Note di Matematica Vol. X - n. 2, 235-246 (1990)

ON THE FIXED POINTS OF THE LIE ALGEBRAS ASSOCIATED WITH A FREE GROUP PRESENTATION

RAINER ZERCK

Introduction. Linear methods are an important tool in group theory. A standard procedure is to associate Lie algebras $\mathrm{t}\,H = \bigoplus_{c\geq 1} \tau_c H/\tau_{c+1} H$ with suitable central subgroup series $\{\tau_c H | c \geq 1\}$ of the group H under consideration. Thus the commutation in groups may be expressed in terms of (bi)linear forms. Conditions of various kinds on the subgroup series lead to different Lie algebra structures. In this paper we are concerned with three types of central subgroup series (two of them being connected with a prime number p), which have the common property that they carry free groups H into free Lie algebras of several type. We prove (Theorems 1 and 2) that for any group G given by a free presentation $1 \longrightarrow N \longrightarrow F \longrightarrow G \longrightarrow 1$ the fixed point subalgebras of the three Lie algebras mentioned form a free subalgebra (G acts on $\tau_c N/\tau_{c+1} N$ by conjugation). For showing that in the two p-cases we can choose, in fact, the same free generating set we state a more general result (Theorem 5) concerning the fixed points of free restricted Lie algebras with respect to a homogeneous group action which is of independent interest. The centres of the groups $F/\tau_{c+1} N$ are described (Theorems 3 and 4).

1. CENTRAL SUBGROUP SERIES

First we collect some basic facts about central subgroup series and associated Lie algebras for which we refer to [4], [9], [10].

For an arbitrary non-trivial group H a normal subgroup series

$$H = \tau_1 H \geq \tau_2 H \geq \dots \tag{1}$$

is called *central* if

$$(\tau_c H, \tau_e H) \leq \tau_{c+e} H \quad \text{for } c, e \geq 1, \tag{2}$$

where $(\tau_c H, \tau_e H)$ is that normal subgroup of H which is generated by all commutators $(g, h) = g^{-1}h^{-1}gh$ with $g \in \tau_c H$, $h \in \tau_e H$.

The associated graded abelian group

$$\mathrm{t}\,H = \bigoplus_{c\geq 1} \mathrm{t}_c H = \bigoplus_{c\geq 1} \tau_c H/\tau_{c+1} H$$

carries the structure of a graded Lie ring (that is a graded Lie algebra over the integers) by linearly extending the commutation

$$[g\tau_{c+1}H, h\tau_{e+1}H] := (g,h)\tau_{c+e+1}H$$

for $g \in \tau_c H$, $h \in \tau_e H$.

For a prime p a central series (1) is called *restricted p-central* if

$$(\tau_c H)^p \subseteq \tau_{pc} H$$

for $c \geq 1$. Then $\mathrm{t}\,H$ is a restricted Lie algebra over the prime field $\mathbf{F}_p$ of characteristic p with respect to

$$^{[p]} : \tau_c H/\tau_{c+1} H \longrightarrow \tau_{cp} H/\tau_{cp+1} H : h\tau_{c+1} H \longrightarrow h^p \tau_{cp+1} H.$$

Remind that a restricted Lie algebra over a field k of prime characteristic p is a k-Lie algebra $\mathfrak{L}$ over k together with a map

$$^{[p]} : \mathfrak{L} \longrightarrow \mathfrak{L}$$

which fulfils the following conditions:

$$(\lambda a)^{[p]} = \lambda^p \cdot a^{[p]}$$

(3) $$[a^{[p]}, b] = (\mathrm{ad}\ a)^p(b)$$

(4) $$(a+b)^{[p]} = a^{[p]} + b^{[p]} + \Lambda_p(a,b)$$

for $a, b \in \mathfrak{L}$, $\lambda \in k$. Here, as usually, $(\mathrm{ad}\ a)(b) = [a,b]$. The expression $\Lambda_p(a,b)$ is a linear combination of right-normed Lie brackets of length p with entries a and b. A detailed discussion of these matters may be found in [3], Chap. 1, Sect. 11 or in the exercises 19 and 20 in [4], Chap. 1, Sect. 1. In the latter the restricted Lie algebras are referred to as p-Lie algebras.

