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ON ASYMPTOTICALLY NORMABLE FRÉCHET SPACES

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Dedicated to the memory of Professor Gottfried Köthe

Let E be a Fréchet space, $\| \ \|_1 \leq \| \ \|_2 \leq \ldots$ a fundamental system of seminorms on E and $U_k = \{x \in E : \|x\|_k \leq 1\}$ for every k. E is called *asymptotically normable*, if there is a k_o such that for every $k \geq k_o$ there is a p so that the seminorms $\| \ \|_{k_o}$ and $\| \ \|_k$ define equivalent topologies on U_p. It is easy to see that in this case $\| \ \|_{k_o}$ is in fact a norm.

This class of spaces appears in investigations about the structure of Fréchet spaces and about the behaviour of their operators as a natural counterpart of the class of quasi-normable spaces introduced by Grothendieck [4]. While the quasi-normable Fréchet spaces E are those which admit an Ω-type condition (see[7]) and for which there exists a nontrivial Fréchet space F with $\mathrm{Ext}^1(F, E) = 0$ (see [8], [9], [13], [14]), the asymptotically normable Fréchet spaces E are those which admit a DN-type condition (see [13] and below) and for which there exists a nontrivial Fréchet space F with $\mathrm{Ext}^1(E, F) = 0$. Nontrivial here could mean: an infinite dimensional nuclear Köthe space.

In [7] it is shown that the quasi-normable spaces are the quotient spaces of standard spaces of the form

$$\lambda(A, E) = \left\{ x = (x_j)_j \in E^{\mathbb{N}} : \|x\|_k = \sum_j \|x_j\|_k a_{j,k} < +\infty \text{ for all } k \right\}$$

where E is a Banach space and $A = (a_{j,k})$ a matrix with $0 < a_{j,k} \leq a_{j,k+1}$ for all k and

$$\sum_j \frac{a_{j,k}}{a_{j,k+1}} < +\infty.$$

We show that the asymptotically normable spaces are the subspaces of these standard spaces. They are the smallest class of Fréchet spaces which contains the nuclear Köthe spaces with continuous norm, the Banach spaces and is closed under ε-tensor products and subspaces.

The main tool for that is Theorem 3.3. For Schwartz spaces asymptotic normability coincides with countable normability in the sense of Gelfand-Shilov. We show by an example that this even for Montel spaces is not the case.

1. We use the standard terminology of the theory of locally convex spaces as in [6]. For the theory of Ext and the splitting conditions we refer to [9], [13], [14]. By $(\| \ \|_k)$ we always denote an increasing sequence of seminorms defining the topology of a Fréchet space E and

$U_k = \{x \in E : ||x||_k \leq 1\}$. In this section we collect some simple results in order to see the concept of asymptotic normability in a proper perspective.

Let $\varphi : (0, \infty) \to (0, \infty$, be a strictly increasing function. We say E satisfies (DN_φ) if we have a k_o with the property that for every k there is a p and $C > 0$ so that the inequality

$$||x||_k \leq C\varphi(r)||x||_{k_o} + \frac{1}{r}||x||_p$$

holds for every $x \in E$ and $r > 0$ (see [11], [13]). In the case $\varphi(r) = r$ this condition, denoted by (DN), was introduced in [12] to the characterize initially the subspaces of the space (s) of rapidly decreasing sequences. Subsequently it was proved in [15] that every space of type (DN) is isomorphic to a subspace of $\ell_\infty(I)\widehat{\otimes}_\pi(s)$ for some index set I.

Consider also the following condition: there is a k_o such that for every $k \geq k_o$ we have $p \geq k_o$ with the following property: every sequence in E which is Cauchy with respect to $|| \ ||_p$ and converges to 0 with respect to $|| \ ||_{k_o}$ even converges to 0 with respect to $|| \ ||_k$.

This condition is equivalent to E being countably normable is the sense of Gelfand and Shilov (see [3]). Since, in view of Grothendieck [4], Lemma, p. 74, it is obvious that is differs from the condition of asymptotic normability only in so far as we have to replace «Cauchy» by «bounded», we have:

Proposition 1.1. *Every asymptotically normable Fréchet space is countably normable.*

Moreover, from [13], Lemma 5.6, and an adaption of [4], p. 176 f., we obtain

Proposition 1.2. *The following are equivalent for a Fréchet space E:*

(1) E is asymptotically normable.

(2) There is k_o such that for every k there is a p such that for every $\varepsilon > 0$ we can choose $M > 0$ with $|| \ ||_k \leq M|| \ ||_{k_o} + \varepsilon|| \ ||_p$.

(3) E has property (DN_φ) for some φ.

