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ABSOLUTELY CONTINUOUS INVARIANT MEASURES FOR EXPANDING MAPS OF A d-DIMENSIONAL UNIT CUBE IN R^d AND THEIR ENTROPY

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Abstract. *In this paper we «construct» invariant measures with respect to expanding maps F of the d-dimensional cube in R^d. A lower bound for entropy is given for these maps and some examples are shown.*

0. INTRODUCTION

In this paper we analize the construction of a.c. invariant measures with respect to expanding maps F of the d-dimensional unit cube

$$Q = \{\underline{x} = (x_1, \ldots, x_d) \in R^d : 0 \leq x_i \leq 1, i = 1, \ldots, d\}.$$

It is well known the existence of invariant measures with respect to expanding maps on the unit interval [0,1] and absolutely continuous with respect to the Lebesgue measure ([2], [3], [4], [5], [6]); this work deals with the generalization to the d-dimensional case.

Indeed, in the special case when F is an expanding endomorphism on a finite dimensional manifold, it is known that it is a finite-to-one factor of the full one-sided k-shift, where $2 \leq k < \infty$ is the degree of the map F ([9,10]), and so there exists a unique ergodic F-invariant measure of maximal entropy (see also [11], [12]). However, this fact does not allow to construct explicitely the invariant measure.

Recently, in [13] the existence of a.c. invariant measures for piecewise expanding C^2 maps in R^d was proved, by using generalized bounded variation arguments, in a similar manner as Lasota and Yorke did in [3] for the one-dimensional case.

In this work, we use a different approach, that is the theory of Gibbs measures; we run over the construction of Gallavotti [2] of the invariant measure for one-dimensional expanding maps on the unit interval with the modifications of our case. Thus, the invariant measure turns out to be the Gibbs measure associated to a Markov partition of the dynamical system.

In section 1. we give some notations and definitions and construct the invariant measure; in section 2. we state and prove some bounds to the entropy of the dynamical system (for the one-dimensional case, see [1]); in section 3. we show some examples of bidimensional maps F : $[0,1] \times [0,1] \to [0,1] \times [0,1]$ which belong to the class of considered transformations.

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1. CONSTRUCTION OF THE INVARIANT MEASURE

We consider a map $F : [0,1]^d \to [0,1]^d$ with the following properties:

(1) let $\{Q_\sigma\}_{\sigma=0,\ldots,m} \subset [0,1]^d$ be a partition of $[0,1]^d$, that is a finite sequence of closed subsets such that $\cup_{\sigma=0,\ldots,m} Q_\sigma = [0,1]^d$, $\mathrm{int}(Q_\sigma) \cap \mathrm{int}(Q_{\sigma'}) = \emptyset$, $\sigma \neq \sigma'$ and let $\{F_\sigma\}\sigma = 0,\ldots,m$ be a finite sequence of Holder-continuous functions of order $1+\epsilon$, $\epsilon > 0$;

$$F_\sigma : Q_\sigma \to [0,1]^d \equiv Q,\ \sigma = 0,\ldots,m.$$

Indeed, we mean that F_σ is a C^1 map, and its derivatives $\partial_i(F_\sigma)_j$ are Holder-continuous functions of order at least ϵ.

For $x \in [0,1]^d$, we set $F(x) = F_\sigma(x)$ if $x \in \mathrm{int}(Q_\sigma)$; if $x \in \partial Q_\sigma \cap \partial Q_{\sigma'}$ we put $F(x) = F_\sigma(x)$ or $F(x) = F_{\sigma'}(x)$ (here $Q_{\sigma'}$ is a subset of the partition adiacent to Q_σ).

(2) For each $\sigma = 0,\ldots,m$ F_σ is a map of Q_σ onto $[0,1]^d$.

(3) $F : [0,1]^d \to [0,1]^d$ is an *expanding map*, that is real numbers $\lambda > 1$ and $\delta > 0$ exist such that $\mathrm{d}(F(x),F(y)) \geq \lambda \mathrm{d}(x,y)$ whenever $\mathrm{d}(x,y) < \delta$, where d is the euclidean metric.

We call $F : [0,1]^d \to [0,1]^d$ an EHS map (expanding, Holder-continuous of order $1+\epsilon$, surjective) if the previous conditions (1)-(2)-(3) are satisfied for F.

Remark 1.1. Indeed, condition (3) for some iterate of F is implied by any of the following equivalent but more easily verifiable conditions:

(3') $\forall x \in [0,1]^d$ all the eigenvalues of the Jacobian matrix $J_F(x)$ have modulus greater than 1.

(3") $\forall x \in [0,1]^d$ all the eigenvalues of the jacobian matrix of F^{-1} have modulus smaller than 1.

(3"') $|\partial((F_\sigma)^{-n})_j/\partial x_i| < 1$ for some n and all possible σ, i, j.

Remark 1.2. A C^1 map $F : X \to X$ of a compact, finite dimensional differentiable manifold X is called an expanding endmorphism if there are numbers $c > 0$, $\lambda > 1$ and a Riemannian metric on X for which $\| DF_x^n(v) \| \geq c\lambda^n \| v \|$, $\forall\ n > 0$, $\forall x \in X$, for all tangent vectors v. Here, DF_x denotes the tangent map of F at x.

If a different metric is choosen, than the same condition holds with a new choise of the constant c, and it is well known that one can choose a metric ρ for which $c = 1$ and so $\| DF_x^n(v) \|_\rho \geq \lambda \| v \|_\rho$, $\forall x$, $\forall n$, for all tangent vectors v (here $\| \ \|_\rho$ is the norm induced by the metric ρ).

Then, there exists $\delta > 0$ such that $\rho(x,y) < \delta \Rightarrow \rho(F(x),F(y)) \geq \lambda\rho(x,y)$.

So, if F is an expanding endomorphism, F turns out to be an expanding map in a suitable metric. In the special case when ρ is the euclidean metric, we have an expanding endomorphism on $[0,1]^d$ which is also an expanding map.

