ 0_{1,2} & -1 \end{bmatrix}, \sigma_1 \to \begin{bmatrix} -1 & 0_{1,2} \\ 0_{2,1} & A_2(y) \end{bmatrix}, \sigma_2 \to \begin{bmatrix} y & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

6) $1/1^2$. Degree 3.

$$\sigma_0 \to \begin{bmatrix} -1(y) & 0_{1,2} \\ 0_{2,1} & A_2(y) \end{bmatrix}, \sigma_1 \to \begin{bmatrix} A_1(y) & 0_{2,1} \\ 0_{1,2} & -1 \end{bmatrix}, \sigma_2 \to \begin{bmatrix} y & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

7) 1/2. Degree 3.

$$\sigma_0 \rightarrow \begin{vmatrix} x & 0_{1,2} \\ 0_{2,1} & A_0(y) \end{vmatrix}, \sigma_1 \rightarrow \begin{vmatrix} A_1(y) & 0_{2,1} \\ 0_{1,2} & x \end{vmatrix}, \sigma_2 \rightarrow \begin{vmatrix} y & 0 & \\ & & \\ 0 & 0 & -1 \end{vmatrix}$$

8) $0/1^3$. Dcgree 1.

$$\sigma_0 \rightarrow -1, \sigma_1 \rightarrow -1, \sigma_2 \rightarrow -1.$$

9) 0/2 . 1. Dcgree 2.

$$\sigma_0 \rightarrow A_2(-1), \sigma_1 \rightarrow \begin{bmatrix} x & 0 \\ 0 & -1 \end{bmatrix}, \sigma_2 \rightarrow \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}.$$

10) 0/3. Dcgree 1.

$$\sigma_0 \rightarrow x, \sigma_1 \rightarrow x, \sigma_2 \rightarrow -1.$$

Let us come to thc multiplicities. We switch to [24], now. Given an irreducible representation φ of $H_{\mathscr{C},\chi}$, let χ_φ be the character afforded by φ and let m_φ be the multiplicity of φ when φ is viewed as a component of $H(\Gamma)$. Of course, $m_\varphi = 0$ is allowed. We also recall that $H(\Gamma)$ consists of lincar mappings of V_Γ. Let χ_0 be the (character of) the index reprcscntation. For every elcment w of thc Coxeter group W of type C_3, we define

$$\lambda(w) = \sigma_{i_n}\sigma_{i_{n-1}} \ldots \sigma_{i_1}$$

whcrc

$$w = r_{i_1} r_{i_2} \ldots r_{i_n}$$

is a reprcscntation of w as a reduccd word of W with respect to a given system (r_0, r_1, r_2) of gcncrators of W. The elcment $\lambda(w)$ of $H(\Gamma)$ does not depend on the representation choscn for w, once whcn (r_0, r_1, r_2) is givcn. Let us define $[\chi_\varphi, \chi_\varphi]$ as follows:

(2) $$[\chi_\varphi, \chi_\varphi] = \sum_{w \in W} \frac{\chi_\varphi(\lambda(w))\chi_\varphi(\lambda(w^{-1}))}{\chi_0(\lambda(w))}$$

(scc [24], (13)). Let w_0 bc the non trivial clcmcnt of the center of W, let γ be the number of chambcrs of Γ (i.e., thc dimcnsion of V_Γ) and let α be the Ott-Liebler number of Γ (§1.6), so that we havc

(3) $$\gamma = \frac{(1+x+x^2)(1+x^2y)(1+xy)(1+x)(1+y)}{1+\alpha}$$

(see (1). (2) and (3) of §1.6). Then we have:

$$m_\varphi[\chi_\varphi,\chi_\varphi] = \gamma\left[\chi_\varphi(\lambda(1)) + \frac{\alpha\chi_\varphi(\lambda(w_0))}{x^6y^3}\right] \tag{4}$$

This relation has been proved by **Ott ([24], (26))** in the particular case of $x = y$, but that proof can be extended to the general case. Relation **(4)** already appears in [13] with $\alpha = 0$ (corresponding to the case of buildings). Rees and Schariau have extensively used **(4)** in [44]. They did not know ihe work by Hoefsmit, and they made all computations only **for** the reflection representation (by the way, Ott **did** the same in [24]). Thus, they had troubles with the case of $x = y^2$, where the reflection representation does not help us so much. Actually, we know all possible irreducible representations of $H(\Gamma)$: we have just to look for them among the **10** irreducible representations of $H_{\mathcal{C}\chi}$. We can compute $[\chi_\varphi,\chi_\varphi]$ and m_φ using (2), **(4)** and **(3).** Computing $[\chi_\varphi,\chi_\varphi]$ is a very tiresome job, but it can be done. As far as we know, Liebler [58] has been the first one to apply the method described here so that to get the list of the multiplicities of the irreducible representations of $H(\Gamma)$. Here is that list:

$3/0$	$m_\varphi = 1$
$2\cdot 1/0$	$m_\varphi = \dfrac{(1+x^2y)(1+x)(x^3+\alpha)}{x(x+y)(1+\alpha)}$
$1^3/0$	$m_\varphi = \dfrac{(1+xy)(1+x^2y)(x^6+\alpha)}{(x^2+y)(x+y)(1+\alpha)}$
$2/1$	$m_\varphi = \dfrac{(1+x+x^2)(1+xy)(x^2y-\alpha)}{x(x+y)(1+\alpha)}$
$1^2/1$	$m_\varphi = \dfrac{(1+x^2y)(1+x+x^2)(x^4y-\alpha)}{x(x^2+y)(1+\alpha)}$
$1/1^2$	$m_\varphi = \dfrac{(1+xy)(1+x+x^2)(x^4y^2+\alpha)}{x(x+y)(1+\alpha)}$
$1/2$	$m_\varphi = \dfrac{(1+x^2y)(1+x+x^2)(x^2y^2+\alpha)}{x(x^2+y)(1+\alpha)}$
$0/1^3$	$m_\varphi = \dfrac{x^6y^3-\alpha}{\alpha+1}$
$0/2\cdot 1$	$m_\varphi = \dfrac{(1+x)(1+x^2y)(x^3y^3-\alpha)}{x(x+y)(1+\alpha)}$
$0/3$	$m_\varphi = \dfrac{(1+xy)(1+x^2y)(y^3-\alpha)}{(x+y)(x^2+y)(1+\alpha)}$

Multiplicities must **be** non-negative integers. ***So,*** by the fifth or the seventh of these relations we have:

Proposition 8. xd ***divides*** a***, where*** $d = g.c.d.(x^2, y)$.