So the class of asymptotically normable spaces is the «union» of all classes (DN_φ) and, by means of [13], Theorem 5.5, has the properties concerning $\mathrm{Ext}^1(\cdot,\cdot)$ mentioned in the introduction.

Also from [13] we take the following supplement to Proposition 1.1.

Proposition 1.3. *A Fréchet-Schwartz space is asymptotically normable if and only if it is countably normable.*

The question whether these classes coincide in general we shall answer in the negative in the next section.

Before we do this let us recall that a Fréchet space E is called locally normable (see [10]) if there is a k_o such that on every bounded set B the topologies induced by E and by $\|\ \|_{k_o}$ coincide. Obviously every asymptotically normable space is locally normable. On the other hand every Fréchet-Montel space admitting a continuous norm is locally normable. Hence, in view of Proposition 1.3, every Fréchet-Schwartz space admitting a continuous norm but not countably normable (see [2], [16]) gives an example of a locally normable space which is not asymptotically normable (see also Section 2).

2. In this section we shall consider Köthe sequence spaces. Our main aim will be to construct a Köthe space $\lambda(B)$ which is Montel, admits a continuous norm, but still is not asymptotically normable. In particular, $\lambda(B)$ will be an example of a locally normable Köthe space which is not asymptotically normable.

By Proposition 1.3, $\lambda(B)$ cannot be a Schwartz space. In this context we would like to mention that the well-known example constructed by Köthe ([6];§31, 5) of a Montel space which is not a Schwartz space, is asymptotically normable by the following Proposition which is an immediate consequence of Proposition 1.2 and gives a characterization of asymptotically normable Köthe spaces in terms of the defining matrix.

Proposition 2.1. *A Köthe space $\lambda(A)$ is asymptotically normable if and only if there is k_o such that for every k we have p so that for every $\varepsilon > 0$ there is an $M > 0$ with*

$$a_{j,k} \leq M a_{j,k_o} + \varepsilon a_{j,p}, \quad j \in \mathbb{N}.$$

Example 2.2. *We define*

$$b_{i,j,v;k} = \begin{cases} (ijv)^k & \text{for } k \leq v \\ i^v (jv)^k & \text{for } v < k \leq j + v \\ i^{k-j}(jv)^k & \text{for } j + v < k \end{cases}$$

Then the Köthe space $\lambda(B)$ is Montel, admits a continuous norm but is not asymptotically normable.

Proof. Suppose for some k there is a p such that for every $\varepsilon > 0$ there is $M > 0$ with

$$b_{i,j,v;k+1} \leq M b_{i,j,v;k} + \varepsilon b_{i,j,v;p}$$

for all $(i, j, k) \in \mathbb{N}^3$. We may assume $p > k+1$, set $v = k+1$, $j = p - v$ and obtain

$$(ijv)^{k+1} \leq M(ijv)^k + \varepsilon i^{k+1}(jv)^p.$$

For $\varepsilon < (jv)^{k+1-p}$, we let i tend to infinity and get a contradiction. Hence the Köthe space $\lambda(B)$ is not asymptotically normable by Proposition 1.2.

Suppose we fix k_o and let $I \subset \mathbb{N}^3$ be an index set such that for all k we have

$$b_{i,jv;k} \leq \rho(k) b_{i,j,v;k_o}$$

for some $\rho(k) > 0$ and $(i, j, v) \in I$. If we show that I must be finite, then we have that $\lambda(B)$ is a Montel space ([6]; §30,9, (1)). Since the exponent of i is increasing with k, we have $jv \leq \rho(k_o + 1)$ for all $(i, j, v) \in I$. For any given j_o, v_o let $k > j_o + v_o + k_o$. If $(i, j_o, v_o) \in I$, we get

$$i^{k-j_o}(j_o v_o)^k \leq \rho(k) i^m (j_o v_o)^{k_o}$$

where m is some integer which is strictly less than $k-j_o$ in all cases. Hence $\{i : (i, j_o, v_o) \in$ $\in I\}$ is finite and so I must also be finite.

3. This section is devoted to the proof of our main theorem. First we state without proof a specialized and slightly modified version of a result of Vogt and Wagner ([17], Lemma 2.2).