Remark 1.3. A special case of an expanding onto-map on $[0,1]^d$, which is holder-continuous of order α is given by:

$$F(\underline{x}) = F(x_1,\dots,x_d) = (S_1(x_1),\dots,S_d(x_d)),\ \underline{x} \in [0,1]^d \tag{1.1}$$

where:

a) $S_j : [0,1] \to [0,1]$ are expanding, holder-continuous maps of order α on the unit interval, for $j = 1,\dots,d$. That is: let $K^j_{\sigma_{ij}}$, $j = 1,\dots,d$; $\sigma_{i_j} = 0,1,\dots,m_j, m_j \in N$ be closed intervals such that:

$$\bigcup_{\sigma_{i_j}=0}^{m_j} K^j_{\sigma_{ij}} = [0,1],\ j = 1,\dots,d \qquad \text{and}$$

$$K^1_{\sigma_{i_1}} \times K^2_{\sigma_{i_2}} \times \dots \times K^d_{\sigma_{i_d}} = Q_\sigma,\ \sigma = \sigma_{i_1 i_2 \dots i_d} \in \{0,\dots,m\},$$

$$m = [\prod_{j=1}^{d}(m_j+1)] - 1;$$

we suppose that there exist d numbers $\lambda_1,\dots,\lambda_d > 1$ such that $\prod_{j=1}^d \lambda_j = \lambda$ and for every $j = 1,\dots,d$ there exist m_j functions $f^j_0, f^j_1,\dots,f^j_{mj} \in C^\alpha(K^j_{\sigma_{ij}})$, $f^j_{\sigma_{ij}} : K\sigma_{i_j} \to [0,1]$ such that:

a.1) $S_j(u) = f^j_{\sigma_{i_j}}(u)$ if $u \in \operatorname{int}(K^j_{\sigma_{ij}})$

a.2) $|S'_j(u)| > \lambda_j$ for every $u \in [0,1]$, $j = 1,\dots,d; \sigma_{ij} = 1,\dots,m_j$

b) the map S_j is onto $[0,1]$, $j = 1,\dots,d$.

Now we state and proof the main theorem of section 1.

Theorem 1.1. *If $F : [0,1]^d \to [0,1]^d$ is an EHS map, then there exists a measure μ on $[0,1]^d$ which is F-invariant and absolutely continuous with respect to the Lebesgue measure on $[0,1]^d$. Furthermore, μ is ergodic (and mixing).*

Proof. We recall that this theorem is the analogous of Proposition XXXIV in [2] for maps of the unit interval, so the proof and the notations are similar.

We consider $\{0,\dots,m\}^{Z+}$ and the «code»

$$X : \underline{\sigma} \to X(\underline{\sigma}) = \bigcap_{j=0}^{\infty} F^{-1} Q_{\sigma j},$$

where $\underline{\sigma} = (\sigma_j) \in \{0,\ldots,m\}^{Z+}$, $\sigma_j = 0,\ldots,m$, $j \in Z^+$, is such that $\sigma_j = \overline{\sigma}$ if $F_j(x) \in Q_{\overline{\sigma}}$, $x \in [0,1]^d$.

It is easy to see that:

$$Q^{0\ldots N}_{\sigma_0\ldots\sigma_N} = \bigcap_{j=0}^{N} F^{-1} Q_{\sigma_j} \equiv Q_{\sigma_0} \cap F^{-1} Q_{\sigma_1} \cap \ldots$$
$$\ldots \cap F^{-N} Q_{\sigma_N} = \varphi_{\sigma_0}\varphi_{\sigma_1}\ldots\varphi_{\sigma_N}([0,1]^d),$$

where: $\varphi_\sigma : [0,1]^d \to Q_\sigma$ is the inverse function of $F_\sigma, \sigma = 0,\ldots,m$. Moreover $Q^{0\ldots N}_{\sigma_0\ldots\sigma_N}$ is a compact set in $[0,1]^d$ with non empty interior.

Thus $X(\underline{\sigma}) = \bigcap_{j=0}^{+\infty} F^{-1} Q_{\sigma_j} = \bigcap_{N=0}^{+\infty} Q^{0\ldots N}_{\sigma_0\ldots\sigma_N}$ is a compact non empty set in $[0,1]^d$, because an infinite intersection of decreasing compact sets with non empty interior is a compact non empty set, and also:

$$\operatorname{diam}(Q^{0\ldots N}_{\sigma_0\ldots\sigma_N}) \le \lambda^{-N} \max_j(\operatorname{diam} Q_{\sigma_j}) = \text{const}\,\lambda^{-N},$$

since F is expanding.

Therefore the code X is an holder-continuous function and: $\mathrm{d}(X(\underline{\sigma}),X(\underline{\sigma}')) < \lambda^{-\nu}$, if ν is the greatest integer for which $\sigma_j = \sigma'_j$, $j = 0,1,\ldots,\nu$. So, if we define the metric in $\{0,1,\ldots,m\}^{Z+}$: $\mathrm{d}(\underline{\sigma},\underline{\sigma}') = e^{-\nu}$, we have:

$$\mathrm{d}(X(\underline{\sigma}),X(\underline{\sigma}')) \le \mathrm{d}(\underline{\sigma},\underline{\sigma}')^{\log\lambda} \tag{1.2}$$

and $X(\underline{\sigma})$ consists of only one point.

The Lebesgue measure m is codified by the code X in a measure ν on $\{0,\ldots,m\}^{Z+}$:

$$\nu(E) = m(X(E)) \qquad \forall E \in \mathscr{B}(\{0,\ldots,m\}^{Z+})$$

which is isomorphic to $m \ mod\ 0$. Moreover the measure ν is the gibbs measure on Z^+ with potential Φ such that:

$$\Phi_R(\underline{\sigma}_R) = 0 \quad \text{if} \quad R \neq \{a,\ldots,a+\rho\}\ \forall a, \forall\rho > 0 \tag{1.3}$$

$$\Phi_{\{a,\ldots,a+\rho\}}(\sigma_0\ldots\sigma_\rho) = A(\sigma_0\ldots\sigma_\rho 00\ldots00\ldots) - A(\sigma_0\ldots\sigma_{\rho-1}00\ldots00\ldots)$$

where $A(\sigma) = -\log|\det J_{\varphi_{\sigma_0}}(X(\sigma_0\sigma_2\ldots))|$ and J_{φ_σ} is the Jacobian matrix of $\varphi_\sigma : Q \to Q_\sigma$.

