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ON THE FUNDAMENTAL THEOREM OF OVERLAYS

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Abstract. *We show that the fundamental theorem of overlays of Fox ([1],[2]) is true without the assumption of metrizability of the base space.*

0. INTRODUCTION

The usual definition of covering spaces gives nice results only if the base space is locally connected and semilocally 1-connected. In order to avoid such assumptions Fox suggested to replace covering spaces by what he called overlays. For a metrizable base space, an overlay is nothing else but a fibre bundle (in the sense of Steenrod's book) which has a discrete fibre and allows a coordinate bundle whose coordinate transformations are (constant) elements of the permutation group of the fibre.

Fox established a natural bijection between the set of (isomorphism classes of) overlays over a connected metrizable base space with a given (discrete) fibre and the direct limit of the sets of representations (= conjugacy classes of homomorphisms) from $\Pi_1(U)$ to the group of permutations of the fiber where U runs through the system of neighborhoods of the base space in some ANR into which one embeds it.

We prove a generalization of this result to arbitrary (only connected) base spaces by replacing $\Pi_1(U)$ by $\Pi_1(\nu\mathbf{U})$ where $\mathbf{U}$ runs through the open coverings of the base space and $\nu\mathbf{U}$ is the nerve of $\mathbf{U}$.

It should be noted that for paracompact base spaces an overlay is still essentially a fibre bundle (see remark 3.3) but for a general base space this hasn't been proved yet. Therefore we operate not with fibre bundles but neverthless we give below a formulation for overlays which is in close resemblance to fibre bundle theory.

1. DEFINITION OF PRINCIPAL OVERLAYS

Let X be a topological space, G a discrete group and $\mathbf{U}$ an open covering of X. We define a principal coordinate overlay on $\mathbf{U}$ with group G to be a function assigning to every pair (U,V) with $U \in \mathbf{U}, V \in \mathbf{U}, U \cap V \neq \emptyset$ an element $g_{UV} \in G$ (transition element between U and V) such that

$$g_{UV} \cdot g_{VW} = g_{UW} \qquad \text{if } U \cap V \cap W \neq \emptyset .$$

(It is then clear that $g_{UU} = id_G$ and $g_{VU} = g_{UV}^{-1}$). We denote such a principal coordinate overlay by $g_{\mathbf{U}}$.

We call two principal coordinate overlays $g_{\mathbf{U}}$ and $g'_{\mathbf{U}}$ (on $\mathbf{U}$ with group G) strictly-equivalent on $\mathbf{U}$, if there exist element $\lambda_U \in G$ (for $U \in \mathbf{U}$, $U \neq \emptyset$) such that

$$g'_{UV} = \lambda_U^{-1} \cdot g_{UV} \cdot \lambda_V \qquad \text{for } U, V \in \mathbf{U} \text{ , } U \cap V \neq \emptyset .$$

We denote the set of strict-equivalence classes of principal coordinate overlays on $\mathbf{U}$ with group G by $OV(X, \mathbf{U}; G)$.

A principal coordinate overlay $g_{\mathbf{U}}$ can be restricted to a refinement $\mathbf{V} \geq \mathbf{U}$ if a certain specification of the refinement $\mu : \mathbf{V} \to \mathbf{U}$ ($V \subset \mu(V)$ for $V \in \mathbf{V}$) is given:

For $V_1, V_2 \in \mathbf{V}$, $V_1 \cap V_2 \neq \emptyset$, let us set $h_{V_1 V_2} = g_{U_1 U_2}$, where $U_1 = \mu(V_1)$ and $U_2 = \mu(V_2)$. Then these transition elements constitute a principal coordinate overlay $h_{\mathbf{V}}$ on $\mathbf{V}$. If $\mu' : \mathbf{V} \to \mathbf{U}$ is another specification and $h'_{\mathbf{V}}$ is given by $h'_{V_1 V_2} = g_{U'_1 U'_2}$, where $U'_1 = \mu'(V_1)$ and $U'_2 = \mu'(V_2)$, then $h_{\mathbf{V}}$ and $h'_{\mathbf{V}}$ are strict-equivalent on $\mathbf{V}$: Because, if we take $\lambda_V = g_{UU'}$ (with $V \in \mathbf{V}, U = \mu(V)$, $U' = \mu'(V)$), then we have

$$g_{U'_1 U'_2} = g_{U'_1 U_1} \cdot g_{U_1 U_2} \cdot g_{U_2 U'_2}$$

i.e.