By the fourth of the tenth relation we have:

Proposition 9. $a \leq m^2 y$, ***where*** $m = \min(x, y)$.

Finally:

Proposition 10. ***We have*** $H(\Gamma) \cong H_{\mathscr{C}, \mathfrak{X}}$ ***iff*** $\alpha < m^2 y$, ***where*** $m = \min(x, y)$.

(Of course, $\cong$ here means isomorphism of abstract algebra). Indeed, if $a < m^2 y$, we have $m_\varphi > 0$ in any case, so that none of the simple factors of $H_{\mathscr{C}, \mathfrak{X}}$ is lost when we pass to $H(\Gamma)$. The following conjecture now looks quite sensible:

Conjecture 2. ***Let*** $x > 1$. ***The geometry*** Γ ***is a building iff*** $H(\Gamma) \cong H_{\mathscr{C}, \mathfrak{X}}$. ***That is, either*** $\alpha = 0$ ***(***Γ ***is a building) or*** $a = m^2 y$ ***(where*** $m = \min(x, y)$***).***

We remark that the statement of conjecture **2** is true for finite ordinary C_3-geometries with parameters of known type (Theorem 2). But it may **be** false if $x = 1$ is allowed. Indeed every anomalous C_3-geometry with $x = 1 < y$ is a counterexample to that statement (and a lot of such geometries exist: see [40]).

3.3. Finite ordinary C_3-geometries admitting parameters of known type

Most of what we say in this paragraph rests on the fact that the multiplicities m_φ **(93.2)** must be non-negative integers.

Let Γ be a finite ordinary C_3-geometry admitting parameters **of** known type. Case **(1)** of § 1.5 has already been settled by Theorem 1. In cases **(2)**, **(3)**, (5) and **(6) of** § 1.5 very easy and short computations, exploiting the divisibility conditions stated in Proposition 6 and **8,** show that Γ is either a building ($a = 0$) or fat ($a = x^2 y$). The reader may find details in **$4** of [27]. We also remark that a divisibility condition even weaker than that stated in Proposition 8 would **be** sufficient here: xd' divides a (where $d' = g.c.d.(x, y)$), **as** we can see by the relation for the multiplicity **of** the reflaction representation 2/1. On the other hand, we have $x \leq y$ in flat geometries ($1.6, remarks following Proposition 5). Then Γ is a building in case (6). However thick C_3-buildings have been classified by Tits in [55], and none of them has parameters **as** in case (6). Hence case (6) is impossible. We remark that this conclusion could also **be** got directly, exploiting the formula for m_φ in the case of $1^3/0$.

If Γ is fat, then the set of lines through two distinct points a, b **of** Γ is an ovoid in the residue Γ_a of a. Indeed there are exactly $xy + 1$ lines through a and b and no two of them are coplanar. Generalized quadrangles of order (x, x^2) have no ovoids ([36], 1.8.3). Hence Γ is a building in case (3).

Cases (7) and (8) are shown to be impossible by elementary computations: in each of them some of the multiplicities m_φ cannot be a non-negative integer ([27], §4).

In case (4), $a = 0$ and $a = y^3$ are the only surviving possibilities ([27], §4).

Furthermore, $H_4(x)$ has no ovoids ([36], 3.4.1(iii)). Then, if Γ is flat in case (5), we have $\Gamma_a \not\cong H_4(x)$ for every point a of Γ. However $H_4(z)$ is the only known generalized quadrangle of order (t^2, t^3), where $t^2 = x$ (see [36] and [35]). Hence Γ_a cannot be of any of the known types if Γ is flat in case (5).

The part of Theorem 2 conceming the rank 3 case is proved. As for (i) of Theorem 3, we recall that $W(x)$ has noovoids if x is odd ([35], 3.4.1(i); note that $Q_4(x)$ has a lot of ovoids if x is even). We are done.

Case (2) has been the first to be settled (by Ott [24], but using an argument fairly different from the one sketched here). Next, cases (3), (5) and (6) have been solved by Rees and Scharlau [44]. The rest appeared in [27].

Remark 1. The geometry Γ does not admit any homogeneous point in (iii) of Theorem 2, by Proposition 7. Other strange properties of Γ can be discovered in this case, but they are not yet sufficient to give us any contradiction.

Remark 2. If Γ is as in (i) of Theorem 2, then the planes and the lines of Γ form a linear space $L(\Gamma)$ with $(x^2+1)(x+1)$ points (planes of Γ), $(x^2+1)(x^2+x+1)$ lines and parameters (x, x^2+x) (see [57]), as if $L(\Gamma)$ were a 3-dimensional projective space of order x.

We remark that $L(\Gamma)$ is a 3-dimensional projective space iff Γ is obtained from a maximal set of points exterior to a Klein quadric as in [39] (see also §5.3). The «if» part of this claim is trivial. Let us prove the «only if» part. Let $L(\Gamma)$ be the system of points and lines of $PG(3, x)$ (hence x is a prime power). Then the planes of Γ form one of the two families of planes of $Q_5^+(x)$ and the lines of Γ are the points of $Q_5^+(x)$. The set of lines in Γ_a, where a is any point of Γ, is the set of lines of a generalized quadrangle Γ_a^* (dual of Γ_a), embedded in $PG(3, x)$. The generalized quadrangle Γ_a^* is classicai ([36], chp. 4), hence it is of type $W(x)$ ((i) of Theorem 3). It is well known that the set of lines of a generalized quadrangle of type $W(x)$ embedded in $PG(3, x)$ is the set of points of a hyperplane section $H \cap Q_5^+(x)$ of $Q_5^+(x)$ by a hyperplane H of the projective geometry $PG(5, x)$ in which $Q_5^+(x)$ is naturally embedded. Given a point a of Γ, let H_a be the hyperplane of $PG(5, x)$ defining Γ_a^* as a hyperplane section of $Q_5^+(z)$ and let $f(a)$ be the pole of H_a with respect to the quadratic form defining $Q_5^+(x)$ in $PG(5, x)$. If $a \neq b$, then $H_a \cap H_b \cap Q_5^+(x)$ consists of the x^2+1 lines of Γ through the points a, b and is an ovoid both in $H_a \cap Q_5^+(x)$ and in $H_b \cap Q_5^+(x)$. Now it is not so difficult to check that $X = \{f(a) \mid a \text{ point of } \Gamma\}$ is a maximal exterior set with respect to $Q_5^+(X)$ and that Γ is isomorphic with the geometry obtained from X as in [39].