The third type of subgroup series we are interested in satisfies the conditions

(5) $$(\tau_c H)^p \subseteq \tau_{c+1} H$$

for $c \geq 1$ and is called the *p-central series.*

In this case the map

$$\omega : \tau_c H/\tau_{c+1} H \longrightarrow \tau_{c+1} H/\tau_{c+2} H : h\tau_{c+1} H \longrightarrow h^p \tau_{c+2} H$$

is well defined. If $(p, c) \neq (2, 1)$, then ω turns out to be a group homomorphism. Hence, if $p \neq 2$ and (5) is fulfilled then $\mathfrak{t}\, H$ is an $\mathbf{F}_p[\omega]$-Lie algebra, where ω is dealt with as a transcendent element over $\mathbf{F}_p$. As an endomorphism of a graded Lie algebra ω is of degree 1:

$$\omega(\mathfrak{t}_c H) \subseteq \mathfrak{t}_{c+1} H.$$

The *lower* central series, the *lower* restricted p-central series and the *lower* p-central series of a group H are defined inductively by

$$\gamma_1 H = \kappa_1 H = \lambda_1 H = H \text{ and}$$

$$\gamma_c H = (\gamma_{c-1} H, H)$$

$$\kappa_c H = \sqcap(\gamma_i H)^{p^k} \text{ (the product is taken over all pairs } (i, k) \text{ with } ip^k \geq c)$$

$$\lambda_c H = (\lambda_{c-1} H, H)(\lambda_{c-1} H)^p =$$

$$= (\gamma_1 H)^{p^{c-1}} (\gamma_2 H)^{p^{c-2}} \ldots (\gamma_{c-1} H)^p (\gamma_c H),\ p \neq 2,$$

respectively.

We introduce the following notations:

(6) $$\mathfrak{g}_c H = \gamma_c H/\gamma_{c+1} H \qquad \mathfrak{g}\, H = \bigoplus_{c\geq 1} \mathfrak{g}_c H$$

(7) $$\mathfrak{k}_c H = \kappa_c H/\kappa_{c+1} H \qquad \mathfrak{k}\, H = \bigoplus_{c\geq 1} \mathfrak{k}_c H$$

(8) $$\mathfrak{l}_c H = \lambda_c H/\lambda_{c+1} H \qquad \mathfrak{l}\, H = \bigoplus_{c\geq 1} \mathfrak{l}_c H \ (p \neq 2)$$

Clearly, $\mathfrak{g}$, $\mathfrak{k}$ and $\mathfrak{l}$ may be viewed as functors from the category of groups to the categories of graded Lie rings, graded restricted $\mathbf{F}_p$-Lie algebras and graded $\mathbf{F}_p[\omega]$-Lie algebras with ω of degree 1, respectively.

These functors carry free objects into free objects. More precisely, let $F(X)$ be the free group on X, k a commutative ring, $L^k(X)$ and $R^k(X)$ the free k-Lie algebra and the free restricted k-Lie algebra (in this case k is assumed to be a field of prime characteristic p) over k on X, respectively. Then we have

Theorem 0.

$$\mathfrak{g}\, F(X) = L^{\mathbb{Z}}(X^{\gamma}) \quad \text{with } X^{\gamma} = \{x\gamma_2 F(X) \mid x \in X\}$$

$$\mathfrak{k}\, F(X) = R^{\mathbf{F}_p}(X^{\kappa}) \quad \text{with } X^{\kappa} = \{x\kappa_2 F(X) \mid x \in X\}$$

$$\mathfrak{l}\, F(X) = L^{\mathbf{F}_p[\omega]}(X^{\lambda}) \quad \text{with } X^{\lambda} = \{x\lambda_2 F(X) \mid x \in X\}\ (p \neq 2).$$

The first part of the Theorem is the classical result of Magnus [11] and Witt [16]. The case $\mathfrak{k}$ is treated by Lazard [10], Theorem (6.5). The $\mathfrak{l}$ -assertion is due to Skopin [15] and Lazard [10], pp. 138/139. It got the form we cited here in the note [1] of Andozhskij and Tsvetkov. A short explanation may also be found in the paper [6] of Bryant and Kovács.

2. RESULTS

Assume the non-trivial group G to be given by the presentation

$$1 \longrightarrow N \longrightarrow F \longrightarrow G \longrightarrow 1$$

with F a non-cyclic free group. Then by Schreier's Theorem, the normal subgroup N is also free with some free generating set X : $N = F(X)$. We henceforth assume that $N \notin \{F, 1\}$.

The graded Lie algebras $\mathfrak{g}\, N$, $\mathfrak{k}\, N$ and $\mathfrak{l}\, N$ $(p \neq 2)$ are endowed with a homogeneous (right) G-module structure by setting

$$(u\tau_{c+1} N)^{wN} := w^{-1} u w \tau_{c+1} N$$

for $c \geq 1$, $w \in F$, $u \in \tau_c N$, $\tau \in \{\gamma, \kappa, \lambda\}$. Note that this G-action is compatible with the associated Lie algebra structure, i.e. for $g \in G$, $c, e \geq 1$, $\tau \in \{\gamma, \kappa, \lambda\}$, $u \in \tau_c H$, $v \in \tau_e H$ we have

$$[u\tau_{c+1} N, v\tau_{e+1} N]^g = [(u\tau_{c+1} N)^g, (v\tau_{e+1} N)^g].$$

Moreover,

$$((u\kappa_{c+1} N)^{[p]})^g = ((u\kappa_{c+1} N)^g)^{[p]}$$

and

$$(9) \qquad (\omega(u\lambda_{c+1} N))^g = \omega((u\lambda_{c+1} N)^g) \quad (p \neq 2).$$

The direct sums (6), (7), (8) turn out to be G-module decompositions for $H = N$.