Indeed, if

$$\underline{\sigma}' = (\sigma'_0 \dots \sigma'_N \sigma_{N+1} \dots) = (\sigma'_0 \dots \sigma'_N \sigma_{N^c})$$
$$\underline{\sigma}'' = (\sigma''_0 \dots \sigma''_N \sigma_{N+1} \dots) = (\sigma''_0 \dots \sigma''_N \sigma_{N^c}),$$
$$x' = X(\underline{\sigma}') \in Q, x'' = X(\underline{\sigma}'') \in Q,$$

we have:

$$\frac{\nu(\sigma'_0 \dots \sigma'_N / \sigma_{N^c})}{\nu(\sigma''_0 \dots \sigma''_N / \sigma_{N^c})} =$$

$$(1.4) \qquad = \lim_{M\to\infty} \frac{\nu(\sigma'_0 \dots \sigma'_N / \sigma_{N+1} \dots \sigma_M)}{\nu(\sigma''_0 \dots \sigma''_N / \sigma_{N+1} \dots \sigma_M)} = \lim_{M\to\infty} \frac{m(\bigcap_{j=0,\dots,M} F^{-j} Q_{\sigma' j})}{m(\bigcap_{j=0,\dots,M} F^{-j} Q_{\sigma'' j})} =$$
$$= \lim_{M\to\infty} \frac{m(\varphi_{\sigma'_0} \dots \varphi_{\sigma'_N} \varphi_{\sigma_{N+1}} \dots \varphi_{\sigma_M}(Q))}{m(\varphi_{\sigma''_0} \dots \varphi_{\sigma''_N} \varphi_{\sigma_{N+1}} \dots \varphi_{\sigma_M}(Q))}.$$

Now, we consider the compact set $E_M = \varphi\sigma_{N+1} \dots \varphi\sigma_M(Q)$; it is easy to see that:

$$m(\varphi_{\sigma_0} \dots \varphi_{\sigma_N}(\varphi_{\sigma_{N+1}} \dots \varphi_{\sigma_M}(Q)) = m(\varphi_{\sigma_0} \dots \varphi_{\sigma_N}(E_M)) =$$
$$= \int_{E_M} \mathrm{d}x \prod_{j=0}^{N} |\det(J_{\varphi_{\sigma j}}(\varphi_{\sigma_{j+1}} \dots \varphi_{\sigma_N}(x)))|$$

for the chain rule, and so, by (1.4):

$$\frac{\nu(\sigma'_0 \dots \sigma'_N / \sigma_{N^c})}{\nu(\sigma''_0 \dots \sigma''_N / \sigma_{N^c})} =$$

$$(1.5) \qquad = \lim_{M\to\infty} \frac{\int_{E_M} \mathrm{d}x \prod_{j=0}^{N} |\det(J_{\varphi_{\sigma' j}}(\varphi_{\sigma' j+1} \dots \varphi_{\sigma' N}(x)))|}{\int E_M \mathrm{d}x \prod_{j=0}^{N} |\det(J_{\varphi_{\sigma'' j}}(\varphi\sigma''_{j+1} \dots \varphi\sigma''_N(x)))|}.$$

Finally, by the Vitali-Lebesgue theorem (see e.g. [8]), since $\mathrm{diam}(E_M) < \lambda^{-M-(N+1)}$, we obtain

$$\frac{\nu(\sigma'_0 \dots \sigma'_N / \sigma_{N^c})}{\nu(\sigma''_0 \dots \sigma''_N / \sigma_{N^c})} =$$

$$(1.6) \qquad = \prod_{j=0}^{N} \frac{|\det(J_{\varphi\sigma'_j}(\varphi_{\sigma'_{j+1}} \dots \varphi_{\sigma'_N}(X(\sigma_{N+1} \dots))))|}{|\det(J_{\varphi\sigma''_j}(\varphi_{\sigma''_{j+1}} \dots \varphi_{\sigma''_N}(X((\sigma_{N+1} \dots))))|} =$$

$$= \prod_{j=0}^{N} \frac{|\det(J_{\varphi_{\sigma'_j}}(X(\sigma'_{j+1}\dots\sigma'_N\sigma_{N+1}\dots)))|}{|\det(J_{\varphi\sigma_j{}''}(X(\sigma_{j+1}{}''\dots\sigma_N{}''\sigma_{N+1}\dots)))|} =$$

$$= \exp - \sum_{j=0,\dots,N} \{-\log|\det(J_{\varphi_{\sigma'_j}}(X(\sigma'_{j+1}\dots)))|+$$

$$+ \log|\det(J_{\varphi_{\sigma''_j}}(X_{\sigma''_{j+1}}\dots)))|\} =$$

$$= \exp - \sum_{j=0,\dots,N} \{A(\tau^j\underline{\sigma}') - A(\tau^j\underline{\sigma}'')\} =$$

$$= \exp \sum_{j\geq 0} \{A(\tau^j\underline{\sigma}') - A(\tau^j\underline{\sigma}'')\},$$

since $\sigma_i'' = \sigma_i''$, for $i > N$.

Now, we write $A(\underline{\sigma}) = A(0\dots0\dots0\dots) + \sum_{k\geq 0}\Phi_k(\sigma_0\dots\sigma_k)$, where the sum is the «cylindrical expansion» of the function $A(\underline{\sigma})$ by means of functions Φ_k which only depend on the first k coordinates σ_i, that is:

$$\Phi_k(\sigma_0\dots\sigma_k) = A(\sigma_0\sigma_k 00\dots0\dots) - A(\sigma_0\dots\sigma_{k-1}00\dots0\dots);$$

thus, we have:

$$(1.7)\qquad \begin{aligned} &\frac{\nu(\sigma'_0\dots\sigma'_N/\sigma_{N^c})}{\nu(\sigma_0{}''\dots\sigma_N{}''/\sigma_{N^c})} = \\ &= \exp - \sum_{j=0}^{+\infty}\sum_{k=0}^{+\infty}\{\Phi_k(\sigma'_j\dots\sigma'_{j+k}) - \Phi_k(\sigma''j\dots\sigma''_{j+k})\}. \end{aligned}$$

