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TETRAHEDRON MANIFOLDS AND SPACE FORMS

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Dedicated to my father

In an earlier paper [5] I have constructed two interesting concave «tetrahedra» representing minimally the compact Euclidean space forms $E^3/\mathbf{P4}_1$ and $E^3/\mathbf{P6}_1$. The method can be generalized in a natural way and we shall construct an infinite series of tetrahedra $\mathcal{P}$ with suitable face identifications (Fig. 1), so that each of them represents a manifold M_{mn} with fundamental group

$$(1) \qquad G_{mn} = (a, b - a^{m-1}b^{-1}a^{-1}b^{-1} = 1 = b^{n-1}a^{-1}b^{-1}a^{-1}).$$

The group G_{mn} is generated by the homeomorphisms a and b, identifying the faces of $\mathcal{P}$ in pairs. The relations in (1) will be in connection with the two equivalence classes of edge segments of $\mathcal{P}$, induced by the face identifications. For the natural parameters m, n the inequalities $3 \leqq n \leqq m$ are required.

In the cases $(m, n) = (4, 4)$ and $(6, 3)$ the fundamental group is isomorphic to the crystallographic group $\mathbf{P4}_1$ and $\mathbf{P6}_1$, respectively, and the homeomorphisms $M_{44} \cong E^3/\mathbf{P4}_1$, $M_{63} \cong E^3/\mathbf{P6}_1$ hold, as it has been shown in [5].

We remark that G_{mn} is known as Threlfall's binary polyhedral group of symbol $\langle m, n, 2\rangle$ if it is of finite order $4\left(\frac{1}{m}+\frac{1}{n}-\frac{1}{2}\right)^{-1}$ in three cases [2, Ch. 6.5]. The manifold $M_{33} = S^3/G_{33}$ will be the spherical octahedron space (Fig. 3). $M_{43} = S^3/G_{43}$ and $M_{53} = S^3/G_{53}$ are also spherical space forms [9, Satz 9], [10, §11 (1930)]. The first is realizable by identifications of a truncated cube (with Schläfli symbol (8,8,3) see also in [6]). The second is nothing but the famous Poincaré dodecahedron space (Fig. 4). These spherical space forms, in another desscription, are known also from the classification of spaces of constant positive curvature [13, Ch. 7].

In Section 3, Th. 1, we shall prove that the other manifolds M_{mn} cannot be space forms. Thus we get a simple infinite series of connected compact manifolds which cannot wear a complete Riemannian structure of constant curvature.

It turns out in Sect. 4, Th. 2, that our manifolds M_{mn} all are Seifert fibre spaces [9]. The last infinite series are covered by $\widetilde{SL_2\mathbb{R}}$ the universal covering of real 2×2 matrices with determinant 1. This is the 6^{th} Thurston's geometry with a locally homogeneous complete Riemannian metric [8], [11]. So, we have found a minimally presenting fundamental domain (in the sense of [5]) for each Seifert fibre space M_{mn}.

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For the sake of completeness I give a tetrahedron series for the spherical lens spaces $L((m-1)(n-1)-1, n-1) \cong S^3/H_{mn}$ constructed in a similar way (Fig. 5). Here the finite cyclic groups H_{mn} are represented by

$$(2) \qquad H_{mn} = (a, b - a^{m-1}b^{-1} = 1 = b^{n-1}a^{-1}), \ 3 \leqq n \leqq m.$$

I thank prof. E. B. Vinberg and the Referee of the former version of this note for calling my attention to the classical works of Threlfall and Seifert [9], [10].

This note also intends to illustrate an algorithmic method (implemented on computer as well) searching for manifolds by fundamental domains [6].

1. THE CONSTRUCTION OF THE MANIFOLD M_{mn}

In Figure 1 we have described the Schlegel diagram of $\mathcal{P} \sim M_{mn}$ to be constructed. This is a simplicial complex. We start with a 3-simplex $\mathcal{P} := A_0A_1A_2A_3$ spanned by the linearly independent vectors A_i $(i = 0, 1, 2, 3)$ of the real 4-space $\mathbb{R}^4$. As usual the inner points of $\mathcal{P}$ are represented by vectors $X = x^0A_0 + x^1A_1 + x^2A_2 + x^3A_3 =: x^iA_i$ where the barycentric coordinates $x^0, x^1, x^2, x^3 > 0$ and $x^0 + x^1 + x^2 + x^3 = 1$. On the edge A_0A_1 we introduce additional vertices $B_s = \frac{r}{m-2}A_0 + \frac{s}{m-2}A_1 (r + s = m-2; 0 \leqq r, s \in \mathbb{Z})$ to get a subdivision into $m-2$ edge segments. On A_2A_3 we analogously introduce the vertices $C_0 := A_2, C_1, \dots, C_{n-3}, C_{n-2} := A_3$ to get $n-2$ segments. Furthermore, we take the midpoints $A_{12}, A_{20}, A_{03}, A_{13}$ as additional vertices of $\mathcal{P}$. Moreover, let us consider also the barycentres S_i of the faces opposite to A_i and the barycentre $S = \frac{1}{4}(A_0 + A_1 + A_2 + A_3)$ of $\mathcal{P}$. Now we define the homeomorphisms a and b by

$$(3) \qquad \begin{aligned} a : A_{20}A_0B_1 \dots B_{m-3}A_1A_{12}A_2(S_3) =: f_{a^{-1}} \longmapsto \\ \longmapsto A_0B_1B_2 \dots A_1A_{13}A_3A_{30}(S_2) =: f_a; \end{aligned}$$

$$(4) \qquad \begin{aligned} b : A_{02}A_2C_1 \dots C_{n-3}A_3A_{30}A_0(S_1) =: f_{b^{-1}} \longmapsto \\ \longmapsto A_2C_1C_2 \dots A_3A_{31}A_1A_{12}(S_0) =: f_b \end{aligned}$$

linearly piece by piece. That means, e.g.