The G-modules $\mathfrak{g}_c N$, $\mathfrak{k}_c N$ and $\mathfrak{l}_c N$ are called the *cth relation module*, the *cth restricted p-relation module* and the *cth p-relation module*, respectively.

For a (right) G-module M write M^G for the set of those elements of M which are fixed by all elements of G. We are interested in describing the Lie algebras $(\mathfrak{g}\, N)^G$, $(\mathfrak{k}\, N)^G$ and $(\mathfrak{l}\, N)^G$.

Theorem 1. *For all $c \geq 1$ we can choose a set $Y_c \subseteq \gamma_c N \setminus \gamma_{c+1} N$ such that $Y^\gamma := \cup_{c\geq 1} Y_c^\gamma$ with $Y_c^\gamma := \{y\gamma_{c+1}N | y \in Y_c\}$ is a free generating set of the free Lie ring $(\mathfrak{g}\, N)^G$, that is, $(\mathfrak{g}\, N)^G = L^{\mathbf{Z}}(Y^\gamma)$.*

In [18] this Theorem was shown in the particular case of finite X and finite G.

The subalgebras $(\mathfrak{k}\, N)^G$ and $(\mathfrak{l}\, N)^G$ may be characterized in a similar manner. The precise result is as follows.

Theorem 2. *For all $c \geq 1$ we can choose a set $Z_c \subseteq \gamma_c N \setminus \gamma_{c+1} N(\gamma_c N)^p$ such that for $Z_c^\kappa := \{z\kappa_{c+1}N | z \in Z_c\}$ and $Z^\kappa := \cup_{c\geq 1} Z_c^\kappa$ we have $(\mathfrak{k}\, N)^G = R^{\mathbf{F}_p}(Z^\kappa)$. Furthermore, if $p \neq 2$ then for $Z_c^\lambda := \{z\lambda_{c+1}N | z \in Z_c\}$ and $Z^\lambda := \cup_{c\geq 1} Z_c^\lambda$ we get $(\mathfrak{l}\, N)^G = L^{\mathbf{F}_p[\omega]}(Z^\lambda)$.*

We shall obtain that the Lie algebras $(\mathfrak{g}\, N)^G$, $(\mathfrak{k}\, N)^G$ and $(\mathfrak{l}\, N)^G$ are, in fact, trivial or not finitely generated.

Theorem 3. *(i) The following assertions are equivalent:*

(ia) G is infinite

(ib) $(\mathfrak{g}\, N)^G = 0$

(ic) $(\mathfrak{k}\, N)^G = 0$

(id) $(\mathfrak{l}\, N)^G = 0$.

(ii) If G is finite, then the Lie algebras $(\mathfrak{g}\, N)^G$, $(\mathfrak{k}\, N)^G$ and $(\mathfrak{l}\, N)^G$ are not finitely generated.

The $\mathfrak{g}$-part of (ii) is treated in [18] and generalized to free Lie algebras over any principal ideal domain which are equipped with a homogeneous group action by Bryant [5]. The equivalence of (ia) and (ib) is essentially due to Auslander and Lyndon [2]. It was also proved in [2] that $(N/N')^G$ is precisely the centre of the quotient F/N'. Here we provide a generalization of this result. By $\zeta(H)$ we denote the centre of the group H.

Theorem 4. *Let $c \geq 1$. Then*

(i) $\zeta(F/\gamma_{c+1}N) = (\gamma_c N/\gamma_{c+1}N)^G$

(ii) $\zeta(F/\kappa_{c+1}N) = (\kappa_c N/\kappa_{c+1}N)^G$

(iii) $\zeta(F/\lambda_{c+1}N) = (\lambda_c N/\lambda_{c+1}N)^G$ $(p \neq 2)$

(o) These centres are non-trivial if and only if G is finite.

The proof of Theorem 2 is based on a result concerning the fixed points of free restricted Lie algebras which we shall state separately.

Let V be a vector space with basis X over a perfect field k of prime characteristic p. Assume G to be a non-trivial subgroup of $GL(V)$. The G-action on $L_1^k(X) = R_1^k(X) = V$ may be naturally extended to the whole algebras $L^k(X)$ and $R^k(X)$. The subalgebras

$L^k(X)^G$ and $R^k(X)^G$ are homogeneous with respect to the natural graduation of $L^k(X)$ and $R^k(X)$. By theorems of Shirshov [13] and Witt [17] these subalgebras are also free with homogeneous free generating sets. We shall add to this knowledge by proving the following.