Therefore, the measure ν is the Gibbs measure on $\{0,\dots,m\}^{Z+}$ with potential Φ defined in (1.3), we have only to verify that the series $\sum_{k\geq 0}\Phi_k(\sigma_0,\dots,\sigma_k)$ converges. Indeed we have:

$$(1.8)\qquad \begin{aligned} &|\Phi_k(\sigma_0,\dots,\sigma_k)| = |\log|\det J_{\varphi\sigma 0}(X(\sigma_1\sigma_2\dots\sigma_k\underline{0}))|+ \\ &-\log|\det J_{\varphi\sigma 0}(X(\sigma_1\sigma_2\dots\sigma_{k-1}\underline{0}))|| \leq \\ &\leq [\inf{}_{u\in Q}|\det J\varphi_{\sigma 0}(u)|]^{-1}|(\det J\varphi_{\sigma 0}(X(\sigma_1\sigma_2\dots\sigma_k\underline{0}))|+ \\ &-|\det J_{\varphi\sigma 0}(X(\sigma_1\sigma_2\dots\sigma_{k-1}\underline{0}))|)| \leq \\ &\leq [\inf|\det J\varphi_{\sigma 0}(u)|]^{-1}\cdot C_{J\varphi\sigma 0}\,\|X(\sigma_1\sigma_2\dots\sigma_k\underline{0})+ \\ &-X(\sigma_1\sigma_2\dots\sigma_{k-1}\underline{0})\|^{\epsilon}, \end{aligned}$$

$C_{J_{\varphi_\sigma 0}}$ is a positive constant.

The last inequality follows by the fact that $\det J_{\varphi_\sigma}(u)$ turns out to be an holder-continuous function of order ϵ, as it is easy to see (naturally, with modulus equal to some suitable constant $C_{J_{\varphi_\sigma}} > 0$).

Thus we have:

$$(1.9) \qquad |\Phi_k(\sigma 0 \ldots \sigma_k)| < C\lambda^{-(k-1)\epsilon}$$

which implies:

$$\| \Phi \|_1 = \sum_{\substack{X \ni 0 \\ X \subset \{0,\ldots,n\}}} (\operatorname{diam} X) \sup_{\underline{\sigma}} |\Phi_X(\underline{\sigma})| < \infty$$

and so the Gibbs measure ν on $\{0,\ldots,m\}^{Z+}$ is uniquely determinated by its conditional probabilities (see [2]). Moreover, the Gibbs measure $\overline{\nu}$ with potential Φ on $\{0,\ldots,m\}^Z$ (i.e. on bilater sequences) restricted to the σ-field $\mathscr{B}(Z^+)$ is absolutely continuous with respect to $\overline{\nu}$ and it is invariant under τ (this follows by the τ-invariance of $\overline{\nu}$), where τ denotes the shift on $\{0,\ldots,m\}^Z$ such that $\tau(\underline{\sigma}) = F(X(\underline{\sigma}))$.

Finally, the measure μ on Q defined by:

$$(1.10) \qquad \mu(E) = \overline{\nu}(X^{-1}(E)) \qquad \forall E \in \mathscr{B}(Q)$$

turns out to be F-invariant and absolutely continuous with respect to the Lebesgue measure m on Q, as it easily follows by the properties of Gibbs measures (see [2]). This completes the construction of the invariant measure.

The ergodicity and mixing of the measure μ can be proved as in [2].

2. BOUNDS FOR THE ENTROPY

We recall (see sect. 1) that:

$$(2.1) \qquad \| \Phi \| = \sum_{X \ni 0} \sup_{\underline{\sigma}} |\Phi_X(\underline{\sigma})| = \sum_{X \ni 0} \| \Phi_X \| < \| \Phi \|_1 < \infty$$

$$(2.2) \qquad \begin{aligned} \Phi_k(\sigma_0 \ldots \sigma_k) &= A(\sigma_0 \ldots \sigma_k 000 \ldots) - A(\sigma_0 \ldots \sigma_{k-1} 000), \\ A(\sigma) &= A(000\ldots) + \sum_{k \geq 0} \Phi_k(\sigma_0 \ldots \sigma_k) \end{aligned}$$

and

$$|\Phi_k(\sigma_0 \ldots \sigma_k)| \leq \frac{\sup_\sigma C_{J_{\varphi_\sigma}}}{\inf_{\sigma,x} |\det J_{\varphi_\sigma}(x)|} \lambda^{-\epsilon(k-1)}. \tag{2.3}$$

If:

$$C = \frac{\sup_\sigma C_{J_{\varphi_\sigma}}}{\inf_{\sigma,x} |\det J_{\varphi_\sigma}|} \tag{2.4}$$

from (2.3) we have:

$$\sum_{k=0}(k+1)|\Phi_k(\sigma_0 \ldots \sigma_k)| \leq C \cdot f(\lambda, \epsilon) \tag{2.5}$$

and therefore $\| \Phi \|_1 \leq C \cdot f(\lambda, \epsilon)$, where

$$f(\lambda, \epsilon) = \lambda^\epsilon \left(\frac{\lambda^\epsilon}{\lambda^\epsilon - 1} + \frac{\lambda^\epsilon}{(\lambda^\epsilon - 1)^2} \right) = \frac{\lambda^{3\epsilon}}{(\lambda^\epsilon - 1)^2}, \tag{2.6}$$

$C_{J_{\varphi_\sigma}}$ is the holderianity modulus of the map $x \to |\det J_{\varphi_\sigma}(x)|$, $x \in Q$; λ is the minimum expansion coefficient.

Now we can enunciate the main result of this section:

Theorem 2.1. *Let T be an EHS map of the* d*-dimensional unit cube into itself (see sect. 1). Let C be the constant defined in (2.4). If:*

$$C < \frac{1}{2 f(\lambda, \epsilon)} \log(m+1), \tag{2.7}$$

where $m+1$ is the number of elements of the partition of $[0,1]^{\mathrm{d}}$; if $h(T)$ is the entropy of the transformation T (see [7]), then $h(T)$ is above and below bounded:

$$0 < \log(m+1) - 2 \| \Phi \| < h(T) < \log(m+1). \tag{2.8}$$

We will prove only the left side of (2.8); the right one follows from general properties (for the proof see e.g. [1] or [7]).

Proof. Consider the following limit:

$$\rho(\Phi) = \lim_{N \to \infty} N^{-1} \log \sum_{\sigma \in \{0, \ldots, n\}^T_{\Lambda_N}} \exp - \sum_{R \subset \Lambda_N} \Phi_R(\sigma_R), \tag{2.9}$$

where T is a compatibility $(m+1) \times (m+1)$ matrix and $\Lambda_N = \{1, \ldots, N\}$; $\rho(\Phi)$ is called «pressure».