$$(5) \qquad \begin{aligned} a : A_{20}A_0S_3 \longmapsto A_0B_1S_2, \ xA_{20} + yA_0 + zS_3 \longmapsto \\ \longmapsto xA_0 + yB_1 + zS_2 (x, y, z \geqq 0; \ x + y + z = 1). \end{aligned}$$

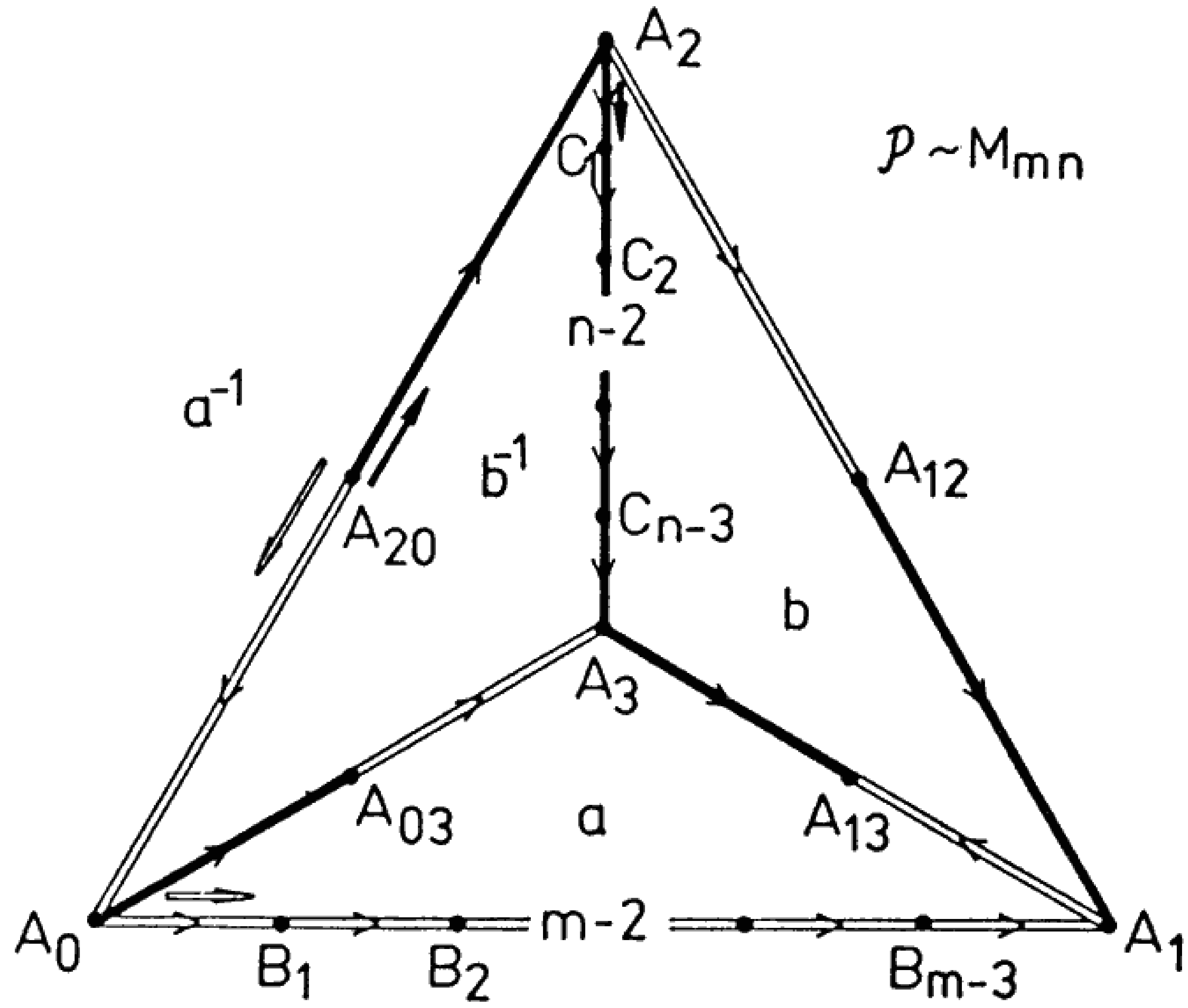

Figure 1

In Fig. 1 we have indicated the two equivalence classes of edge segments by ⤜ and →. The vertices belong to one equivalence class. The faces are denoted by their suffix (without f's). Observe that the orientations, say on the faces $f_{a^{-1}}$ and f_a induced by the homeomorphism a, are opposite with respect to a fixed orientation of the surface of $\mathscr{P}$. The same is true for b. This is why M_{mn} is orientable, and a, b are said to be topological screw motions.

2. THE FUNDAMENTAL GROUP G_{mn} AND THE UNIVERSAL COVERING $\widetilde{M}_{mn}$

We are going to describe the fundamental group of M_{mn} which will be denoted by G_{mn}. At the same time we shall construct one atlas, consisting of two charts for the points of M_{mn}, and the universal covering space of M_{mn} which will be denoted by $\widetilde{M}_{mn}$. The method is standard, we only sketch it in our case [1, 9.8], [3], [6], [7], [10].

Take a directed edge segment $A_{20}A_0$ and the face $f_{a^{-1}}$ containing it on the boundary. These are identified by the map a with A_0B_1 and f_a, respectively. Then we continue with A_0B_1 and $f_{a^{-1}}$ by the following cyclic scheme

$$\text{(6)}\qquad \begin{aligned} &⤜(A_{20}A_0, f_{a^{-1}}) \xmapsto{a} (A_0B_1, f_a); (A_0B_1, f_{a^{-1}}) \xmapsto{a} (B_1B_2, f_a); \ldots \\ &\ldots; (A_{30}A_3, f_a) \xmapsto{a^{-1}} (A_2A_{12}, f_{a^{-1}}); (A_2A_{12}, f_b) \xmapsto{b^{-1}} (A_{20}A_0, f_{b^{-1}}). \end{aligned}$$

This provides the relator $a^{m-1}b^{-1}a^{-1}b^{-1}$ defined to be 1 the neutral element of G_{mn} being described. Similarly, if we start with segment $A_{02}A_2$ from the class → we get the relation $b^{n-1}a^{-1}b^{-1}b^{-1} = 1$ and the presentation (1).