Thus, if we succeeded to force $L(\Gamma)$ to be a projective space, we would have proved that the A_7-geometry is the only surviving possibility in (i) of Theorem 2 **(See § 1.4).**

What about the other «possible» flat case (namely, (ii) of Theorem 2)? Does some contradiction arisc from the existence of many ovoids in residues of points?

Remark 3. By Theorem 2 and Basic Theorem B we immediately obtain that all ordinary finite C_n-geometries with parameters of known type are simply connected. However it would be nice to find a direct proof of this fact.

3.4. The case of rank $n \geq 4$.

Let Γ bc a finite ordinary C_n-geometry admitting parameters x, y of known type and let $n \geq 4$. By what we have already seen in 93.3 and by Proposition 3, we immediately obtain that Γ is a building in ali cases but whcn $x = y^2$, whcre $\alpha = y^3$ might hold in some C_3-residues of Γ. Anyway, thc following lcmma is proved in [31]:

Lemma 4. *Let Γ be a finite ordinary C_4-geometry admitting parameters x, y where $x > y$ and let us assume that, for every point a of Γ, we have either $\alpha(a) = 0$ or $\alpha(a) = y^3$ (where $\alpha(a)$ is the inner local Ott-Liebler number of Γ at a). Then Γ is a building (hence $\alpha(a) = 0$ in any case).*

The proof consists of a long series of computations involving inner and outer local Ott-Licblcr numbers and a non trivial rcsult by Liebler [19] is used, concerning Hecke algebras of finite C_4-geometrics. The rcadcr is rcferred to [31] for deiails.

What rcmaincs to prove of Theorem **2** easily follows from Lemma **4** and Basic Theorems A and B.

Wc remark that thc following improvement of Proposition **3** immediately follows from Lemma **4:**

Proposition 3 bis. *Let Γ be a finite ordinary C_n-geometry where $n \geq 4$ and let us assume that the statement of conjecture 2 of $3.2 holds in all C_3-residues of Γ. Then Γ is a building.*

4. PARAMETERS OF SEMI-CLASSICAL TYPE

In this section Γ will bc a C_n-geomctry admitting parameters of semi-classical type $x = p^h$ and $y = p^k$ (p prime, $x > 1$).

By [36] **(1.2.2)** we have that, if $h < k$, thcn $k = (1+\lambda)\mu$ and $h = \lambda\mu$ for suitable positive integers λ and μ. If $h > k$, we obtain $h = (1+\lambda)\mu$ and $k = \lambda\mu$ (λ, μ as above, but now $\lambda = 0$ is allowed).

Let $n = 3$. Then $y = g.c.d.(x^2, y)$. Hence zy divides a, by Proposition 8. But we cannot go so much further exploiting the machinery of §3.2.

Let $n = 4$. Let us fix a flag $F = (a, r, u)$ of type $(0, 1, 3)$ in Γ. Given a point $b \in \sigma_0(r) - \{a\}$, let v and w be a solid and a plane, respectively, both incident with b and such that $u * w * v * r$ and $r \notin \sigma_1(w)$. Then u and v are non concurrent lines of the generalized quadrangle Γ_{ar}, residue of the flag (a, r). Let $b' \neq b$ be another point in $\sigma_0(r) - \{a\}$ and let v', w' be chosen in the residue of b' similarly as v, w in the residue of b. We have $v \neq v'$. Indeed, if $v = v'$, then $w = w'$ by $(LL)_2^*$ of §1.2 (we have $n - 2 = 2$), hence $r * w(= w')$ in Γ_u and this contradicts our choice of w. Then we have:

$$\sum_{b \in X} \alpha(b) \leq xy^2$$

where $\alpha(b)$ is the inner local Ott-Liebler number of Γ at b and $X = \sigma_0(r) - \{a\}$. Indeed xy^2 is the number of lines of Γ_{ar} (i.e., solids of Γ through r) that are not concurrent with u in Γ_{ar}. Let $\alpha^* = \min(\alpha(b) | b \in X)$. From the above we have $x\alpha^* \leq xy^2$. Then $\alpha^* \leq y^2$ and either $\alpha^* = 0$ or $x \leq y$, because xy divides $\alpha(b)$ for every point b of Γ (see the beginning of this paragraph).

If $\alpha^* = 0$, then Γ has parameters of classical type, because some of the C_3-residues of Γ is a building (Proposition 4). Hence, Γ is a building by Theorem 2.

Let us assume that $\alpha^* > 0$. Then $x \leq y$. We can also assume $x < y$ (otherwise Γ is a building by Theorem 2). Then we have $k = (1+\lambda)\mu$ and $h = \lambda\mu$ for suitable positive integers λ, μ. If $\lambda = 1$, then $y = x^2$ and Γ is a building by Theorem 2. Let us assume $\lambda > 1$. We have $\alpha^* = \overline{\alpha} p^{(2\lambda+1)\mu}$ for some positive integer $\overline{\alpha}$, because zy divides α^*. Let us set $t = p^\mu$, so that $x^2y + 1 = t^{3\lambda+1} + 1$, $x^2 + x + 1 = t^{2\lambda} + t^\lambda + 1$ and $(xy + 1)(y + 1) = t^{3\lambda+2} + t^{2\lambda+1} + t^{\lambda+1} + 1$. It is easily seen that the g.c.d. of $t^{2\lambda} + t^\lambda + 1$ and $t^{3\lambda+2} + t^{2\lambda+1} + t^{\lambda+1} + 1$ divides $t^2 - t + 1$. Then $\overline{\alpha} t^{2\lambda+1} + 1 (= \alpha^* + 1)$ divides $(t^2 - t + 1)(t^{3\lambda+1} + 1)$, by Proposition 6. Hence, $\overline{\alpha} t^{2\lambda+1} + 1$ divides $(t^2 - t + 1)(t^\lambda - \overline{\alpha})$. As $\alpha^* \leq y^2$ and $\lambda > 1$, we have $\overline{\alpha} < t^\lambda$. Therefore $\overline{\alpha} t^{2\lambda+1} + 1 < t^{\lambda+2}$. Hence $t^{2\lambda+1} < t^{\lambda+2}$. This contradicts our assumptions on λ.