The existence of the limit (2.9) is proved in [2], 20, propos. XXXVII.

We need a lemma also (see [2] propos. XXXVIII and coroll. XXXIX).

Lemma 2.1. *In the hypotheses of theorem 2.1, if μ is a Gibbs measure on $\{0,\dots,m\}^Z$ with potential Φ, then we have*

$$\rho(\Phi) \le h(T) - \mu(A_\Phi), \tag{2.10}$$

where $A_\Phi(\underline{\sigma}) = \sum_{x \ni 0} \Phi_\chi(\sigma_\chi)/|\chi|$.

Observe that:

$$\begin{aligned} |\mu(A_\Phi)| = |\int A_\Phi(\underline{\sigma}) \mathrm{d}\mu(\underline{\sigma})| \le |\int \sum_{\chi \ni 0} |\Phi_\chi(\sigma_\chi)| \mathrm{d}\mu(\sigma)| \le \\ \le \| \Phi \| \le \| \Phi \|_1 \le Cf(\lambda, \epsilon) \end{aligned} \tag{2.11}$$

(see also [2]). Since Φ is invariant with respect to translation, we have:

$$\begin{aligned} \sum_{R \subset \Lambda_N} \Phi_R \sigma(R) \le \| \sum_{R \subset \Lambda_N} \Phi_R \sigma(R) \| \le \sum |\Phi_R(\sigma_R)| \le \\ \le \sum_{\chi \in \Lambda_N} \sum_{R \ni \chi} |\Phi_R(\sigma_R)| \le N \| \Phi \| \end{aligned} \tag{2.12}$$

and hence:

$$\sum_{\sigma_0,\dots,\sigma_N=0}^{m-1} \exp\{-N \| \Phi \|\} \le \sum_{\sigma_0,\dots,\sigma_N} \exp\{-\sum_{R \subset \Lambda_N} \Phi_R(\sigma_R)\} \tag{2.13}$$

So, we obtain:

$$(m+1)^N \exp\{-N \| \Phi \|\} \le \sum_{\sigma_0,\dots,\sigma_N} \exp\{-\sum_{R \subset \Lambda_N} \Phi_R(\sigma_R)\}. \tag{2.14}$$

But, if we consider the logarithms of both sides of (2.14), divide by N and we calculate the limits for $N \to \infty$, we have, from (2.9):

$$\log(m+1) - \| \Phi \| \le \rho(\Phi). \tag{2.15}$$

Hence we have, from (2.10) and (2.11):

$$\mu(A_\Phi) + \log(m+1) - \| \Phi \| \le h(T). \tag{2.16}$$

Now, we know, from (2.5) and (2.6), that:

$$(2.17)\qquad \begin{aligned} \| \Phi \| &= \sum_{\chi \ni 0} \| \Phi_\chi \| = \sum_{\chi \ni 0} \sup_{\underline{\sigma}} | \Phi_\chi(\underline{\sigma}) | \le \\ &\le \sum_{K=0}^{\infty} (K+1) | \Phi_K(\sigma_0 \dots \sigma_K) | \le C f(\lambda, \epsilon), \end{aligned}$$

where C and $f(\lambda, \epsilon)$ are defined by (2.4), (2.6).

On the other hand, from (2.11) we have:

$$(2.18)\qquad \mu(A_\Phi) \ge - \| \Phi \| .$$

and, therefore, (2.16) can be written:

$$(2.19)\qquad h(T) \ge \log(m+1) - 2 \| \Phi \| .$$

Note that (2.17) and the hypothesis (2.7) assure the right side of (2.19) is greater than 0. Thus we have completely proved (2.8) and theorem (2.1).

3. SOME EXAMPLES IN R^2

Consider the markov-partition $\mathcal{S}$ on $[0,1] \times [0,1]$ for the EHS map $T : Q \to Q$; let $r+1$ be the number of the elements of $\mathcal{S}$: in Theorem 2.1 we proved that, if:

$$(3.1)\qquad C < \frac{1}{2 f(\lambda, \epsilon)} \log(r+1),$$

then:

$$(3.2)\qquad 0 < \log(r+1) - 2 \| \Phi \| \le h(T).$$

In this section we will give some examples of EHS maps in $Q \subset R^2$ that satisfy the condition (3.1) of theorem (2.1).

Example 3.1. Let $A = (a_{ij}) i, j = 1, 2$ be a 2×2 matrix with integer entries a_{ij}, such that all the eigenvalues of A have absolute value greater than one.

Then, the map $T : Q \to Q$ defined by

$$(3.3)\qquad T(x, y) = A(x, y) \ (\text{mod}.\ 1), (x, y) \in Q,$$

is an EHS map in Q. If we introduce the equivalence in $Q : (x,y) \sim (x',y') \iff \exists$ integers h,k such that $x - x' = h$, $y - y' = k$, we can consider the map T defined in (3.3) as a map on the torus $\mathbf{T}^2, T : \mathbf{T}^2 \to T^2$.

If $|\det A| = n \neq 1$, the map T is not injective; $\forall \Psi \in \mathbf{T}^2$ n distinct points $\varphi_1, \varphi_2, \dots, \varphi_n$ of $\mathbf{T}^2$ exist such that $T\varphi_i = \Psi$, $i = 1, \dots, n$. So, we can construct a partition of $\mathbf{T}^2$ with sets $Q_0, Q_1, \dots, Q_{n-1} \subset \mathbf{T}^2$ such that $T(Q_i) = \mathbf{T}^2$, $i = 0, \dots, n-1$.