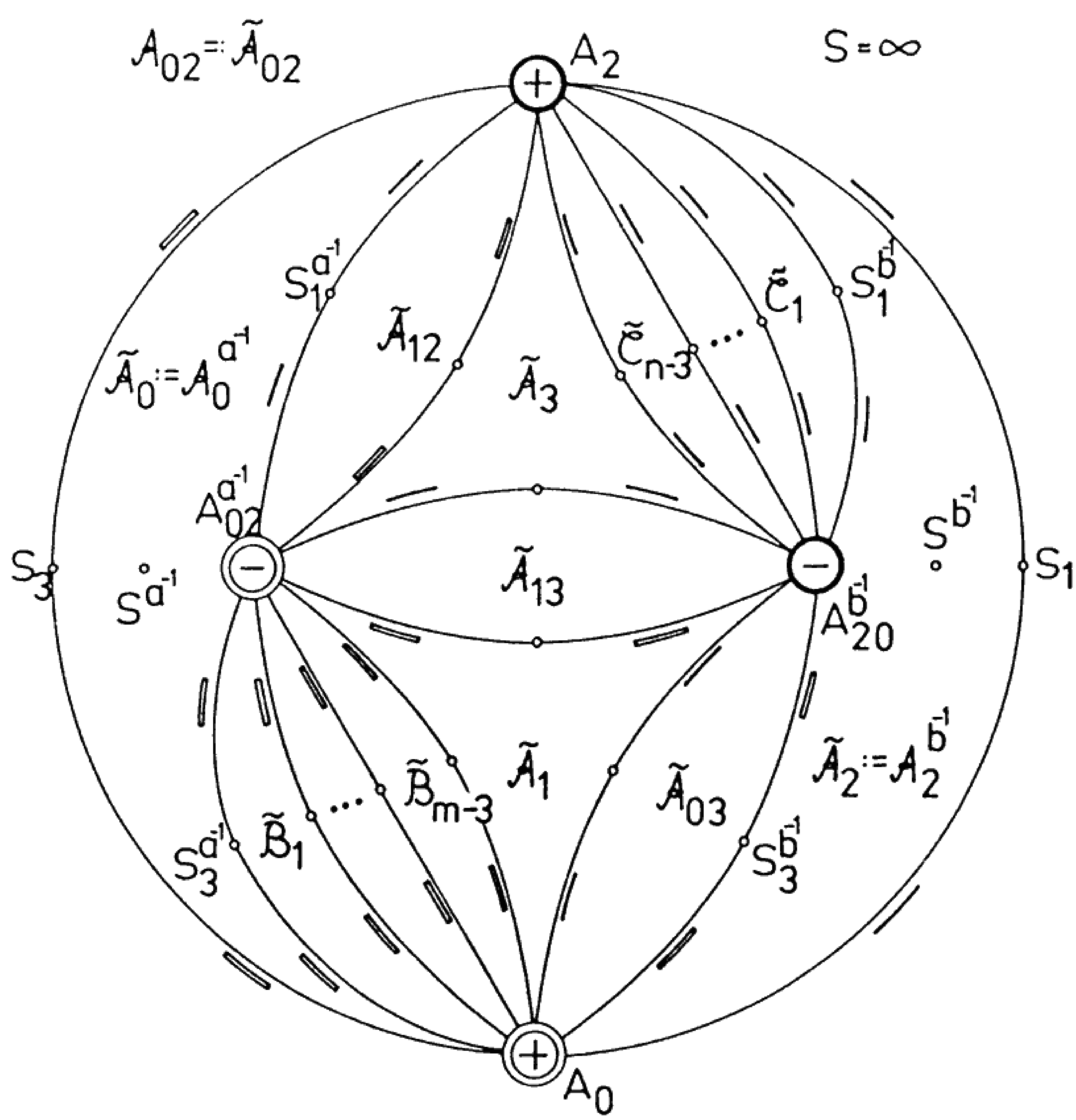

Figure 2

As all the vertices of $\mathscr{P}$ are G_{mn}-equivalent to A_{20}, we construct a chart $\mathscr{C}_1$ for M_{mn} around A_{20}, as the generating homeomorphisms a and b dictate gluing the vertex domains of $\mathscr{P}$. In Fig. 2 we have designed this chart $\mathscr{C}_1$ as a simplicial complex, where the surface of $\mathscr{C}_1$ (its Schlegel diagram) is combinatorially described on the base of Fig. 1. The vertex domain $\mathscr{A}_{02}$ of A_{02} as a spherical digon is fixed by four subsimplices of $\mathscr{P}$. $A_{20}A_0S_1S$ rests on the directed segment $A_{20}A_0$ ⋟ (⊕ in Fig. 2) and on the face $A_{20}A_0S_1 \subset f_{b^{-1}}$ (this is signed by $-$ at A_0S_1 near the face inside $\mathscr{P}$). Now the barycentre S of $\mathscr{P}$ is the ∞ point of the figure plane. The subsimplex $A_{20}A_0S_3S$ rests on the face $A_{20}A_0S_3 \subset f_{a^{-1}}$ (this signed by $=$ at A_2S_3 inside $\mathscr{P}$; the above signs $=$ and $-$ outside $\mathscr{P}$ would refer to f_a and f_b, respectively). Analogously, $A_{02}A_2S_1S$ rests on $A_{02}A_2$ —>— (⊕ in Fig. 2) and on the face $A_{02}A_2S_1 \subset f_{b^{-1}}$, the subsimplex $A_{02}A_2S_3S$ rests on $A_{02}A_2S_3 \subset f_{a^{-1}}$.