$$h'_{V_1 V_2} = \lambda_{V_1}^{-1} \cdot h_{V_1 V_2} \cdot \lambda_{V_2}$$

Likewise, if $g_{\mathbf{U}}$ and $g'_{\mathbf{U}}$ are strict-equivalent principal coordinate overlays on $\mathbf{U}$, then the restrictions $h_{\mathbf{V}}$ and $h'_{\mathbf{V}}$ to $\mathbf{V}$ (by μ) are also strict-equivalent on $\mathbf{V}$: If we set $\lambda'_V = \lambda_U (V \in \mathbf{V}, U = \mu(V))$, then we have

$$g'_{U_1 U_2} = \lambda_{U_1}^{-1} \cdot g_{U_1 U_2} \cdot \lambda_{U_2} \text{ i.e. } h'_{V_1 V_2} = (\lambda'_{V_1})^{-1} \cdot h_{V_1 V_2} \cdot \lambda'_{V_2} .$$

Now we get a well-defined map

$$p_{\mathbf{UV}} : OV(X, \mathbf{U}; G) \to OV(X, \mathbf{V}; G)$$

by taking any representative and by restricting it with respect to any specification of the refinement $\mathbf{V} \geq \mathbf{U}$. These maps constitute a direct system of sets on the directed set of all open coverings of X.

We call two principal coordinate overlays $g_{\mathbf{U}}$ on $\mathbf{U}$ and $g'_{\mathbf{W}}$ on $\mathbf{W}$ (with group G) equivalent, if there is a common refinement $\mathbf{V}$ of $\mathbf{U}$ and $\mathbf{W}$ ($\mathbf{V} \geq \mathbf{U}, \mathbf{W}$) such that the restrictions of $g_{\mathbf{U}}$ and $g'_{\mathbf{W}}$ to $\mathbf{V}$ are strict-equivalent on $\mathbf{V}$.

We define a principal overlay on X (with group G) to be an equivalence class of principal coordinate overlays. Then a principal overlay can be viewed as an element of the limit of the above direct system. Denoting the set of principal overlays on X with group G by $OV(X;G)$, we can write

$$OV(X;G) = \varinjlim_{\mathbf{U}} OV(X,\mathbf{U};G).$$

Remark. $OV(X;G)$ can easily be made into a (pointed)set-valued cofunctor on the category of topological spaces.

2. THE COVERING GROUP

We now give a construction which associates a group with a covering of a topological space. This group can be seen to be isomorphic to the fundamental group of the nerve of the covering, but the form given here is very suitable for classification of overlays ([3]).

Let X be a connected topological space and $\mathbf{U}$ an open covering of X. We call a finite sequence of sets

$$U_0, U_1, U_2, \ldots, U_{n-1}, U_n \qquad (U_i \in \mathbf{U} \text{ for } i = 0, 1, \ldots, n)$$

satisfying the condition

$$U_i \cap U_{i+1} \neq \emptyset \qquad \text{for } i = 0, 1, \ldots, n-1$$

a chain (in $\mathbf{U}$). We write a chain as $U_0U_1U_2 \ldots U_{n-1}U_n$. We call the chain closed if $U_n = U_0$.

Note that for any two sets from $\mathbf{U}$ there is a chain beginning with the one and ending with the other, because X is connected.

We define two operations which can be performed on a chain and which we call elementary homotopies:

a) Removal of a term V from a portion UVW of a chain, provided that $U \cap V \cap W \neq \emptyset$.

b) Insertion of a term V into a portion UW of a chain, provided that $U \cap V \cap W \neq \emptyset$.