Theorem 5. *Every homogeneous free generating set Y of $L^k(X)^G$ is also a free generating set of $R^k(X)^G$, that is, if $L^k(X)^G = L^k(Y)$, then $R^k(X)^G = R^k(Y)$.*

3. FIXED POINTS OF FREE RESTRICTED LIE ALGEBRAS

In this section we prove Theorem 5. Let k be a perfect field of prime characteristic p and V be a non-trivial vector space with basis X. For the following facts concerning the structure of $L^k(X)$ and $R^k(X)$ we refer to [3] and [4].

We will work in the free associative non-commutative k-algebra $A^k(X)$ which may be described as the free k-module with the basis consisting of all (associative non-commutative) words on X (including the empty word: 1) and the multiplication is given by distributively continuing the juxtaposition of words. By the theorem of Poincaré, Birkhoff and Witt the free k-Lie algebra $L^k(X)$ may be characterized as the k-Lie subalgebra (with respect to the Lie multiplication $[a,b] = ab - ba$) inside of $A^k(X)$ which is generated by X. Moreover, the free restricted k-Lie algebra $R^k(X)$ is the closure of $L^k(X)$ in $A^k(X)$ with respect to p-exponentiation. In order to construct k-bases of $L^k(X)$ and $R^k(X)$ the notation of the free magma on X is introduced as the set $\Gamma(X)$ of all non-empty non-associative non-commutative monomials on X with multiplication given by (non-associative) juxtaposition.

Assume any $x \in X$ to be of degree 1. Then $\Gamma(X)$ is the disjoint union of the subsets $\Gamma_c(X)$ consisting of all monomials of degree c, $c \geq 1$. Denote by $\mathfrak{X}^{\Gamma}$ a Hall set contained in $\Gamma(X)$. Let $\mathfrak{X}^{\Gamma}_c = \mathfrak{X}^{\Gamma} \cap \Gamma_c(X)$.

Consider the map $\sigma_X : \Gamma(X) \longrightarrow L^k(X)$ which is given inductively by $\sigma_X(x) = x$ and $\sigma_X(uv) = [\sigma_X(u), \sigma_X(v)]$. Then the Hall set $\mathfrak{X}^{\Gamma}$ yields a k-basis $\mathfrak{X}^L = \{\sigma_X(u) | u \in \mathfrak{X}^{\Gamma}\}$ of $L^k(X)$ (the *Hall basis*) and a k-basis $\mathfrak{X}^R = \{(\sigma_X(u))^{p^r} | u \in \mathfrak{X}^{\Gamma}, r \geq 0\}$ of $R^k(X)$. Let $\mathfrak{X}^L_c = \mathfrak{X}^L \cap L^k_c(X) = \sigma_X(\mathfrak{X}^{\Gamma}_c)$ and $\mathfrak{X}^R_c = \mathfrak{X}^R \cap R^k_c(X)$. The algebras $A^k(X)$, $L^k(X)$ and $R^k(X)$ decompose into the direct sums of their homogeneous components:

$$A^k(X) = \bigoplus_{c \geq 0} A^k_c(X).$$

$$L^k(X) = \bigoplus_{c \geq 1} L^k_c(X) \tag{10}$$

and

$$R^k(X) = \bigoplus_{c \geq 1} R^k_c(X).$$

The sets $\mathfrak{X}_c^L$ and $\mathfrak{X}_c^R$ are k-bases of $L_c^k(X)$ and $R_c^k(X)$, respectively.
In the sequel we will make use of the following facts.

Lemma 1.
(i) $[a,b] \in L^k(X)$ *for* $a,b \in R^k(X)$.
(ii) $\wedge_p(a,b) \in L^k(X)$ *for* $a,b \in R^k(X)$.
(iii) Let $a,b \in L^k(X)$ *and* $n \geq 1$. *Then* $(a+b)^{p^n} = a^{p^n} + b^{p^n} + d_1^{p^{n-1}} + d_2^{p^{n-2}} + \ldots + d_n$ *with suitable* $d_i \in L^k(X)$.
(iv) Each element a *of* $R^k(X)$ *can be uniquely expressed in the form* $a = a_0 + a_1^p + a_2^{p^2} + \ldots + a_n^{p^n}$ *with suitable* $n \geq 0$ *and* $a_i \in L^k(X)$.

Proof. For showing (i) it is sufficient to take $a = u^{p^r}$ and $b = v^{p^s}$ with $u,v \in \mathfrak{X}^L$, $r,s \geq 0$. If $r = s = 0$, then clearly $[a,b] \in L^k(X)$. If one of the exponents r, s does not vanish, say $r \geq 1$, then $[a,b] = [(u^{p^{r-1}})^p, v^{p^s}] = (\text{ad}\ (u^{p^{r-1}})^p)(v^{p^s}) = (\text{ad}\ (u^{p^{r-1}})^p(v^{p^s})$ by (3) and induction yields the desired result.