Now we form the a^{-1}-image $\mathscr{A}_0^{a^{-1}} =: \widetilde{\mathscr{A}}_0$ of the vertex domain $\mathscr{A}_0$ which shall consists of 6 simplices. We see in (6) that

$$(A_{20}A_0, f_{a^{-1}}) \overset{a}{\longmapsto} (A_0B_1, f_a).$$

Thus we define $S^{a^{-1}}$ in $\mathbb{R}^4$ (the sum of its coordinates, with respect to the basis A_i, equals to 1) as the a^{-1}-image of the barycentre S of $\mathscr{P}$, just arbitrarily, so that the simplex $A_{20}A_0S_3S^{a^{-1}}$ joins $\mathscr{P}$ from outside, along $A_{20}A_0S_3$. Then, in the sense of (5), we linearly define

$$a^{-1} : A_0B_1S_2S \longmapsto A_{20}A_0S_3S^{a^{-1}}.$$

Analogously define the a^{-1}-images of the other 5 A_0-simplices by prescribing $S_1^{a^{-1}}$, $A_{02}^{a^{-1}}$ ($\ominus$ in Fig. 2) and $S_3^{a^{-1}}$ in order to get non-degenerated simplices, non-overlapping the previous ones.

Finally, we get a surface diagram of $\mathscr{C}_1$ (Fig. 2) with 4 nodes $A_0 \oplus, A_2 \oplus, A_{02}^{a^{-1}} \ominus$, $A_{20}^{b^{-1}} \ominus$, according to directions of the segment classes $\twoheadrightarrow$ and $\rightarrow$. Furthermore, we have $m+n+2$ «countries» corresponding to the vertex domains. The boundaries of countries are signed by $=$, $-$ according to the generators a, a^{-1} and b, b^{-1}, respectively. Observe that if we go around either $\oplus$ or $\ominus$, we can read off the same cycle relation $a^{-(m-1)}b\,a\,b = 1$. We similarly find $b^{-(n-1)}a\,b\,a = 1$ if we do that around $\oplus$ or $\ominus$. These relations are equivalent to those of (1). These facts enable us to close the procedure at constructing the image $\widetilde{\mathscr{A}}_{13}$ of the vertex domain $\mathscr{A}_{13}$ by defining the image of the barycentre S as follows

$$(7) \qquad S^{b^{-1}a^{-1}b^{-1}a^{-1}} = S^{b^{-n}} = S^{a^{-m}} = S^{a^{-1}b^{-1}a^{-1}b^{-1}}.$$

The second chart for M_{mn} will be

$$(8) \qquad \mathscr{C}_2 := \mathscr{P} \cup \mathscr{P}^{a^{-1}} \cup \mathscr{P}^{b^{-1}}$$

where the additional polyhedron $\mathscr{P}^{a^{-1}}$, joining at $f_{a^{-1}}$, is

$$(9) \qquad \mathscr{P}^{a^{-1}} := A_{20}A_0S_3S^{a^{-1}} \cup A_0B_1S_3S^{a^{-1}} \cup \ldots \cup A_2A_{20}S_3S^{a^{-1}}$$

and the analogous polyhedron $\mathscr{P}^{b^{-1}}$ at $f_{b^{-1}}$ is defined with the b^{-1}-image of the barycentre S of $\mathscr{P}$. Hence we get an atlas

$$(10) \qquad \operatorname{Int} \mathscr{C}_1 \cup \operatorname{Int} \mathscr{C}_2$$

providing a ball-like neighbourhood for every point of M_{mn}. Indeed, M_{mn} is a manifold.

We briefly describe the universal covering $\widetilde{M}_{mn}$. First we form the Cartesian product $\mathscr{P} \times G_{mn}$ and define the relation $(X, g) \sim (Y, h)$ iff either

(i) $g = h$ in G_{mn}, $X = Y$ in $\mathscr{P}$ or

(ii) $X \in f_{a^{-1}}$, $Y = X^a$, $g = a\,h$ for a generator a of G_{mn}.

This relation extends to an equivalence relation $\star$ on $\mathscr{P} \times G_{mn}$ by defining

$$(X, g) \star (Y, h) \tag{11}$$

iff for some (X_j, g_j) we have

$$(X, g) = (X_1, g_1) \sim (X_2, g_2) \sim \ldots \sim (X_r, g_r) = (Y, h). \tag{12}$$

The equivalence class containing (X, g) is denoted by $\langle X, g\rangle$ and the quotient space is denoted by $\langle \mathscr{P}, G_{mn}\rangle$. We can see that M_{mn}, i.e. $\mathscr{P}$ with identifications, may be considered as an orbit space $\langle \mathscr{P}, G_{mn}\rangle / G_{mn}$ where the group G_{mn} acts, by the second component, discontinuously and freely on the simply connected manifold $\langle \mathscr{P}, G_{mn}\rangle =: \widetilde{M}_{mn}$. This is just the universal covering of M_{mn}.

3. MANIFOLD M_{mn} AS A POSSIBLE SPACE FORM

By the theory [13] every space form is an orbit space $\mathscr{M}/G$ where $\mathscr{M}$ is one of the classical simply connected spaces of constant curvature, and G is an isometry group acting discontinuously and freely on $\mathscr{M}$. G is just isomorphic to the fundamental group of $\mathscr{M}/G$.

Now our polyhedron $\mathscr{P} \sim M_{mn}$ ought to be realized either in the spherical 3-space S_3 or in E_3 or in the Bolyai-Lobachevskian hyperbolic space H_3. The face pairings a and b ought to be screw motions, in metric sense, generating the isometry group G_{mn}, which should act discontinuously and freely on the corresponding space $\mathscr{M}^3$ of constant curvature.