Now fix a set $U_0 (\neq \emptyset)$ from $\mathbf{U}$ and consider the set of all closed chains beginning and ending with U_0. We call two closed chains $U_0U_1U_2 \ldots U_nU_0$ and $U_0U_1'U_2' \ldots U_m'U_0$ homotopic, if they can be converted into each other by a finite number of elementary homotopies. This gives clearly an equivalence relation and we denote the equivalence class of a chain $U_0U_1U_2 \ldots U_nU_0$ by $[U_0U_1U_2 \ldots U_nU_0]$ and the set of equivalence classes by $\Pi_{U_0}(\mathbf{U})$.

We define the following composition of the elements of $\Pi_{U_0}(\mathbf{U})$:

$$[U_0U_1U_2 \ldots U_nU_0] \cdot [U_0U_1'U_2' \ldots U_m'U_0] = [U_0U_1U_2 \ldots U_nU_0U_1'U_2' \ldots U_m'U_0].$$

This is well-defined and makes $\Pi_{U_0}(\mathbf{U})$ into a group. $[U_0U_0]$ is the neutral element and $[U_0U_nU_{n-1}\dots U_2U_1U_0]$ is the inverse of $[U_0U_1U_2\dots U_nU_0]$. The dependence on the base-set U_0 is such as the dependence of the fundamental group on the base-point. If V_0 is another base-set and $U_0W_1W_2\dots W_mV_0$ is a chain from U_0 to V_0, then there is an isomorphism

$$\Pi_{U_0}(\mathbf{U}) \to \Pi_{V_0}(\mathbf{U})$$

given by $[U_0U_1U_2\dots U_nU_0] \mapsto [V_0W_m\dots W_1U_0U_1\dots U_nU_0W_1\dots W_mV_0]$.

A covering groupoid like the fundamental groupoid can also be constructed by defining in the same way the homotopy for chains with common first and last terms (but not necessarily closed) and then taking the composition of two matching chain classes $[U_0W_1\dots W_mV_0]$ and $[V_0Y_1\dots Y_kZ_0]$ as $[U_0W_1\dots W_mV_0Y_1\dots Y_kZ_0]$. We can then write the above isomorphism as

$$[U_0U_1\dots U_nU_0] \mapsto \mathbf{k}^{-1}[U_0U_1\dots U_nU_0]\mathbf{k} \text{ with } \mathbf{k} = [U_0W_1\dots W_mV_0].$$

We call this isomorphism the conjugation by $\mathbf{k}$.

A still slightly more general composition of chain classes can be defined. Given two classes $[U_0W_1\dots W_mV_0]$ and $[V_1Y_1\dots Y_kZ_0]$ with $V_0 \cap V_1 \neq \emptyset$, then the composition

$$[U_0W_1\dots W_mV_0]\cdot[V_1Y_1\dots Y_kZ_0] = [U_0W_1\dots W_mV_0V_1Y_1\dots Y_kZ_0]$$

is well defined. This operation makes some computations easier.

Let $\mathbf{V}$ be a refinement of $\mathbf{U}$ and $\mu : \mathbf{V} \to \mathbf{U}$ a specification of the refinement. Now choosing a base-set $V_0 \in \mathbf{V}$ we can define a homomorphism

$$\mu_\star : \Pi_{V_0}(\mathbf{V}) \to \Pi_{\mu(V_0)}(\mathbf{U})$$

by $[V_0V_1\dots V_nV_0] \mapsto [\mu(V_0)\mu(V_1)\dots\mu(V_n)\mu(V_0)]$.