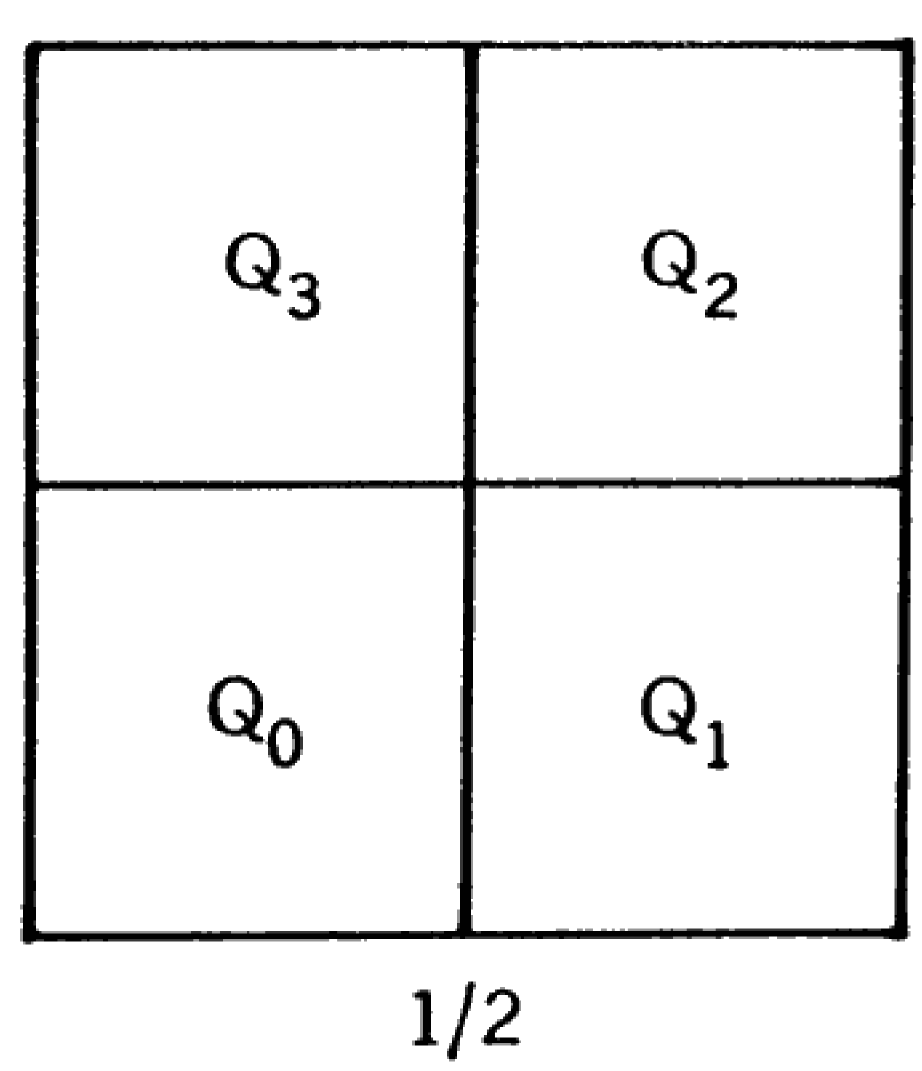
Since the Jacobian matrix of T, $J(x,y) \equiv A$ is constant, the potential Φ is zero, so we obtain by (3.2): $h(T) = \log n$, as it has to be.

The map (3.3) is the correspondent one in Q of an expanding piecewise linear map on $[0,1]$.

As an explicit example, we consider the matrix:

$$A = \begin{pmatrix} 2 & \epsilon \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} + \begin{pmatrix} 0 & \epsilon \\ 0 & 0 \end{pmatrix}, \text{ with } 0 < \epsilon < 2;$$

the eigenvalues of A are $\lambda_1 = \lambda_2 = 2$, so it is expandig. The partition of $\mathbf{T}^2$ associated to A is represented in the picture below:



The sets $T(Q_i)$, $i = 0, \dots, 3$ as subsets of R^2 are parallelogram of sides $1, \sqrt{1 + \epsilon^2/4}$; indeed on the torus $T(Q_i) = \mathbf{T}^2$. So $T = A$ is an EHS map.

Every map on the torus as the map (3.3) is called a linear expanding endomorphism of the torus $\mathbf{T}^2$; we note that such a map is an expanding map, too. We know that homotopic expanding endomorphisms are topologically conjugate; moreover if $F : \mathbf{T}^n \to \mathbf{T}^n$ is an expanding endomorphism, then F is topologically conjugate to a linear expanding endomorphism on $\mathbf{T}^n$ (see [10]).

Thus, since linear expanding endomorphisms of $\mathbf{T}^n$ have an invariant measure absolutely continuous with respect to the Lebesgue measure on the torus, we recover an a.c. measure invariant with respect to any expanding endomorphism of T^n. The expanding endomorphisms of $\mathbf{T}^n$ are examples of expanding maps on $\mathbf{T}^n$ in a suitable metric.

However, the so obtained invariant measure cannot be explicitely known, because one does not know the «factor» of conjugation.

Example 3.2. We consider the maps:

$$\begin{aligned} &T_{ij} = (x\sqrt{x} + mx - i, y\sqrt{y} + ny - j), \\ &i = 0, \ldots, m;\ j = 0, \ldots, n,\ m > 1,\ n > 1; \\ &T_{ij} : X \times Y \to X \times Y;\ X = [0,1] = Y; \end{aligned} \tag{3.4}$$

We will construct a map T which is the «cartesian product» of two maps:

$$\begin{aligned} &T_1 : X \to [0,1];\ T_2 : Y \to [0,1]; \\ &T = T_1 \times T_2 : X \times Y \to X \times Y, \text{ that is} \\ &T(x,y) = (T_1(x), T_2(y)),\ (x,y) \in Q = [0,1]^2. \end{aligned} \tag{3.5}$$

We say $h(x) = x\sqrt{x} + mx$ and observe that $h(x)$ is monotone increasing and $h(1) = 1 + m \geq 3$.