If we find in any $\mathscr{M}^3$ an isometry group G_{mn} with its generators a and b such that (1) is a faithful presentation for G_{mn} then we can construct the fundamental polyhedron $\mathscr{P}$ with great freedom. We choose a suitable point A_{20} «between the screw axes» of a and b (e.g. on a shortest transversal to both). Then we form the images $A_0 := A_{20}^a$, $A_2 := A_{20}^b, \ldots, A_{13} := A_{20}^k$ with

$$k := a^m = b^n = (a\,b)^2 = (b\,a)^2 \tag{13}$$

as (1) and Fig. 1-2 dictate, so we get the vertices of $\mathscr{P}$. A suitable S_3 (on the axis of a) and its a-image $S_2 := S_3^a$, then S_1 (on the axis of b) and the b-image $S_0 := S_1^b$ will be the «midpoints» of the faces $f_{a^{-1}}, f_a, f_{b^{-1}}, f_b$, respectively. The faces themselves will be unions of triangles, e.g.

$$f_a := S_2 A_0 B_1 \cup S_2 B_1 B_2 \cup \ldots \cup S_2 A_{03} A_0. \tag{14}$$

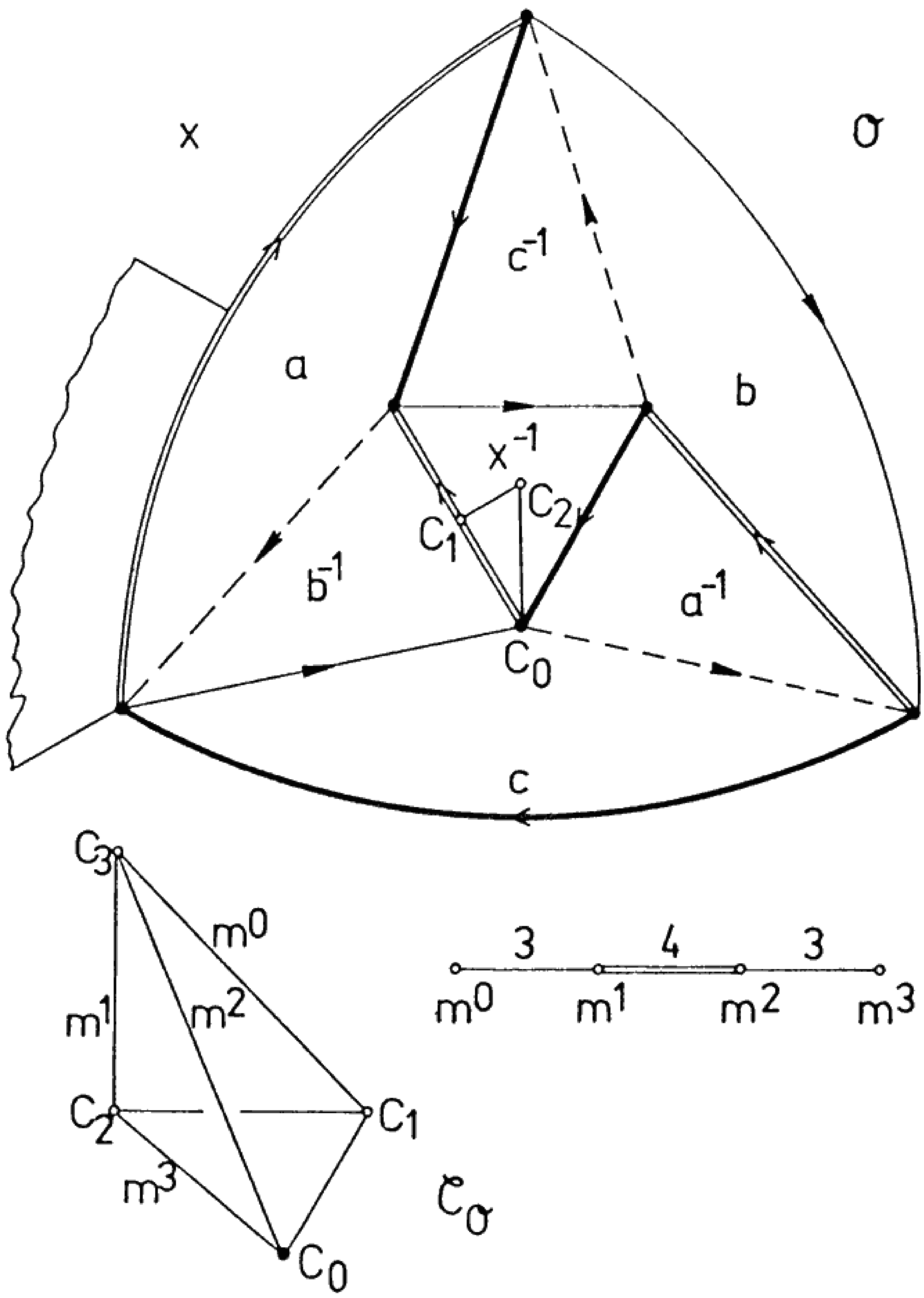

Figure 3

Finally, we shall have a concave polyhedron $\mathscr{P}$ bounded by these faces.

In [5] we have given this construction for the Euclidean groups $\mathbf{P4_1}$ and $\mathbf{P6_1}$. Both are generated by screw motions with axes parallel to the translation k in (13). Then ab and so $ba = a^{-1}(ab)a$ are half screw motions. In the Euclidean cases (13) implies $\frac{1}{m} + \frac{1}{n} = \frac{1}{2}$ providing only $(m, n) = (4, 4)$ and $(6, 3)$.

If $\frac{1}{m} + \frac{1}{n} > \frac{1}{2}$ then G_{mn} is finite group, and we get well known spherical space forms. In (13) $k = -1$ will be the involutive screw motion in the spherical space S^3.