For another specification $\mu' : \mathbf{V} \to \mathbf{U}$ we have a commutative diagram

$$\begin{array}{ccc} & \overset{\mu_\star}{\nearrow} & \Pi_{\mu(V_0)}(\mathbf{U}) \\ \Pi_{V_0}(\mathbf{V}) & & \downarrow \mathbf{k}_\star \\ & \underset{\mu'_\star}{\searrow} & \Pi_{\mu'(V_0)}(\mathbf{U}) \end{array}$$

where $\mathbf{k}_\star$ is the conjugation by $\mathbf{k} = [\mu(V_0)\mu'(V_0)]$.

Now with the help of a chain from $\mu(V_0)$ to the formerly selected $U_0 \in \mathbf{U}$ we can define a composite homomorphism

$$\Pi_{V_0}(\mathbf{V}) \to \Pi_{\mu(V_0)}(\mathbf{U}) \to \Pi_{U_0}(\mathbf{U})$$

and this is well-defined up to conjugation in $\Pi_{U_0}(\mathbf{U})$.

It is convenient to work in the category $\mathbf{D}$, whose objects are groups and whose morphisms are homomorphisms up to conjugation in the image group. If we fix for every covering of X a base-set, we get an inverse system in this category:

$$\check{\Pi}(X) = \{\Pi_{U_0}(\mathbf{U}),\ \Pi_{V_0}(\mathbf{V}) \to \Pi_{U_0}(\mathbf{U}) \quad \text{for } \mathbf{U} \leq \mathbf{V}\}.$$

(It is practical to fix the base-sets but one could allow the same covering to appear in the inverse system with all possible base-sets by a little change of the indexing set of the inverse system. Conceiving $\check{\Pi}(X)$ as an object of pro-$\mathbf{D}$ (Mardesic and Segal [5]) this construction could be made functorial, but we use below only the representation of this inverse system, which is clearly functorial).

Let $I(X,\mathbf{U};G) = \mathbf{D}(\Pi_{U_0}(\mathbf{U}),G)$ (i.e. $I(X,\mathbf{U};G)$ is the pointed set of all homomorphism $\Pi_{U_0}(\mathbf{U}) \to G$ up to conjugation in G) and

$$\check{I}(X;G) = \varinjlim_{\mathbf{U}} I(X,\mathbf{U};G).$$

$\check{I}(X;G)$ can be seen to be a pointed-set-valued cotravariant functor of X. (For a map $f : X' \to X$ we can take inverse-image covering $f^{-1}(\mathbf{U})$ of a given covering $\mathbf{U}$ of X and then construct a homomorphism $\Pi(f^{-1}(\mathbf{U})) \to \Pi(\mathbf{U})$ as for a refinement within X, yielding a map $\check{I}(X;G) \to \check{I}(X';G)$).

3. THE CLASSIFICATION OF PRINCIPAL OVERLAYS

We give first a classification of strict-equivalence classes of principal coordinate overlays on a covering $\mathbf{U}$ by representation of the covering group $\Pi(\mathbf{U})$ in G. The general classification follows immediately from this case.

Definition 3.1. *Let $g_{\mathbf{U}}$ be a principal coordinate overlay on a covering $\mathbf{U}$ of a connected topological space X and $\Pi_{U_0}(\mathbf{U})$ the covering group of $\mathbf{U}$. We define a function*

$$u : \Pi_{U_0}(\mathbf{U}) \to G$$

by $u([U_0U_1U_2\dots U_{n-1}U_nU_0]) = g_{U_0U_1}\cdot g_{U_1U_2}\cdots g_{U_{n-1}U_n}\cdot g_{U_nU_0}$.

u is well-defined and a homomorphism. We call this homomorphism the characteristic class of the principal coordinate overlay $g_{\mathbf{U}}$.