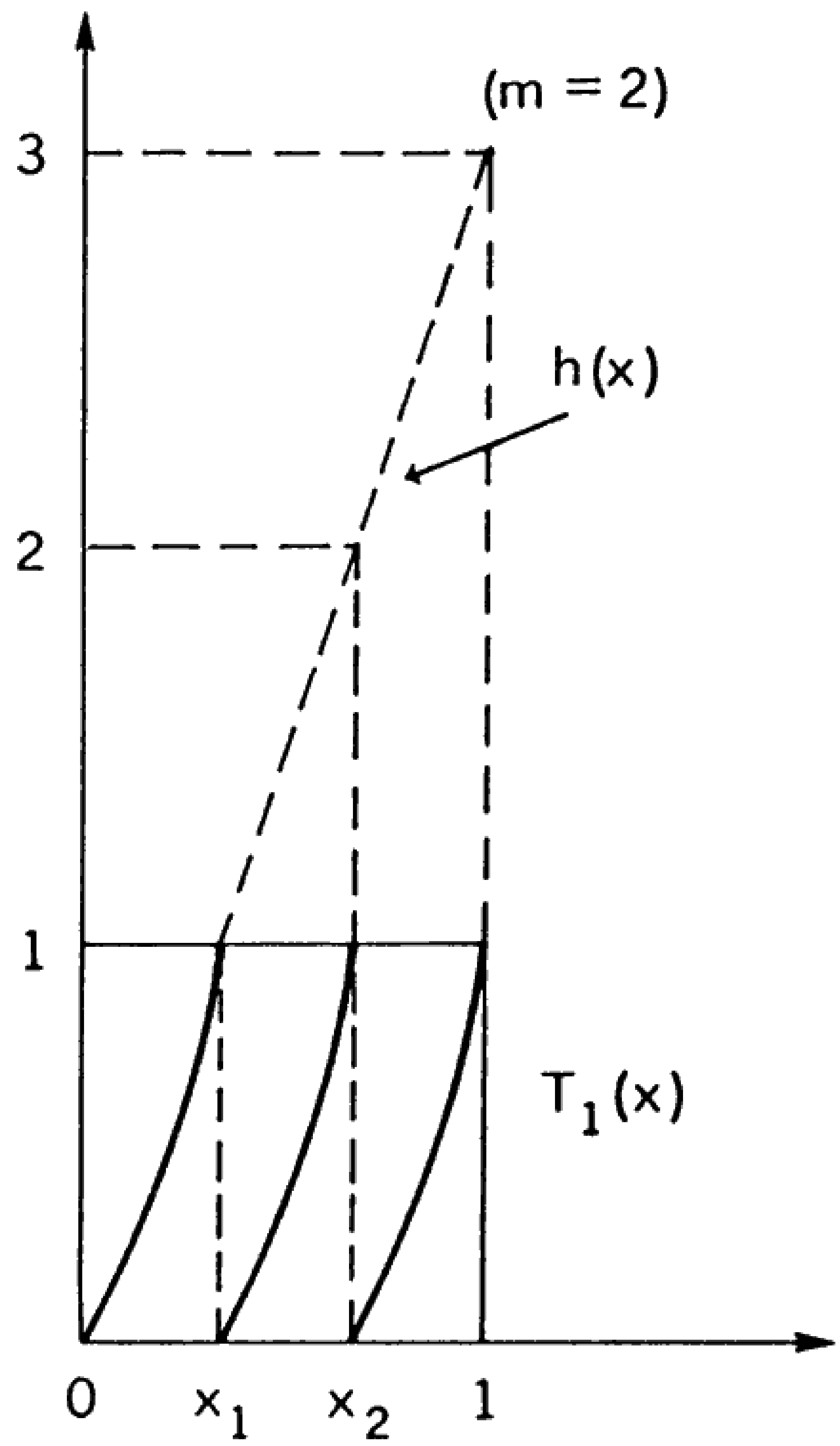
Then, we set $x_1 = h^{-1}(1)$, $x_2 = h^{-1}(2)$, $x_3 = h^{-1}(3) \ldots x_m = h^{-1}(m)$, $x_{m+1} \equiv 1 = h^{-1}(m+1)$.

So we obtain a partition of $X = [0,1]$ by means of the intervals:

$$P_i = [0, x_{i+1}) \subset [0,1],\ i = 0, 1, \ldots, m. \tag{3.6}$$

In the intervals $P_0, P_1, \ldots, P_m$ we define $m+1$ functions:

$$\begin{aligned} &f_0(x) = x\sqrt{x} + mx \\ &f_1(x) = x\sqrt{x} + mx - 1 \\ &f_2(x) = x\sqrt{x} + mx - 2 \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ &f_m(x) = x\sqrt{x} + mx - m, \end{aligned} \tag{3.7}$$



$f_i(x) : P_i \to [0,1]$, $i = 0, \ldots, m$, $f_i(x)$ is surjective; $f_i(x) = x\sqrt{x} + mx + i$ (see (3.4) and also the figure, for $m = 2$). In the same way we define $n+1$ intervals R_j, $j = 0, \ldots, n$ as in (3.6) and $n+1$ functions $g_j(y) : R_j \to [0,1]$ as in (3.7), substituting x by y and m by n.

Then we define a map $T : Q \to Q$ such that

$$T(x,y) \doteq T_{ij}(x,y) \doteq (f_i(x), g_j(y)) \text{ if } (x,y) \in P_i \times R_j;$$

the family $\{P_i \times R_j\}$, $i = 0, \ldots, m$; $j = 0, \ldots, n$ forms a partition of Q. We can consider T as a map: $\mathbf{T}^2 \to \mathbf{T}^2$ defined by:

$$T(x,y) = (x\sqrt{x} + mx, y\sqrt{y} + ny) \text{ (mod. 1)}, (x,y) \in T^2.$$

We observe that T is holder-continuous of order $1 + \epsilon$ with $\epsilon = 1/2$.

In fact T is $C^1(Q)$ and $\partial_i T_j$ is holder continuous of order 1/2 (this follows by the fact that the function $x\sqrt{x}$ is holder-continuous of order 1/2: as it is easy to see $|x\sqrt{x} - x'\sqrt{x'}| \leq |x - x'|^{1/2}$).

Now, let $\mathcal{S}$ be the Q partition obtained above and $\mathcal{S}_k = P_{ik} \times Q_{jk}$ an element of $\mathcal{S}$ (for some $i_k = 0, 1, \ldots, m$; $j_k = 0, 1, \ldots, n$), $(k = 0, \ldots, M \times N; M = m+1; N = n+1)$.

Let $F_k(x,y) = T_{i_k\, j_k}(x,y)$ (see (3.4))

$$F_k(x,y) : S_k \to [0,1] \times [0,1],\ F_k(x,y) = (f_{i_k}(x), g_{j_k}(y)).$$

Let $\varphi_k(x,y)$ be the inverse function of F_k, such that:

$$\varphi_k(x,y) : [0,1] \times [0,1] \to S_k.$$

Every function F_k is at least λ-expanding, and

$$\text{(3.8)} \qquad \begin{aligned} \det J_{F_k}(x,y) &= \begin{vmatrix} (3/2)\sqrt{x} + m & 0 \\ 0 & (3/2)\sqrt{y} + n \end{vmatrix} = \\ &= ((3/2)\sqrt{x} + m)((3/2)\sqrt{y} + n) \geq mn \geq 4, \end{aligned}$$

because $m \geq 2$, $n \geq 2$.