The octahedron space $\mathscr{O}$ has been illustrated in Fig. 3 where the characteristic simplex $\mathscr{C}_{\mathscr{O}} - = C_0C_1C_2C_3$ with Coxeter diagram are also drawn. C_3, C_2, C_1 are centres of the body, of a face, an edge, respectively, C_0 is a vertex of $\mathscr{O}$. The planes m^i, opposite to C_i, are characterized by the nodes of the diagram. For istance, m^2 and m^3 are connected with a branch labelled by $n_{23} = 3$, indicating the face angle $\pi/3$ [12]. Thus $\mathscr{O}$ has face angles

$2\pi/3$. Identifying the opposite faces of $\mathcal{O}$ by screw motions x, a, b, c with «translational» component $2 \cdot C_2C_3 = \pi/3$ and with «rotational» component $\pi/3$, we get 4 edge classes and the presentation of the fundamental group $G_{\mathcal{O}}$ by

$$(15) \qquad G_{\mathcal{O}} := (x, a, b, c - 1 = b\,a\,x^{-1} = c\,b\,x^{-1} = a\,c\,x^{-1} = a\,b\,c).$$

Expressing $x = ba, c = b^{-1}a^{-1}$ and changing $b^{-1} \longleftrightarrow b$, we find that $G_{\mathcal{O}} \cong G_{33}$ is just the binary tetrahedral group consisting of 24 elements.

We also know that the Threlfall binary octahedral group G_{43} of order 48 can be faithfully presented by (1) in S^3 and so is the binary icosahedral group G_{53} of order 120.

$M_{53} = S^3/G_{53}$ is just the Poincaré dodecahedron space $\mathcal{D}$ described in Fig. 4, indicating the generating screew motions u, v, w, x, y, z with «rotation and translation components» of $\pi/5$. We get 10 edge classes providing the defining relations:

$$\succ\!\!= uyx^{-1} = 1, \rightarrowtail vzx^{-1} = 1, \longrightarrow wux^{-1} = 1,$$
$$1 = yvx^{-1} = zwx^{-1} = vu^{-1}z = wv^{-1}u = yw^{-1}v = zy^{-1}w = uz^{-1}y.$$

This presentation can be reduced by eliminating $w = u^{-1}v$, $y = v^{-1}u^{-1}v$, $z = v^{-1}u^{-1}vu$, $x = u^{-1}vu$. We have

$$(16) \qquad G_{\mathcal{D}} = (u, v - 1 = u\,v\,u\,v^{-1}u^{-1}v^{-1} = v\,u^{-2}v\,u\,v^{-1}u).$$

Introducing new generators $a := u, b := u^{-2}v^{-1}$ (i.e. $u = a$, $v = b^{-1}a^2$), we just obtain $G_{\mathcal{D}} \cong G_{53}$.

Now we formulate our

THEOREM 1. *The identified tetrahedron* $\mathcal{P} \sim M_{mn}$ *is a compact manifold for every* $3 \leqq n \leqq m \in \mathbb{Z}$. *Its fundamental group is isomorphic to* $G_{mn} \cong \langle m, n, 2\rangle$ *with presentation (1).* M_{mn} *is not homeomorphic to a space form except for the cases* $(m, n) = (3,3), (4,3), (5,3), (6,3), (4,4)$.

Proof. It remains to prove that G_{mn} cannot be isometry group in the hyperbolic space H^3 in cases $\frac{1}{m} + \frac{1}{n} < \frac{1}{2}$. We start with the following equivalent presentation

$$(17) \qquad G_{mn} = (a, b - a^m = b^n = (a\,b)^2 = (b\,a)^2),\ 3 \leqq n \leqq m \in \mathbb{Z}.$$

We know from the classification of motions in H^3 that we should have a screw motion k in (13), i.e. in the relations of (17). Then the screw motions a and b should have the same screw axis, and G_{mn} would be cyclic group. But then G_{mn} could not have a compact fundamental domain. Q.E.D.