Therefore, Γ is a building.

Theorem 4 easily follows from this and from Basic Theorem A and B.

5. FLAG-TRANSITIVITY

In this section we give a sketch of the proofs of Theorem 5 and 6. We will use results on flag-transitive projective planes, on generalized quadrangles and properties of primitive or 2-transitive permutation groups.

5.1. Flag-transitive projective planes

It has been conjectured that a finite projective piane with a flag-transitive collineation group must be desarguesian. A weaker version of this conjecture has been proved by Kantor [14], using the classification of primitive groups of odd degree. More precisely, we have:

Theorem 7. ***(Kantor [14], Theorem A). Let π be a finite projective plane of order x and let G be a flag-transitive collineation group of π. Then one of the following holds:***

(i) The plane π is desarguesian and G contains $PSL(3,x)$ in its natural action on π.

(ii) G is a Frobenius group of order $(x+1)(x^2+x+1)$, x is even and x^2+x+1 is prime.

We remark that π may be desarguesian in (ii) only if $x = 2$ or 8 ([8], 4.4.16). In this case we have $G = \mathrm{Frob}(21)$ or $G = \mathrm{Frob}(73.9)$, respectively, and these possibilities actuaily occur in $PG(2,2)$ and $PG(2,8)$.

Let us give a short sketch of the proof by Kantor for Theorem 7.

The following properties of collineation groups of a finite projective plane π of order x are known.

(1) ([8], 4.1.9). Let o be an involutorial collineation of π. Then either a is a central collineation or il pointwise fixes a Baer subplane of π.

(2) ([8], 4.4.10). If H is a point-transitive collineation group of π containing a non-trivial central collineation, then π is desarguesian and H contains $PSL(3,x)$ in its naturai action on π.

(3) ([8], 2.3.7a). Flag-transitive collineation groups of π are point-primitive and line-primitive.

Moreover, by §§4.4.11-4.4.20 of [8] we have:

(4) Let G be a flag-transitive collineation group of π and let us assume that either π is not desarguesian or $G \not\geq PSL(3,x)$. Then one of the following holds:

(a) x^2+x+1 is prime, x is even and G is a Frobenius group containing a sharply Bag-transitive (Frobenius) subgroup F.

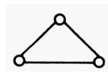
(b) x is a square and either x is even or x is a fourth power.

Eliminating (b) (and forcing $G = F$ in (a)) is the problem solved by Theorem 7. The proof runs as follows.

The group G is point-primitive, by (3). If G has a normal abelian subgroup, then x^2+x+1 is prime and G is a Frobenius group of order $(x+1)(x^2+x+1)$ (Lemma 6.5 of [14]). Thus, we assume that G has no normal abelian subgroups (apart from the trivial one, of course). The group G is primitive of odd degree and primitive groups of odd degree having no normal abelian subgroups are known ([14] or [17]). They have even order. Hence G contains involutions. By (1) and (2) we may assume that each of the involutions of G pointwise fixes a Baer subplane of π. Assuming this, Kantor finds a contradiction for each of the primitive groups of odd degree having no normal abelian subgroups and he proves Theorem 7 in this way.

Something more can be said in (ii) of Theorem 7: by 4.4.4.c of [8] we have $x + 1 \equiv 0$ (mod. 3). Moreover, x cannot be a prime power if $x \leq 3006$, except of course when $x = 2$ or 8, ([8], page 209, footnote 2).

Very deep relations exists between sharply flag-transitive collineation groups of finite projective planes and finite chamber systems $\mathscr{C}$ belonging to the following diagram

$(\tilde{A}_2)$ 

and admitting a group of special automorphisms transitive on the set of chambers of $\mathscr{C}$ (finite flag-transitive $\tilde{A}_2$-chamber systems, for short; flag-transitive triangle geometries in [47]). We are not going to insist on this here. The reader is referred to [47], [53] and [54] (3.3 and 3.4) for details.

5.2. Remarks on flag-transitive generalized quadrangles

All classical generalized quadrangles have flag-transitive automorphism groups. Non classical flag-transitive finite generalized quadrangles also exist. Let H be a piane in $\mathrm{PG}(3,q)$, q even, and let 0 be a hyperoval of H (i.e., a (g + 2)-arc). Let $T_2^*(0)$ be the generalized quadrangie defined by 0 as in 3.1.3 of [36]. The generalized quadrangle $T_2^*(0)$ has parameters $(q-1, q+1)$ and is not classical if $q \neq 2$ (if $q = 2$, then $T_2^*(0)$ is a dual grid). Let G be the stabilizer of 0 in $P\Gamma L(4,$ g) and let K be the pointwise stabilizer of 0 in G, so that $\overline{G}$ = G/K is the action of G on 0. The group K is transitive on the q^2 lines of $T_2^*(0)$ through a given point of 0, and on the points of each of those lines. Therefore, if $\overline{G}$ is transitive on 0, then the group G is flag-transitive in $T_2^*(0)$.

Hyperovals 0 as above, such that $\overline{G}$ is transitive on 0, exist iff $q = 2, 4$ or 16. ([12], page 177). Thus, non classical flag-transitive generalized quadrangles are obtained of order (3, 5) and (15, 17) (or (5, 3) and (17, 15), dually). However none of them can occur as a rank 2 residue in a finite ordinary C_3-geometry (Theorem 2).