Two principal coordinate overlays on $\mathbf{U}$, which are strictly-equivalent on $\mathbf{U}$, give rise to conjugate homomorphisms. So, an element in $I(X,\mathbf{U};G)$ can be associated with a strict-equivalence class of principal coordinate overlays on $\mathbf{U}$, giving a function

$$c_{\mathbf{U}} : OV(X,\mathbf{U};G) \to I(X,\mathbf{U};G)$$

Proposition 3.1. $c_{\mathbf{U}} : OV(X,\mathbf{U};G) \to I(X,\mathbf{U};G)$ *is bijective.*

Proof. 1. Injectivity: We show that two principal coordinate overlays $g_{\mathbf{U}}$ and $g'_{\mathbf{U}}$ on $\mathbf{U}$ having conjugate characteristic classes u, u' are strictly-equivalent on $\mathbf{U}$. Let $u' = \lambda^{-1}\cdot u\cdot\lambda$ with $\lambda \in G$. We must determine $\lambda_U \in G$ for $U \in \mathbf{U}$ $(U \neq \emptyset)$ such that

$$g'_{UV} = (\lambda_U)^{-1}\cdot g_{UV}\cdot\lambda_V \quad \text{for} \quad U,V \in \mathbf{U},\ U\cap V \neq \emptyset.$$

Let us choose for every $U \in \mathbf{U}$ a chain from U_0 to U: $U_0U_1^\star U_2^\star \dots U_s^\star U$. (The in-between terms and the s depend certainly on U, but we choose this notation for simplicity). Let us fix $U \in \mathbf{U}$, $V \in \mathbf{U}$, $U \cap V \neq \emptyset$, denote the chosen chain from U_0 to V by $U_0V_1^\star V_2^\star \dots V_t^\star V$ and consider the closed chain $U_0U_1^\star \dots U_s^\star UVV_t^\star \dots V_1^\star U_0$. We can write

$$u([U_0U_1^\star U_2^\star \dots U_s^\star UVV_t^\star \dots V_2^\star V_1^\star U_0]) = g_{U_0U_1^\star}\cdot g_{U_1^\star U_2^\star}\cdots$$
$$g_{U_s^\star U}\cdot g_{UV}\cdot g_{VV_t^\star}\cdots g_{V_1^\star U_0}$$

and

$$u'([U_0U_1^\star \dots U_s^\star UVV_t^\star \dots V_1^\star U_0]) = g'_{U_0U_1^\star}\cdots g'_{U_s^\star U}\cdot g'_{UV}\cdot g'_{VV_t^\star}\cdots g'_{V_1^\star U_0}$$

Let us set $g_U = g_{U_0U_1^\star}\cdot g_{U_1^\star U_2^\star}\cdots g_{U_s^\star U}$ (resp. for g'). We can then write (by conjugation of u and u')

$$g'_U\cdot g'_{UV}\cdot(g'_V)^{-1} = \lambda^{-1}\cdot g_U\cdot g_{UV}\cdot(g_V)^{-1}\cdot\lambda.$$

Let us now set $\lambda_U = (g_U)^{-1}\cdot\lambda\cdot g'_U$. The last equality then reads $g'_{UV} = \lambda_U^{-1}\cdot g_{UV}\cdot\lambda_V$.

2. Surjectivity: If a homomorphism $u : \Pi(\mathbf{U}) \to G$ is given, then a principal coordinate overlay $g_{\mathbf{U}}$ is to be found with characteristic class conjugate to u.

Let, with notations as above, $\mathbf{k}_U = [U_0U_1^\star U_2^\star \dots U_s^\star U]$ and define $g_{UV} = u(\mathbf{k}_U\mathbf{k}_V^{-1})$. It is easily seen that these g_{UV} constitute a principal coordinate overlay $g_{\mathbf{U}}$ on $\mathbf{U}$. Let us compute the characteristic class u_g of $g_{\mathbf{U}}$.