Assertion (ii) may be immediately deduced from (i) and the fact that $\wedge_p(a,b)$ is a linear combination of Lie brackets with entries a and b.

Assertion (iii) may be easily shown by induction on n and is based on (ii) and (4). The fact (iv) that any $a \in R^k(X)$ may be written as a sum of p^i-th powers of elements of $L^k(X)$ figures up in Lazard's paper [10] as Theorem (6.1). It can be verified by subtracting those summands in the $\mathfrak{X}^L$-decomposition of $a \in R^k(X)$ which are of highest exponent and then applying (iii) and the induction hypothesis. Here we need the field k to be perfect.

For showing the uniqueness-part of (iv) assume $a = a_0 + a_1^p + \ldots + a_n^{p^n} = b_0 + b_1^p + \ldots + b_m^{p^m}$ with $n \geq m$, $a_i, b_i \in L^k(X)$, $a_n \neq 0$, $b_m \neq 0$. By comparing the $\mathfrak{X}^L$-decompositions of a_n and b_m we conclude $m = n$ and $a_n = b_n$. ∎

The *proof of Theorem 5* splits into two parts. Let Y be a homogeneous free generating set of $L^k(X)^G$. Then we have to show:

(a) The restricted Lie algebra S generated (inside of $A^k(X)$) by Y is free on Y.

(b) $R^k(X)^G \subseteq S$.

Consider a Hall set $\mathfrak{Y} \subset \Gamma(Y)$ and define σ_Y and $\mathfrak{Y}^L \subset L^k(Y) \subseteq L^k(X)$ analoguously to σ_X and $\mathfrak{X}^L$.

The set Y is homogeneous with respect to decomposition (10), hence $\mathfrak{Y}^L$ is so. Let $\mathfrak{Y}_c^L := \mathfrak{Y}^L \cap L_c^k(X)$. The k-basis $\mathfrak{Y}^L$ of $L^k(Y)$ may be extended to a k-basis $\mathfrak{Z}^L$ of $L^k(X)$ by adding a suitable number of elements. Assume $\mathfrak{Z}^L$ to be linearly ordered including $\mathfrak{Y}^L$. The Theorem of Poincaré, Birkhoff and Witt yields the set

$$\mathfrak{Z}^A := \{w_1 \ldots w_s | s \geq 0,\ w_j \in \mathfrak{Z}^L,\ w_1 \geq \ldots \geq w_s\}$$

to be a k-basis of $A^k(X)$. In particular,

$$\mathfrak{W}^R := \{w^{p^i} \mid i \geq 0,\ w \in \mathfrak{W}^L\} \subseteq \mathfrak{Z}^A$$

is linearly independent. So the restricted Lie algebra S generated by Y is free. Thus assertion (a) is shown.

We turn to (b). Take $a \in R^k(X)$. By Lemma 1(iv) $a = a_0 + a_1^p + \ldots + a_n^{p^n}$. For $g \in G$ we get $a^g = a_0^g + (a_1^p)^g + \ldots + (a_n^{p^n})^g = a_0^g + (a_1^g)^p + \ldots + (a_n^g)^{p^n}$, hence $a^g = a$ if and only if $a_i^g = a_i$ for all i by the uniqueness-part o f Lemma 1(iv). Thus, if $a \in R^k(X)$ is a fixed point, then $a_i \in L^k(X)^G = L^k(Y)$ and $a \in R^k(Y)$ follows. ∎

Remark. Note that

$$\mathfrak{W}^A := \{w_1 \ldots w_s \mid s \geq 0,\ w_j \in \mathfrak{W}^L,\ w_1 \geq \ldots \geq w_s\} \subseteq \mathfrak{Z}^A$$

is linearly independent, too. Consequently, the associative algebra generated inside of $A^k(X)$ by Y is also free. But $A^k(Y) \neq A^k(X)^G$, in general. For example, if $X = \{x_1, x_2, x_3\}$, char $k \neq 2$ and the symmetric group S_3 acts on V by naturally permuting the $x_i's$, then $\mathfrak{X}_1^L = X$, $\mathfrak{X}_2^L = \{[x_1, x_2], [x_1, x_3], [x_2, x_3]\}$, $\mathfrak{W}_1^L = \{y_1 = x_1 + x_2 + x_3\}$, $\mathfrak{W}_2^L = \emptyset$. But $A_1^k(X)^G = ky_1$, $A_2^k(X)^G = ky_1^2 \oplus k(x_1^2 + x_2^2 + x_3^2)$ and $x_1^2 + x_2^2 + x_3^2 \notin A^k(Y)$, hence $A^k(X)^G$ is not contained in $A^k(Y)$. So assertion (b) is not self-evident.