In $5.1 we have remarked that Frob(21) < PSL(3, 2) and Frob (73.9) < PSL(3, 8) are the only possible examples of subgroups of $P\Gamma L(3,q)$ acting flag-transitively on $\mathrm{PG}(2,q)$ and non containing $PSL(3,q)$. The analogue of this result is known for classical generalized quadrangles. Here are the only possible examples of groups acting flag-transitively on a thick classical generalized quadrangle S but non containing the classical simple group naturally associated with S ([15], Theorem C.7.1): A, actingon $W(2)(\cong Q_4(2))$, $2^4 \cdot A_5$, $2^4 \cdot S_5$ or $2^4 \cdot Frob(20)$ actingon W(3) and $PSL(3,4) \cdot 2$ or $PSL(3,4) \cdot 2^2$ actingon $H_3(3^2)$.

Non surprisingly, an analogue of Theorem 7 is not yet known for generalized quadrangles. Things are even worse. An analogue of (1) of $5.1 can be obtained for generalized quadrangles using §§2.3 and 2.4 of [36], but the conclusions we get are rather weaker than in (1) of $5.1. Simiiar remarks can be made for (2) of § 1.5 (see chps. 8 and 9 of [36]). Finally, the analogue of

(3) is false for generalized quadrangles. Apart from groups of automorphisms of grids, which never can be line-primitive, the groups $2^4 \cdot A_5$, $2^4 \cdot S_5$ and $2^4 . Frob(20)$ are flag-transitive but point-imprimitive in $Q_4(3)$, $PSL(3,4) .2$ and $PSL(3,4) .2^2$ are flag-transitive but point-imprimitive in $Q_5^-(3)$. Flag-transitive but line-imprimitive groups are also easily recognized in the $T_2^*(0)$ examples described **before,** wheri **q = 4 or 16.**

5.3. The flat case

We begin with the description of the construction of flat C_3-geometries by means of maximal exterior sets given by Rees in [39]. A *maximal exterior set* X with respect to $Q_5^+(q)$ is a set of q^2+q+1 points of $PG(5,q)$ such that each line of $PG(5,q)$ joining two distinct points of X does not meet $Q_5^+(q)$. Given a maximal exterior set X with respect to $Q_5^+(q)$ we can define a flat C_3-geometry $\Gamma(X)$ as follows. X is the set of points of $\Gamma(X)$ and the lines of $\Gamma(X)$ are the points of $Q_5^+(q)$. The set of planes of $\Gamma(X)$ is one of the two families of planes of $Q_5^+(q)$. The incidence relation is defined as follows. Every point of $\Gamma(X)$ is incident with all planes of $\Gamma(X)$ A point p and a line r of $\Gamma(X)$ are incident iff r belongs to the polar plane of p with respect to $Q_5^+(q)$.

The geometry $\Gamma(X)$ is flat of type C_3 ([39], 92).

This construction can be generalized to the infinite case, modulo some minor changes (see [39]). But we *are* interested in the finite case here. When $q = 2$, a maximal exterior set with respect to $Q_5^+(2)$ exists and $\Gamma(X)$ is the A_7-geometry (91.4). However, this is the only possibility in the finite case (Thas [60]).

Let Γ be a flat C_3-geometry with parameters x, y. We can define a partial linear space $\Pi(\Gamma) = (\mathscr{P}, \mathscr{L})$ as follows. $\mathscr{P}$ is the set of lines of Γ and $\mathscr{L}$ is the set of point-plane flags of Γ. A line and a point-plane flag of Γ are said to be incident as elements of $\Pi(\Gamma)$ precisely when they are incident in Γ. In short, $\Pi(\Gamma)$ is the point-line system of the linearization of Γ with respect to the centrai node of the diagram ([25], page 317).

Proposition 11. *(Rees [39], (3.3)). Let Γ be a flat C_3-geometry with uniform parameter x. If $\pi(\Gamma)$ is isomorphic to the system of points and lines of $Q_5^+(q)$, then a maximal exterior set X exists with respect to $Q_5^+(q)$ such that $\Gamma \cong \Gamma(X)$*

Now we are ready to prove the part of Theorem 5 of § 1.7 that concems the flat case.

Theorem 8. *([20]). The A_7-geometry is the only flag-transitive flat finite ordinary C_3-geometry.*

We give a sketch of the proof here. Let Γ be a flat finite ordinary C_3-geometry and let x, y be the parameters of Γ. As Γ is flat and ordinary, we have $1 < x \leq y \leq x^2 - x$, by 1.2.5 of [36] and Theorem 2. If $x = 2$, then $y = x = 2$ and Γ is the A_7-geometry by

Lemma 5.14 of [42]. Let us assume $x > 2$ and let A = $Aut(\Gamma)$ be flag-transitive,by way of contradiction.

Using Theorem 7, a theorem of Burnside on permutation groups of prime degree and the classification of 2-transitive permutation groups and exploiting the inequaiity $x > 2$, we can prove the following:

Statement i. Let K **be** the kemel of the action of A = $Aut(\Gamma)$ on the set S_0 of the points of Γ and $\overline{A} = A/K$ **be** the action of A on S_0. Then x is a prime power and $PSL(3,z) \leq \overline{A} \leq P\Gamma L(3,x)$. Moreover, if A, is the stabilizer in A of a plane u of Γ and $\overline{A}_u = A_u/(K \cap A,)$ is the action of A, on the residue Γ_u of u, then $PSL(3,x) \leq \overline{A}_u$ (that is, case (ii) of Theorem 7 never occurs on $\overline{A}_u$).

By statement 1 we easily obtain the following:

Statement 2. If **two** lines r, s of Γ meet in two distinct points, then $\sigma_0(r) = \sigma_0(s)$. That is, Γ is obtained from PG$(2,x)$ repeating its lines $x^2 + 1$ times, counting the plane PG$(2,x)$ itself $(x^2 + 1)(x + 1)$ times and defining the line-plane incidence in a suitable way.

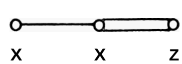
We remark that statement 2 is false in the A, -geometry. Statement 2 essentially depends on the inclusions $\overline{A}_u \geq PSL(3,x) \leq \overline{A} \leq P\Gamma L(3,x)$, which cannot be proved if $x = 2$.