For $\mathbf{k} = [U_0U_1U_2\dots U_nU_0] \in \Pi_{U_0}(\mathbf{U})$ we can also write $\mathbf{k} = [U_0U_0U_1U_1U_2U_2\dots U_nU_nU_0U_0]$, or because of $[U_iU_i] = \mathbf{k}_{U_i}^{-1}\mathbf{k}_{U_i}$,

$$\mathbf{k} = (\mathbf{k}_{U_0}^{-1}\mathbf{k}_{U_0})(\mathbf{k}_{U_1}^{-1}\mathbf{k}_{U_1})(\mathbf{k}_{U_2}^{-1}\mathbf{k}_{U_2})\dots(\mathbf{k}_{U_n}^{-1}\mathbf{k}_{U_n})(\mathbf{k}_{U_0}^{-1}\mathbf{k}_{U_0})$$
$$\mathbf{k} = \mathbf{k}_{U_0}^{-1}(\mathbf{k}_{U_0}\mathbf{k}_{U_1}^{-1})(\mathbf{k}_{U_1}\mathbf{k}_{U_2}^{-1})\dots(\mathbf{k}_{U_n}\mathbf{k}_{U_0}^{-1})\mathbf{k}_{U_0}.$$

Now by applying the given homomorphism u we get

$$u(\mathbf{k}) = u(\mathbf{k}_{U_0}^{-1}) \cdot u(\mathbf{k}_{U_0}\mathbf{k}_{U_1}^{-1}) \cdot u(\mathbf{k}_{U_1}\mathbf{k}_{U_2}^{-1}) \ldots u(\mathbf{k}_{U_n}\mathbf{k}_{U_0}^{-1}) \cdot u(\mathbf{k}_{U_0})$$

$$u(\mathbf{k}) = (u(\mathbf{k}_{U_0}))^{-1} \cdot g_{U_0U_1} \cdot g_{U_1U_2} \cdots g_{U_nU_0} \cdot u(\mathbf{k}_{U_0}).$$

Since $u_g(\mathbf{k}) = g_{U_0U_1} \cdot g_{U_1U_2} \cdots g_{U_nU_0}$, we find that u and u_g are conjugated. (They become even equal if we choose $\mathbf{k}_{U_0} = [U_0U_0]$).

The bijective ccorrespondence $c_{\mathbf{U}} : OV(X,\mathbf{U};G) \to I(X,\mathbf{U};G)$ can be easily seen to be compatible with refinements. That is, for $\mathbf{U} \leq \mathbf{V}$ the following diagram is commutative:

$$\begin{array}{ccc} OV(X,\mathbf{U};G) & \xrightarrow{c_{\mathbf{U}}} & I(X,\mathbf{U};G) \\ \Big\downarrow{\scriptstyle P_{\mathbf{UV}}} & & \Big\downarrow \\ OV(X,\mathbf{V};G) & \xrightarrow{c_{\mathbf{V}}} & I(X,\mathbf{V};G) \end{array}$$

(The unnamed map is given through $\Pi_{V_0}(\mathbf{V}) \to \Pi_{\mu(V_0)}\mathbf{U} \to \Pi_{U_0}(\mathbf{U}) \to G$).

This yields a map $\check{c}$ of direct limits, which is itself bijective: So, we get a classification of principal overlays on X, with the single assumption that X is connected.

Theorem 3.1. *Let X be a connected topological space. Then the map*

$$\check{c} : OV(X;G) = \varinjlim_{\mathbf{U}} OV(X,\mathbf{U};G) \to \varinjlim_{\mathbf{U}} I(X,\mathbf{U};G) = \check{I}(X;G)$$

is bijective.

Remark 3.1. This one-to-one correspondence can be seen to be a natural equivalence if the two sides are viewed as cofunctors X.

Remark 3.2. Since the covering group is isomorphic to the fundamental group $\Pi_1(\nu\mathbf{U})$ of the nerve $\nu\mathbf{U}$ of the covering $\mathbf{U}$, $\check{I}(X;G)$ can be replaced by the more familiar $\varinjlim_{\mathbf{U}} \mathbf{D}(\Pi_1(\nu\mathbf{U}),G)$.

Remark 3.3. It can be shown [4] that, if only numerable coverings of X are allowed, the principal overlays on X (with a discrete group G) and the principal bundles on X (with group G) are in one-to-one (natural) correspondence.

Remark 3.4. The definition of principal overlays by transition elements has of course a couterpart as a space over X with certain properties, which easily can be given.

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