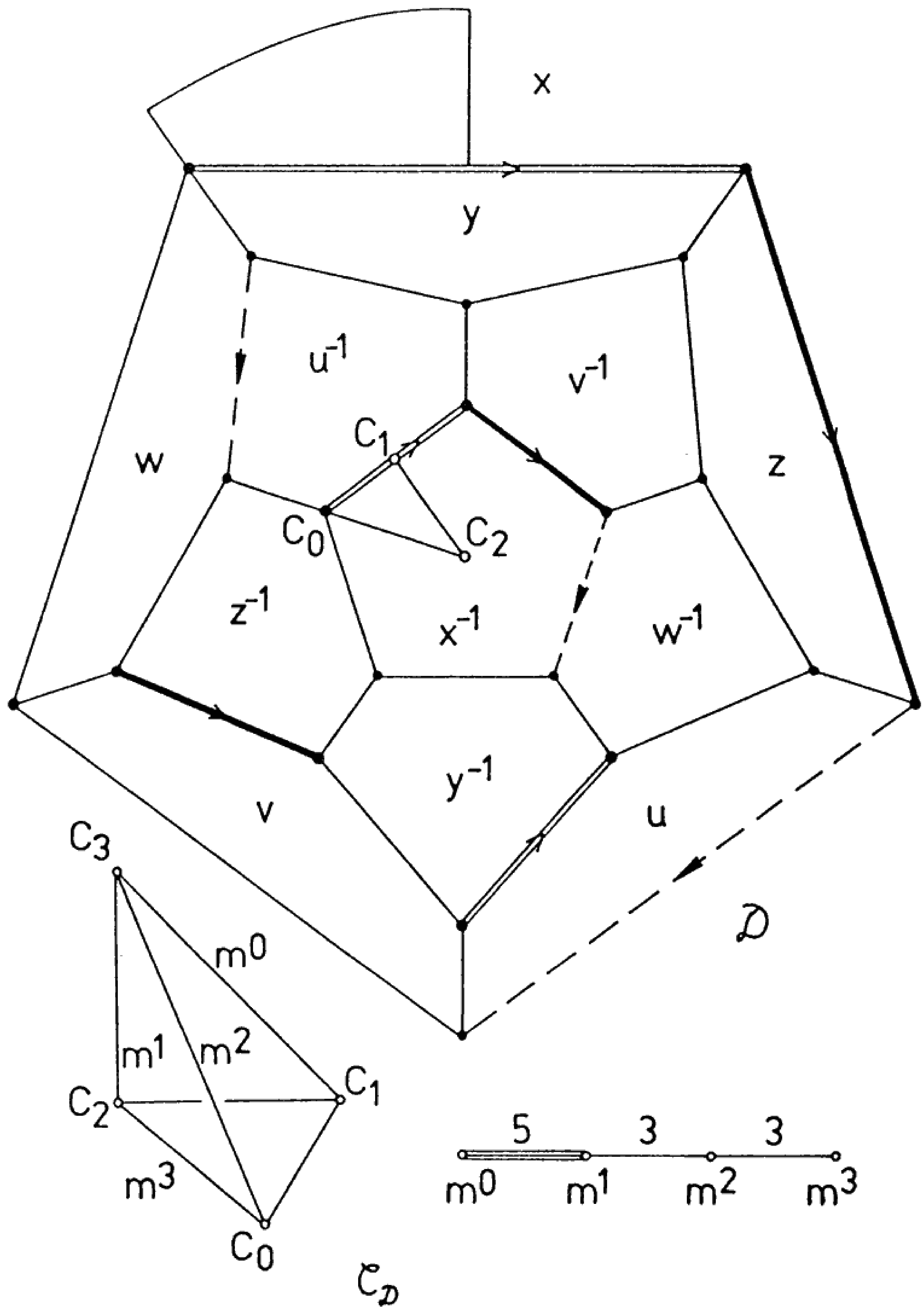

Figure 4

4. M_{mn} AS SEIFERT FIBRE SPACE

For the exceptional manifolds M_{mn} in Th. 1. arises a natural problem, how to describe their topological structures, and wether these M_{mn} can wear locally homogeneous complete Riemannian metric. Discussing about my construction (1988 October), E. B. Vinberg called my attention to the Seifert fibration [9] and the eight geometries of Thurston [11]. I received the same questions from the Referee to the former version of my note (1989 August). After having studied Seifert's classic [9] and the excellent survey paper of P. Scott [8, §3-5], I can formulate the answer as follows:

THEOREM 2. *Each manifold M_{mn} is homeomorphic to a Seifert's fibre space, namely, to* *the circle bundle over a 2-orbifold* $\mathcal{M}^2/[m,n,2]^+$ *where* $\mathcal{M}^2$ *is either the sphere* $S^2(>)$ *or the Euclidean plane* $E^2(=)$ *or the hyperbolic plane* $H^2(<)$ *if* $\frac{1}{m}+\frac{1}{n} \quad \frac{1}{2}$, *respectively. The universal covering space* $\widetilde{M}_{mn}$ *is just the universal cover of the unit tangent bundle* $T_1(\widetilde{\mathcal{M}^2})$ *which is isometric to* $S^3(>)$, $E^3(=)$ *and* $\widetilde{SL_2\mathbf{R}}(<)$, *respectively.*

Here $[m,n,2]$ denotes the Coxeter group generated by reflections in the side lines of the triangle with angles $\frac{\pi}{m}$, $\frac{\pi}{n}$, $\frac{\pi}{2}$; and $[m,n,2]^+$ i s its rotational subgroup of index 2. $T_1(\mathcal{M}^2)$ consists of all the unit vectors at the points of the plane $\mathcal{M}^2$ of constant curvature.

Proof. We do not cite all the arguments from [8], [9], [11]. See particulary pp. 479-480 of [8]. By formulas (13) and (17) we derive that the cyclic group K generated by k is the centre of G_{mn}. Factorizing by K, we get the presentation

$$G_{mn}/K = (\overline{a},\overline{b} - \overline{a}^m = \overline{b}^n = (\overline{a}\,\overline{b})^2 = \overline{1}) \tag{18}$$

which is just isomorphic to $[m,n,2]^+$ and it can be realized as isometry group of $\mathcal{M}^2$ by Th. 2. So we get the orbifold $\mathcal{M}^2/[m,n,2]^+$ as the base of the bundle (Zerlegungsfläche). The cyclic group K characterizes any (regular) circle fibre (the circle of the unit vectors at a point of $\mathcal{M}^2$ different from any rotation centre). G_{mn} acts by isometries on $T_1(\widetilde{\mathcal{M}^2})$ discontinuously and freely and the quotient is $T_1(\mathcal{M}^2/[m,n,2]^+)$ as in Th. 2. The cyclic group K winds up the real line $\mathbf{R}$ about every circle fibre. The isometry $T_1\widetilde{(H^2)} \cong \widetilde{SL_2}\mathbf{R}$ is described, e.g. in [8, §4] and [11]. Q.E.D.