Indeed, if $x = 2$, then we have $\overline{A}$ = A, $> PSL(3,2) = \overline{A}_u$ as further surviving possibility, and this in fact corresponds to the A, -geometry.

Using statements 1 and 2 and exploiting the flag-transitivity again, we can prove the following:

Statement 3. Given a plane **u** and a line $r \notin \sigma_1(u)$, a point $a_{u,r}$ of u is uniquely determined such that the lines incident with u and coplanar with r are precisely the lines of **u** through $a_{u,r}$.

Using statement 3, it is not so difficult to prove that il(Γ) is a rank 3 polar space. The polar space $\Pi(\Gamma)$ is classical by [56] and has parameters x, z as follows



where $z = x^r$ and $r = 0, 1/2, 1, 3/2$ or 2. As the points of $\Pi(\Gamma)$ are the lines of Γ and **as** $y \leq x^2 - x$, we obtain that either $z = 1$ **or** $z^2 = x$, by easy computations. If $x = z^2$ then $y = x^3$ and residues of points of Γ are isomorphic with $H_4(z^2)$. However $H_4(z^2)$ has no ovoids ([36], **3.4.1(iii)**), whereas, given any two distinct points a, b of Γ, the set of lines of Γ through a and b is an ovoid in the residue Γ_a of a. Therefore $z = 1, y = x$ and $\Pi(\Gamma) \cong Q_5^+(x)$.

By Proposition 11, a maximal exterior set exists with respect to $Q_5^+(x)$ such that $\Gamma \cong \Gamma(X)$. Hence $x = 2$ by [60]. The final contradiction is reached and Theorem **8** is proved.

Remark 1. The situation descnbed in statement 2 actually occurs in non finite flat geometries $\Gamma(X)$ where X is a plane of PG$(5,K)$ exterior to $Q_5^+(K)$ (such a plane exists iff K is an ordered field: [39], 2.2ii). It occurs also in all flat C_3-geometries with all thin lines (see [40]).

Are these the only possibiiities? If we were able to obtain statement 3 directly from statement 2, without using groups or finiteness assumptions at ail, then we would be very close to a positive answer. Indeed the reamining part of the proof of Theorem **8** could **be** generaiized in some way. It is worth remarking that properties like that of statement 2 occur in a number of examples of rank $n \geq 4$ (see [32] and [49], $5). Moreover, the A_7-geometry is the only known flat geometry where statement 2 faiis to hold. Thus we might even hope to succeed to prove that statement 2 is a consequence of the flatness whenever $x \neq 2$.

Remark 2. It may be interesting remarking that, if we tried to prove that the system $L(\Gamma)$ of planes and lines of a flat C_3-geometry Γ with uniform parameter x is a projective space (**$3.3,** Remark 2), then we would soon get stuck with statements which, in one form oranother, say the same thing as statement **3.** This is not surprising in view of Remark 2 of 93.3, of Proposition 11 and of the final part of the proof of Theorem **8.**

5.4. The anomalous case

We finish the proof of Theorem 5 of § 1.7. In this paragraph Γ is an anomalous finite ordinary C_3-geometry with parameters x, y and flag-transitive automorphism group $A = Aut(\Gamma)$. S_0, S_1 and S_2 are the sets of points, lines and planes of Γ, respectively. K is the kemel of the action of A on S_0 and $\overline{A} = A/K$ is that action.

For each plane u, iet A_u and N_u be the stabilizer of u in A and the kemel of the action of A_u on the residue Γ_u of u, respectively. Thus, $\overline{A}_u = A_u/N_u$ is the action of A_u on Γ_u. We remark that $A_u \cap K \leq N_u$, but the equality might fail to hold, as far as we know (indeed Γ is not flat). Finally, α is the Ott-Liebler number of Γ.

Proposition **12.** *For every point* u *of* Γ, $\overline{A}_u$ *is a Frobenius group, sharply flag-transitive* ***on*** Γ_u.

That is, (ii) of Theorem 7 occurs for every $u \in S_2$. Indeed, otherwise we have $\overline{A}_u \geq PSL(3,x)$ for every $u \in S_2$, by Theorem 7. Hence $\overline{A}$ is transitive on the set of pairs of distinct collinear points of Γ. Therefore, the number of lines through two distinct collinear points a, b of Γ does not depend on the choice of the collinear pair (a,b). Then Γ is either a building or flat, by Proposition 7. On the other hand, Γ is anomalous by assumption and we have the conuadiction.

Proposition 13. *(Arithmetic conditions). All the following hold:*

(i) x *is even,* $1 + x + x^2$ *is prime and* $x + 1 \equiv 0$ *(mod. 3).*

(ii) $x < y < x^2 - x$.

(iii) $(x + y)(\alpha + 1)$ *divides* $(1 + sy)\left(xy - \frac{\alpha}{x}\right)$ *and* $(x^2 + y)(\alpha + i)$ *divides* $(x^2 y + 1)\left(x^3 y - \frac{\alpha}{x}\right)$. *If* $d = g.c.d.\ (x^2, y)$, *then*

(iii.a) $d^2 < x$.

(iii.b) $(x - 1)d^2 + d \leq y$.

(iii.c) xd *divides* a.

(iii.d) $\alpha + 1$ *divides both* $\frac{xy}{d} - \frac{\alpha}{xd}$ *and* $\frac{Y}{d} + \frac{\alpha^2}{x^2 d}$.

Property (i) follows from Proposition 12 and Theorem 7 (see 4.4.4.c of *[8]* for $x + 1 \equiv 0$ (mod. 3)). Property (ii) follows from Theorem 2, from 1.2.5 of **[36]** and from (iii.b). Property (iii.c) is nothing but Proposition **8.** All remaining properties listed in (iii) arc obtained exploiting (iii.c) and the fact that $1 + x + x^2$ is prime in the divisibility conditions given by the formulas for the multiplicities m_φ of the irreducible representations of thc Hcckc algebra of Γ (53.2); the reader is referred to [29] for details of these computations. Wc warn that thc two divisibility conditions in (iii.d) are equivalent.