4. THE FIXED POINTS OF $\mathfrak{g}N$, $\mathfrak{k}N$ AND $\mathfrak{l}N$

The crucial point in proving the Theorems 3 and 4 is the following assertion.

Lemma 2. *The G-action on $N/\kappa_2 N$ is faithful, that is, the group $G_0 = \{g \in G \mid (u\kappa_2 N)^g = u\kappa_2 N$ for all $u\kappa_2 N \in N/\kappa_2 N\}$ is trivial.*

Proof. Let $h = fN \in G_0$. Denote by F_h the subgroup of F generated by f and N. The group $F_h/\kappa_2 N$ is centre-by-cyclic, hence abelian and the inclusions $F_h' \leq \kappa_2 N = N'N^p$ and $F_h'/N' \leq (N/N')^p = N'N^p/N'$ follow. On the other hand, the group N/F_h' is free abelian as a subgroup of F_h/F_h'. Thus the exact sequence $1 \longrightarrow F_h'/N' \longrightarrow N/N' \longrightarrow N/F_h' \longrightarrow 1$ splits. So F_h'/N' is a direct summand of N/N'. But a non-trivial direct summand cannot be contained in $(N/N')^p$, hence $F_h'/N' = 1$. By [2], Cor. 1.2. this implies that $F_h = N$, hence $G_0 = \{1\}$. ∎

If $f\kappa_2 N \in \zeta(F/\kappa_2 N)$, then for all $u\kappa_2 N \in N/\kappa_2 N$ we have $fu\kappa_2 N = uf\kappa_2 N$, or equivalently, $(u\kappa_2 N)^{fN} = u\kappa_2 N$. Hence, $f \in N$ by Lemma 2 and the inclusion

$$\zeta(F/\tau_{c+1}N) \subseteq N/\tau_{c+1}N \text{ for } c \geq 1 \text{ and } \tau \in \{\gamma, \kappa, \lambda\} \tag{11}$$

follows.

Lemma 3.

$$(12)\qquad \zeta(N/\tau_{c+1}N) = \tau_c N/\tau_{c+1}N.$$

Proof. By (2) $(\tau_c N, N) \subseteq \tau_{c+1}N$ hence $\tau_c N/\tau_{c+1}N$ is a subgroup of the centre.

Now take $u\tau_{c+1}N \in \zeta(N/\tau_{c+1}N)$. Then

$$(13)\qquad (u,v) \in \tau_{c+1}N \text{ for any } v \in N.$$

Assume

$$(14)\qquad u \in \tau_e N \setminus \tau_{e+1}N \text{ with } e \leq c-1.$$

By (13)

$$(15)\qquad \left[u\tau_{e+1}N, v\tau_2 N\right] = (u,v)\tau_{e+2}N = 1\tau_{e+2}N$$

for arbitrary $v \in N$. The (restricted) Lie algebra $\mathfrak{t}\,N$ is free on X^τ (by Theorem 0) and as such may be embedded in the free associative algebra $A^{k^\tau}(X^\tau)$ on X^τ over k^τ by the Theorem of Poincaré, Birkhoff and Witt, where $k^\gamma = \mathbf{Z}$, $k^\kappa = \mathbf{F}_p$ and $k^\lambda = \mathbf{F}_p[\omega]$. From (15) we conclude that $u\tau_{e+1}N$ is in the centre of $A^{k^\tau}(X^\tau)$. But the centre of $A^{k^\tau}(X^\tau)$ is trivial, whenever X^τ contains more than one element. A proof of this is outlined in [12], Problem 5 on page 333. Hence $u\tau_{e+1}N = 1N$, what is a contradiction to (14). Consequently, the element u is contained in $\tau_c N$. ∎

The case $\tau = \gamma$ and the idea of proving Lemma 3 is due to Witt [16].

Now we return to the *proof of the Theorems 3 and 4.* By (11) and (12) we have $\zeta(F/\tau_{c+1}N) \subseteq (\tau_c N/\tau_{c+1}N)$ and, of course, $\zeta(F/\tau_{c+1}N) \subseteq (\tau_c N/\tau_{c+1}N)^G$. The inverse inclusion is obvious. This proves the assertions (i), (ii) and (iii) of Theorem 4.

For finite G the free abelian group $(\mathfrak{g}_1 N)^G$ has at least two generators (see [7]). Hence, $(\mathfrak{t}_c N)^G \neq 0$ for $c \geq 1$, $\mathfrak{t} \in \{\mathfrak{g}, \mathfrak{k}, \mathfrak{l}\}$.

Now assume G to be infinite. Then, by Shmel'kin's Theorem (4.1) in [14], the centre of $F/\tau_{c+1}N$ is trivial. This gives assertion (i) of Theorem 3 and assertion (o) of Theorem 4. It remains to show the Theorems 1, 2, and 3 in the case of a finite group G only.