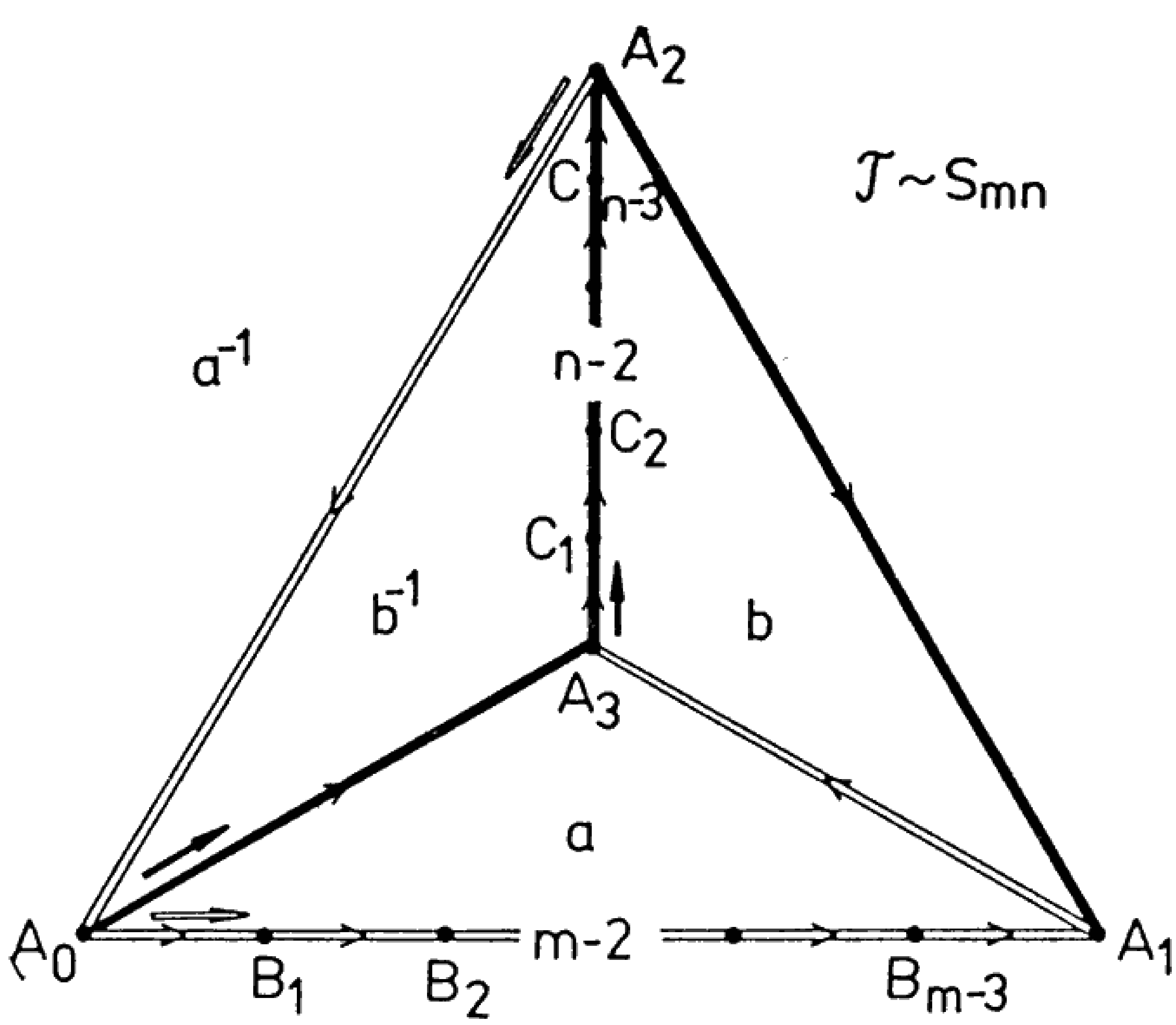

Figure 5

Summarizing, our construction gives a geometric (simplicial) picture for an interesting class of Seifert fibre spaces.

In Fig. 5 I have described another tetrahedron, identified by topological screw motions a and b. This $\mathcal{T} \sim S_{mn}$ analogously leads to the spherical lens space mentioned in the introduction.

REFERENCES

[1] A. F. BEARDON, *The geometry of disscrete groups*, Springer, 1983.

[2] H. S. M. COXETER, W. O. J. MOSER, *Generators and relations for discrete groups*, 4th ed., Ergeb. Math. **14**, Springer, 1980.

[3] B. MASKIT, *On Poincare's theorem for fundamental polygons*, Adv. in Math. **7** (1971) 219-230.

[4] E. MOLNAR, *Twice punctured compact euclidean and hyperbolic manifolds and their twofold coverings*, Colloq. Math. Soc. Jànos Bolyai **46**, North-Holland, 1987, 883-919.

[5] E. MOLNAR, *Minimal presentation of the 10 compact euclidean space forms by fundamental domains*, Studia Sci. Math. Hungar. **22**, (1987), 19-51.

[6] E. MOLNAR, *Polyhedron complexes with simply transitive group actions and their realizations*, Acta Math. Hungar. (to appear).

[7] H. POINCARE, *Mémoire sur les groupes Kleinéens*, Acta Math. **3**, (1893) 49-92; (*Oeuv.* **2**, 258-299).

[8] P. SCOTT, *The geometries of 3-manifolds*, Bull. London Math. Soc. **15**, (1983) 401-487.

[9] H. SEIFERT, *Topologie dreidimensionaler gefaserter Räume*, Acta Math. **60** (1933) 147-288; reprinted as «Topology of 3-dimensional fibered spaces», in: SEIFERT and THRELFALL: *A Textbook of Topology*, Acad. Press, 1980.

[10] W. THRELFALL, H. SEIFERT, *Topologische Untersuchung der Diskontinuitätsbereiche endlicher Bewegungsgruppen des dreidimensionalen sphärischen Raumes*, Math. Ann. **104** (1930) 1-70; **107** (1933) 543-586.

[11] W. P. THURSTON, *Three dimensional manifolds, Kleinian groups and hyperbolic geometry*, Bull. Amer. Math. Soc. **6** (1982) 357-381.

[12] E. B. VINBERG, *Hyperbolic reflection groups*, Uspehi Mat. Nauk **40** (1985) 29-66 (Russian).

[13] J. A. WOLF, *Spaces of constant curvature*, University of Califomia, Berkley, Califomia, 1972.

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