It may be that some way exists to reach **a** contradiction taking all previous propcrties together with the divisibility condition of §1.2.2 of **[36]** and with the well known Bruck-Ryser condition on the order of a finite projective plane. One of the authors has tried to do this by a computer some time ago, testing all values of $x \leq 1000$. It tumed out that nonc of them worked. Therefore:

Proposition 14. *We have* $s > 1000$.

By Proposition 12 and 14 and by 4.4.16 of [8], we immediately have thc following:

Corollary. *For every plane* u *of* Γ, Γ_u *is not desarguesian.*

Thus, as we have already observed in § 1.7 (remarks following the statemcnt of Theorcm 5), x cannot be a prime power if $x \leq 3006$. We remark that, by Proposition 12, the previous Corollary amounts to say that $x \neq 2, 8$. That is, $x = 2$ or $x = 8$ do not fit with Proposition *13.* Of course, this can be checked even «by hand», without using computcrs.

The next step is collecting information on involutions. Unfortunately, the information we have is very weak when y is odd.

Proposition 15. *(Involutions). Let* σ *be an involution of* A *and let* Γ_σ *be the set of elements of* Γ *fixed by* **o**. *Then one of the following holds:*

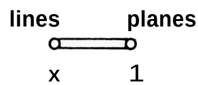
(i) The configuration Γ_σ *consists of exactly one plane* u *and of its residue* Γ_u. *We have* $y \equiv 0$ *(mod. 2) in this case.*

(ii) The configuration Γ_σ consists of a nonempty set of pairwise non coplanar lines, together with all their points. We have $y \equiv 1$ (mod. 2) in this case.

The reader is referred to [29] (Lemmas 6, 7, 9, 10 and 11) for the case of y even. As a by-product of (i), we obtain a similar statement for Sylow 2-subgroups of A in the case of y even ([29], Lemma 9). This provides a very useful geometric interpretation of Sylow 2-subgroups of A in this case.

As for the case of y odd, a second possibility were left open in [29] (Lemmas 6 and 7) besides (ii) above, namely the following one:

(iii) There is point p fixed by σ such that the configuration $(\Gamma_p)_\sigma$ fixed by σ in Γ_p is a grid with parameters x, 1 :



and y is odd.

We rule out this case here.

Let $u_0^+, \ldots, u_x^+, u_0^-, \ldots, u_x^-$ be the two families of planes (lines in Γ_p) of the grid $(\Gamma_p)_\sigma$. Let r be a line through p (point in Γ_p) not belonging to $(\Gamma_p)_\sigma$. For each $i = 0, 1, \ldots, x$, there is a line r_i in u_i^+ through p such that r and r_i are coplanar. The lines $r_0, r_1, \ldots, r_x$ are mutually non coplanar (i.e., they form an $(x+1)$-arc in Γ_p). Let $r' = \sigma(r)$. As r does not belong to (Γ_p), we have $r \neq r'$. The set $(r')^\perp \cap r^\perp$ of the lines through p coplanar with both r' and r contains the lines $r_0, r_1, \ldots, r_x$ and $y - x$ further lines $s_1, \ldots, s_{y-x}$. As $r' = \sigma(r)$, the involution σ fixes $(r')^\perp \cap r^\perp$ and, as it fixes each of the lines $r_0, \ldots, r_x$, it permutes the lines $s_1, \ldots, s_{y-x}$. However y is odd, whereas x is even. Hence σ fixes some of the lines $s_1, \ldots, s_{y-x}$. We have a contradiction, because none of these lines belongs to $(\Gamma_p)_\sigma$. We are done.

The group K is studied in [29] only in the case of y even. It is proved that $|K|$ is odd if $y \equiv 0$ (mod. 2) ([29], Lemma 8). We give a more complete result here.

Proposition 16. *(Properties of K). The group K has odd order and acts as a Frobenius group on each of its orbits on the set of planes of Γ.*

Given planes u, v of Γ, let $K_u = A_u \cap K$ be the stabilizer of u in K and let $K_{uv} = K_u \cap K_v$ be the siabilizer of both u and v in K.

Let u, v belong to the same orbit of K and let $g \in K_{uv}$. As $v \in K(u)$, we have $\sigma_0(u) = \sigma_0(v)$ and a bijection f of $\sigma_1(u)$ onto $\sigma_1(v)$ exists such that $r' = f(r)$ iff $\sigma_0(r) = \sigma_0(r')$ $(r \in \sigma_1(u), r' \in \sigma_1(v))$. Let $r \in \sigma_1(u) - \sigma_1(v)$. For every point a of r a line-plane flag (r_a, u_a) is uniquely determined in Γ_a such that $r * u_a$ and $r_a * v$. It is easily seen that $u_a \neq u_b$ if $a \# b$ $(a, b$ E $\sigma_0(r))$. Moreover, g fixes everything in the residues of u and v, as g E K_{uv}. Hence g fixes r_a for every point a of r and, as it fixes r too, it fixes u_a $(a$ E $\sigma_0(r))$. Therefore there are at least $x+1$ planes through r fixed by g other than u. As g fixes the residue of every plane that it fixes, the configuration elementwise fixed by g in Γ_a (where a is a point of u) is a subquadrangle of order (x, t), where $t \geq x+1$ ([36], §2.4). By §2.2.2 of [36] we obtain $t = y$. Therefore g fixes everything in the residue of a, for every point a of u.

Let now p be a point such that g fixes everything in Γ_p. If b is a point distinct from p and joined with p by two distinct lines s, s', then s and s' are not coplanar and g fixes all planes incident with either s or s'. Therefore g fixes everything in the residue of b, by §2.4 of [36].

Next, let $b \neq p$ be joined with p by precisely one line s. Let w be a plane on s. As $\alpha > 0$, two line-plane flags $(s_p, w_p), (s_b, w_b)$ exist such that $w_p \neq w \neq w_b$, $s_p * w * s_b$, $s_p \notin \sigma_1(p)$, $s_b \notin \sigma_1(b)$, $p * w_p$ and $b * w_b$. It is easily seen that none of w_p or w_b is incident with s. Hence $w_p \neq w_b$, as s is unique line through p and b. Let c be a point incident with both s_p and s_b (we can find c in Γ_w, and we have $c \neq p, b$). Let s'_p, s''_p be the lines through p and c in w_p and w, respectively, and let s'_b, s''_b be those through b and c in w_b and w, respectively. We have $s'_p \neq s''_p$ and $s'_b \neq s''_b$, by $(LL)^*_2$ of §1.2 and because $p \notin \sigma_0(s_p)$ and $b \notin \sigma_0(s_b)$. By the previous argument, g fixes everything in Γ_c because it fixes everything in Γ_p and c is joined with p by two distinct lines. Next, g fixes everything in Γ_b, as it fixes everything in Γ_c and b is joined with c by two lines.