The homogeneous subalgebra $(\mathfrak{g}\,N)^G$ is the kernel of the G-module homomorphism

$$\mathfrak{g}\,N \longrightarrow \mathfrak{g}\,N : a \longrightarrow |G| \cdot a - \sum_{g \in G} a^g.$$

Since $\mathfrak{g}\,N$ is a free abelian group, the subgroup $(\mathfrak{g}\,N)^G$ turns out to be a $\mathbf{Z}$-direct summand of $\mathfrak{g}\,N$. By Witt's Theorem [17] on subrings of free Lie rings, these conditions are sufficient for $(\mathfrak{g}\,N)^G$ to be free with homogeneous free generating set. Thus, Theorem 1 is shown. ∎

Now we turn to the proof of Theorem 2. For this purpose we introduce $\mathfrak{h}_c N := \mathfrak{g}_c N \otimes \mathbf{F}_p = \gamma_c N/\gamma_{c+1} N(\gamma_c N)^p$. Then $\mathfrak{h}\,N = \oplus_{c\geq 1}\mathfrak{h}_c N$ is a free $\mathbf{F}_p$-Lie algebra on $X^\chi := \{x\gamma_2 N(\gamma_1 N)^p |\, x \in X\} = X^\kappa = X^\lambda$. The fixed point subalgebra $(\mathfrak{h}\,N)^G$ is homogeneous. Then by Shirshov's Theorem [13] $(\mathfrak{h}\,N)^G$ is free on a homogeneous subset: $(\mathfrak{h}\,N)^G = L^{\mathbf{F}_p}(Z^\chi)$ with $Z_c \subseteq \gamma_c N \setminus \gamma_{c+1} N(\gamma_c N)^p$, $Z_c^\chi = \{z\gamma_{c+1} N(\gamma_c N)^p |\, z \in Z_c\}$ and $Z^\chi = \cup_{c\geq 1} Z_c^\chi$.

Now consider the map

$$\begin{aligned} \epsilon : \mathfrak{h}\,N = L^{\mathbf{F}_p}(X^\chi) &\longrightarrow \mathfrak{k}\,N = R^{\mathbf{F}_p}(X^\kappa) \\ u\gamma_{c+1} N(\gamma_c N)^p &\longmapsto u\kappa_{c+1} N. \end{aligned}$$

Clearly, ϵ embeds the free $\mathbf{F}_p$-Lie algebra in the corresponding free restricted $\mathbf{F}_p$-Lie algebra. The map ϵ is obviously a G-module homomorphism. Consequently, we are in the position to apply Theorem 5. We get: $(\mathfrak{k}\,N)^G = R^{\mathbf{F}_p}(Z^\kappa)$ with $Z_c^\kappa = \{z\kappa_{c+1} N |\, z \in Z_c\}$ and $Z^\kappa = \cup_{c\geq 1} Z_c^\kappa$.

On the other hand, in case $p \neq 2$ we have

$$\mathfrak{l}\,N = L^{\mathbf{F}_p[\omega]}(X^\lambda) = \bigoplus_{i\geq 0} \omega^i \cdot L^{\mathbf{F}_p}(X^\lambda).$$

By (9), the G-action and the ω-action on $\mathfrak{l}\,N$ commute. Hence

$$\begin{aligned} (\mathfrak{l}\,N)^G &= \bigoplus_{i\geq 0} \omega^i \cdot \left(\left(L^{\mathbf{F}_p}(X^\lambda) \right)^G \right) = \bigoplus_{i\geq 0} \omega^i \cdot L^{\mathbf{F}_p}(Z^\lambda) = \\ &= L^{\mathbf{F}_p[\omega]}(Z^\lambda) \end{aligned}$$

with $Z_c^\lambda = \{z\lambda_{c+1} N |\, z \in Z_c\}$ and $Z^\lambda = \cup_{c\geq 1} Z_c^\lambda$ and this completes the proof of Theorem 2. ∎

Example. Define $\pi : F(t_1, t_2) \longrightarrow G : t_1 \longmapsto \rho; t_2 \longmapsto 1$ to be a presentation of the group $G = \{1, \rho, \rho^2\}$ of order 3. Take $X = \{x_1 = t_2;\ x_2 = t_1^{-1} t_2 t_1;\ x_3 = t_1^{-2} t_2 t_1^2;\ x_4 = t_1^3\}$.

Then $(x_1\gamma_2 N)^\rho = x_2\gamma_2 N$, $(x_2\gamma_2 N)^\rho = x_3\gamma_2 N$, $(x_3\gamma_2 N)^\rho = x_1\gamma_2 N$, $(x_4\gamma_2 N)^\rho = x_4\gamma_2 N$. Suppose $p = 3$. Then

$$|Y_1| = |Z_1| = 2$$
$$|Y_2| = |Z_2| = 1$$
$$|Y_3| = 2,\ |Z_3| = 3$$
$$|Y_4| = 9,\ |Z_4| = 7.$$

The last two lines show that there is no way of choosing a compatible Y and Z.