Hence we can assume $\lambda = mn$.

Then T is an EHS map. We have: (3.9)

$$\det J_{\varphi_k} = \frac{1}{\det J_{F_k}} = \frac{1}{((3/2)\sqrt{x} + m)((3/2)\sqrt{y} + n)} \leq \frac{1}{mn} < 1.$$

From (3.9) we have:

$$\text{(3.9b)} \qquad \begin{aligned} \inf_{\mathcal{S}} |\det J_{\varphi_k}(x,y)| &= \min_{\mathcal{S}} |\det J_{\varphi_k}(x,y)| = \frac{2}{(3+2m)(3+2n)} \\ \sup_{\mathcal{S}} |\det J_{\varphi_k}(x,y)| &= \max_{\mathcal{S}} |\det J_{\varphi_k}(x,y)| = \frac{1}{mn}. \end{aligned}$$

Now we have

$$\text{(3.10)} \qquad \begin{aligned} &|\det J_{\varphi_k}(x,y) - \det J_{\varphi_k}(x',y')| = \\ &= \left| \frac{1}{((3/2)\sqrt{x} + m)((3/2)\sqrt{y} + n)} - \right. \\ &\left. + \frac{1}{((3/2)\sqrt{x'} + m)((3/2)\sqrt{y'} + n)} \right|, \end{aligned}$$

and

$$\max_{\substack{x,x'\in[0,1]\\ y,y'\in[0,1]}} |\det J_{\varphi_k}(x,y) - \det J_{\varphi_k}(x',y')| =$$

(3.11)

$$= \left|\frac{1}{mn} - \frac{2}{(3+2m)(3+2n)}\right| = \frac{(3/2)(m+n+(3/2))}{mn(3+2m)(3+2n)}.$$

Now, from the holderianity condition of the map

$$(x,y) \to |\det J_{\varphi_k}(x,y)|,$$

we have:

(3.12)

$$|\det J_{\varphi_k}(x,y) - \det J_{\varphi_k}(x',y')| \leq$$
$$\leq C_{J\varphi_k} \mathrm{d}((x,y),(x',y'))^{1/2}.$$

Hence:

(3.13)

$$\frac{(3/2)(m+n+(3/2))}{mn(3+2m)(3+2n)} \leq C_{J\varphi_k} \mathrm{d}((x,y),(x',y'))^{1/2}$$

that is:

(3.14)

$$C_{J\varphi_k} \geq \frac{(3/2)(m+n+(3/2))}{mn(3+2m)(3+2n)\mathrm{d}((x,y),(x',y'))^{1/2}}$$

But, as we have the maximum of $\mathrm{d}((x,y),(x',x'))$ for $x=0, y=0$ and $x'=1, y'=1$, we obtain:

(3.15)

$$C_{J\varphi_k} \geq \frac{(3/2)(m+n+(3/2))}{mn(3+2m)(3+2n)(\sqrt{2})^{1/2}}.$$

Now, if we consider that:

(3.16)

$$\sup_k CJ_{\varphi_k} = \max_k CJ_{\varphi_k} = \frac{3/2(m+n+3/2)}{mn(3+2m)(3+2n)(\sqrt{2})^{1/2}},$$

we have, from (2.4), (3.9b) and (3.15)

(3.17)

$$C = \frac{(3/4)(n+m+(3/2))}{(\sqrt{2})^{1/2}mn}.$$

So if we take m, n large enough, we have:

$$C < \frac{\log(r+1)}{2f(mn, 1/2)}, r = M \cdot N; \; M = m+1 \; N = n+1. \tag{3.18}$$

Example 3.3. Let $T_0 : \mathbf{T}^2 \to \mathbf{T}^2$ be the map defined in (3.4), i.e. $T_0(x, y) = (x\sqrt{x} + mx, y\sqrt{y} + ny)$ (mod. 1) and let $\{S_0, S_1, \ldots, S_r\}$ be the partition of $\mathbf{T}^2$ associated to T_0; we consider a «perturbation» of the map T_0, defined by:

$$T\begin{pmatrix} x \\ y \end{pmatrix} = T_0\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 & \epsilon \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \text{ (mod. 1)}, \tag{3.19}$$

where $\epsilon > 0$ is small enough (indeed, we take it smaller than any vertical section of the sets S_i).

If T_0^1 and T_0^2 denotes the «components» of T_0, we have:

$$T(x, y) = (T_0^1(x) + \epsilon y, T_0^2(y)) \text{ (mod. 1)} \tag{3.20}$$

The Jacobian matrix of T is:

$$J(x, y) = \begin{pmatrix} (T_0^1)'(x) & \epsilon \\ 0 & (T_0^2)'(y) \end{pmatrix},$$

its eigenvalues are greater than one.

The map $T \in C^{1+1/2}(\mathbf{T}^2)$.

If $r + 1$ is the number of the elements of the partition S of Q associated to T_0 (see example 3.2), we obtain a partition with respect to T which consists of the same $r + 1$ sets: as for the sets Q_i of the example 3.1, $T(S_i)$ are parallelograms obtained by deformation of rectangles, and on the torus, $T(S_i) = \mathbf{T}^2$. So T is an EHS map.

For the map T (3.17), (3.18) hold, so the conditions of theorem (2.1) are satisfied.

If we consider the map:

$$\tilde{T}\begin{pmatrix} x \\ y \end{pmatrix} = T_0\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 & \propto \\ \beta & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \text{ (mod. 1)}, \tag{3.21}$$

for suitable $\propto, \beta$, an eventual partition of $\mathbf{T}^2$ associated to $\tilde{T}$ cannot have the same number of elements as the one associated to T_0 (because the det. of the Jacobian matrix changes). The previous sets S_i cannot be parts of any partition for which $\tilde{T}$ is an EHS map, because $T(S_i) \subset_{\neq} \mathbf{T}^2$.

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