Then g fixes everything in Γ_b as soon as $b \perp p$. Iterating this argument, g fixes all of Γ. $K_{,,} = 1$.

We have proved in this way that K acts as a Frobenius group on each of its orbits on S_2. Let us prove that K has odd order. Let a be an involution of K, by way of contradiction. Let u be a plane such that $a(u) \neq u$.

Let us assume that u and $a(u)$ are not cocollinear. The planes u and $a(u)$ have the same set of points, because σ E K. For every point a in u (and in $a(u)$), there are $x+1$ planes in Γ_a cocollinear with both u and $a(u)$. As x is even, a fixes at least one of those planes. On the other hand, there are x^2+x+1 points in u and, using $(LL)^*_1$ of $1.2, it is easily seen that, if a, b are distinct points of u and u_a, u_b are planes in Γ_a and Γ_b, respectively, cocollinear with both u and $\sigma(u)$, then $u_a \neq u_b$. Hence there are at least x^2+x+1 planes fixed by a. However this contradicts Proposition 15. Therefore, for every plane u, u and $\sigma(u)$ are cocollinear and, if $u \neq a(u)$, then the line incident with both u and $\sigma(u)$ ($1.2, $(LL)^*_1$) is fixed by a. However it is easily seen that this contradicts Proposition 15 if y is even. Hence y is odd and, for every point a, the lines through a fixed by a form an ovoid in Γ_a. Therefore, given any plane u, for every point a of u there is exactly one line of u through a fixed by

σ. On the other hand, at most one of the lines of u is fixed by o (Proposition 15 (ii)). We have reached a contradiction.

Therefore, K has odd order.

Proposition 16 is proved.

Proposition 16 makes it easier to study Sylow 2-subgroups of A: they can be identified with those of $\overline{A}$, and $\overline{A}$ should be easier to study than A itself.

Another prime number has considerable relevance here besides 2, namely $p = 1 + x + x^2$ (Proposition 13, (i)). We have:

Proposition 17. *([29], Lemmas 2 and* **3).** *The Sylow p-subgroups of A have order p and act semi-regularly on the set of points of* Γ .

So far we go without making any extra assumption on A. If we assume the primitivity of $\overline{A}$, then we have the following partial result (which completes the proof of Theorem 5 of 1.7 in the rank 3 case):

Proposition 18. *([29], Theorem 1.C). If $\overline{A}$ is primitive, then* y *is odd.*

We give just a sketch of the proof.

Let $\overline{A}$ be primitive and let L be its socle. The number n_0 of points of Γ is neither prime nor a proper power (see (1) of §1.6 and the arithmetic conditions of Proposition 13). Therefore L is a nonabelian simple group. As $p = 1 + x + x^2$ is pnme (Proposition 13) and $x > 10^3$ (Proposition 14), the order of L is divisible by a prime factor bigger than 10^6. Then L cannot be sporadic (the classification of finite simple groups is used here, of course). By straightforward computations ([29], proof of Lemma 4) we can see that L cannot be an alternating group either. Hence, we have the following:

Statement 1. L is simple of Lie type.

Therefore L contains involutions. Let y be even. By Proposition 15 (i), we have that L has a strongly embedded subgroup ([29], proof of Lemma 12). Hence, using a theorem of Bender ([3], Theorem 4.24), we obtain that L is one of the following groups: $SL(2, 2^n)$ $(n > 2)$, $PSU(3, 2^n)$ $(n \geq 2)$ or ${}^2B_2(2^{2m+1})$ $(m \geq 1)$. Exploiting this information it is possible to prove that p^2 divides $|A|$, where $p = 1 + x + x^2$ (see [29], end of the proof of Theorem 1). But this contradicts Proposition 17. Therefore,

Statement 2. y is odd.

Remarks. When y is odd, Proposition 15(ii) does not give us so much of information and we are in troubles.

As for the imprimitive case, it is easily seen that, if $\overline{A}$ is imprimitive on S_0, then, given an imprimitivity class X for $\overline{A}$ and a piane u of Γ, we have $|X \cap \sigma_0(u)| \leq 1$. Unfortunately remarks like this do not seem to be very deep. Furthermore, imprimitive flag-transitive automorphism groups of finite thick generalized quadrangles exist (see $5.2) even if they are exceptional phenomena. We can guess from this that assuming the imprimitivity on $\overline{A}$ will hardly give us contradictions for free and some work will be needed to reach any of them.

5.5. The case of rank $n \geq 4$.

In this paragraph Γ is a finite ordinary C_n-geometry with $n \geq 4$ and flag-transitive automorphism group $Aut(\Gamma)$.

Theorem 9. *The geometry Γ is a building.*

This theorem completes the proof of Theorem 5 of $1.7. It appeared in [30]. It can be proved in a number of different ways. A very short proof can be given using Proposition 12, the Corollary of Proposition 14, Proposition 7 and Proposition 2, but here we recall the proof given in [30], which is not long either, does not depend on Theorem 7 or on Proposition 13; it uses a celebrated theorem by Seitz (see [15], Theorem C.7.1). As $n \geq 4$, residues of hyperlines of Γ are desarguesian projective geometries of dimension $n-1 \geq 3$ and x is a prime power. By Seitz's theorem ([15], Theorem C.7.1), the stabilizer A, of a flag F of Γ of type $\{0, 1, \ldots, n-4, n-1\}$ acts on Γ_F as a classical group. Hence all C_3-residues of Γ are either buildings or flat, by the same argument used in the proof of Proposition 12 and by Proposition 7. Therefore Γ is a building, by Proposition 2.

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