Our last duty is to give the *proof of Theorem 3(iii).* If G and X are finite, then $(\mathfrak{g}\, N)^G$ and $(\mathfrak{h}\, N)^G$ (and themselves $(\mathfrak{k}\, N)^G$ and $(\mathfrak{l}\, N)^G$) are not finitely generated by Theorem A of Bryant [5].

If G is finite and X is infinite we consider the relation sequence (see [8])

$$0 \longrightarrow N/N' \longrightarrow I_F \otimes_F \mathbb{Z}G \longrightarrow I_G \longrightarrow 0$$

with $I_G = \ker(\mathbb{Z}G \longrightarrow \mathbb{Z})$ the augmentation ideal of G. Then $(I_G)^G = 0$. Moreover, $(I_G \otimes \mathbf{F}_p)^G = 0$ if p does not divide $|G|$ and $(I_G \otimes \mathbf{F}_p)^G = \mathbf{F}_p \cdot \sum_{g\in G} g$ otherwise. We get the exact sequences

$$0 \longrightarrow (\mathfrak{g}_1 N)^G \longrightarrow (I_F \otimes_F \mathbb{Z}G)^G \longrightarrow 0$$

and

$$0 \longrightarrow (\mathfrak{h}_1 N)^G \longrightarrow (I_F \otimes_F \mathbf{F}_p G)^G \longrightarrow (I_G \otimes \mathbf{F}_p)^G.$$

The ranks of $(I_F \otimes_F \mathbb{Z}G)^G$ and $(I_F \otimes_F \mathbf{F}_p G)^G$ are equal to the rank of the free group F, which is supposed to be infinite. Hence, $(\mathfrak{g}_1 N)^G$ and $(\mathfrak{h}_1 N)^G$ are not finitely generated as abelian groups. This completes the proof of Theorem 3. ∎

REFERENCES

[1] I. V. Andozhskij, V. M. Tsvetkov, *On a series of finite closed p-groups*, Izv. Akad. Nauk SSSR Ser. Mat. **38**:2 (1974) 278-290 (Russian).

[2] M. Auslander, R. C. Lyndon, *Commutator subgroups of free groups*, Amer. J. Math. **77** (1955) pp. 929-931.

[3] Ju. A. Bakhturin, *Identities in Lie algebras*, Moscow 1985 (Russian).

[4] N. Bourbaki, *Groupes et algèbres de Lie*, Chap. I, II, Hermann, 1972.

[5] R. M. Bryant, *On the fixed points of a finite group acting on a free Lie algebra*, J. London Math. SOc. (2) **43** (1991) 215-224.

[6] R. M. Bryant, L. C. Kovacs, *Lie representations and groups of prime power order*, J. London Math. Soc. (2) **17** (1978) 415-421.

[7] W. Gaschütz, *Über modulare Darstellungen endlicher Gruppen, die von freien Gruppen induziert werden*, Math. Z. **60** (1954) 254-286.

[8] P. Hilton, U. Stammbach, *A course in homological algebra*, Graduate Texts in Math. **4**, Springer, 1971.

[9] B. Huppert, N. Blackburn, *Finite groups II*, Grundlehren Math. Wiss. **242**, Springer, 1982.

[10] M. Lazard, *Sur les groupes nilpotents et les anneaux de Lie*, Ann. Sci. Ecole Norm. Sup. **71** (1954) 101-190.

[11] W. Magnus, *Über Beziehungen zwischen höheren Kommutatoren*, J. Reine Angew. Math. **177** (1937) 105-115.

[12] W. Magnus, A. Karrass, D. Solitar, *Combinatorial group theory*, Wiley, 1976.

[13] A. I. Shirshov, *Subalgebras of free Lie algebras*, Mat. Sb. **33** (1953) 441-452 (Russian).

[14] A. L. Shmel'kin, *Wreath products and varieties of groups*, Izv. Akad. Nauk SSSR Ser. Mat. **29** (1965) 149-170 (Russian).

[15] A. I. Skopin, *The factor groups of an upper central series of free groups*, Dokl. Akad. Nauk SSSR Ser. Mat. **74** (1950) 425-428 (Russian).

[16] E. Witt, *Treue Darstellungen Liescher Ringe*, J. Reine Angew. Math. **177** (1937) 152-160.

[17] E. Witt, *Die Unterringe der freien Lieschen Ringe*, Math. Z. **64** (1956) 195-216.

[18] R. Zerck, *On the fixed points of the Lie ring associated with a free presentation of a finite group*, Bull. London Math. Soc. **21** (1989) 381-386.

Received February 28, 1990
Rainer Zerck
Cottbuser Platz 7
0-1150 Berlin, Germany