Lemma 3.1. *Let $A = (a_{i,k:m})$ be a doubly indexed matrix and $A_K = (a_{i,k:m} + a_{i+1,k:m})$. We assume that with suitable sequences $i(m)$ and $s(m) \geq m$ we have the following:*

(1) $$a_{i,k;k} = 1 \qquad \text{for all } i, k.$$

(2) $$a_{i,k;m} \geq a_{i+1,k;m} \qquad \text{for all } i, m \leq k.$$

$$a_{i,k;m} \leq a_{i+1,k;m} \qquad \text{for all } i \geq i(m),\ m > k.$$

(3) $$\lim_i a_{i,k;m} = 0 \qquad \text{for all } m < k.$$

(4) $$\sup_k \sum_i \frac{a_{i,k;m}}{a_{i,k;s(m)}} < +\infty.$$

Then there exists an exact sequence

$$0 \to \lambda(A_K) \to \lambda(A) \to \omega \to 0.$$

In the characterization of quasi-normable Fréchet spaces given by Meise and Vogt [7], it was shown that there are sufficiently many exact sequences of the form $0 \to \lambda(\tilde{A}) \to$ $\to \lambda(A) \to \omega \to 0$ of nuclear Köthe spaces. We will now show that for a given Köthe-Schwartz space $\lambda(B)$ there is another such space $\lambda(A)$ and a surjection of $\lambda(A)$ onto ω whose kernel is isomorphic to $\lambda(B)$.

Proposition 3.2. *For every Köthe-Schwartz space $\lambda(B)$ with a continuous norm, there exists a Köthe-Schwartz space $\lambda(A)$ with a continuous norm and an exact sequence*

$$0 \to \lambda(B) \to \lambda(A) \to \omega \to 0.$$

Further, if $\lambda(B)$ is nuclear, then $\lambda(A)$ is also nuclear.

Proof. We assume without loss of generality $b_{j;1} > 0$ for all j and $\lim_j \frac{b_{j;m}}{b_{j;m+1}} = 0$ for all m.

We determine inductively disjoint, infinite sets of integers $M_k = \{n(i,k) : i = 1, 2, \ldots\}$ with $\cup M_k = \mathbf{N}$ such that the following are true:

$$\frac{b_{n(i,k);m}}{b_{n(i,k);k}} \geq \frac{b_{n(i+1,k);m}}{b_{n(i+1,k);k}} \quad \text{for } m < k \tag{5}$$

$$\frac{b_{n(i,k);m}}{b_{n(i,k);k}} \leq \frac{b_{n(i+1,k);m}}{b_{n(i+1,k);k}} \quad \text{for } m > k,\ i \geq m \tag{6}$$

$$\frac{b_{n(i,k);m}}{b_{n(i,k);m+1}} \leq 2^{-i} \qquad \text{for } m \leq i. \tag{7}$$

For a fixed k, the sequence $(n(i,k))_{i=1,2,\ldots}$ can be chosen inductively, starting with $n(1,k) = \min\left\{\mathbf{N} \setminus \bigcup_{l=1}^{k-1} M_l\right\}$ and such that $\left\{\mathbf{N} \setminus \bigcup_{l=1}^{k} M_l\right\}$ remains infinite. We now set

$$a_{i,k;m} = \begin{cases} \dfrac{b_{n(i,k);m}}{b_{n(i,k);k}} & \text{for } m < k \\[2ex] \dfrac{b_{n(i-1,k);m}}{b_{n(i-1,k);k}} & \text{for } m > k,\ i \geq 2 \\[2ex] 1 & \text{elsewhere}. \end{cases}$$

Since $a_{i,k;k} = a_{i+1,k;k} = 1$, we note that $(a_{i,k;m})$ is increasing in m. Then $(1) \ldots (4)$ of 3.1 Lemma are satisfied and hence we have an exact sequence $0 \to \lambda(A_K) \to \lambda(A) \to \omega \to 0$. One can check easily that $\lambda(A)$ is a Schwartz space or if $\lambda(B)$ is nuclear, then $\lambda(A)$ is also nuclear.

For $m < k$ and $m \geq k$ using (5) and (6) we estimate in different ways and obtain

$$\frac{b_{n(i,k);m}}{b_{n(i,k);k}} \leq a_{i,k;m} + a_{i+1,k;m} \leq 2\,\frac{b_{n(i,k);m}}{b_{n(i,k);k}}$$

for $i > m$. Therefore $\lambda(A_k)$ is isomorphic to $\lambda(B)$ by diagonal transformation.

Theorem 3.3. *A Fréchet space E is asymptotically normable if and only if there is an index set I and a nuclear Köthe space $\lambda(A)$ with a continuous norm such that E is isomorphic to a subspace of $\ell_\infty(I)\widehat{\otimes}_\pi\lambda(A)$.*

Proof. We have a nuclear Köthe space $\lambda(B)$ with a continuous norm such that $(E,\lambda(B)) \in (S_1^*)$ ([13]; Lemma 5.4). By the previous result we construct an exact sequence

$$0 \to \lambda(B) \to \lambda(A) \to \omega \to 0.$$

If E is a Banach space, our theorem is trivial. If E is a proper Fréchet space, then $(E,\lambda(B)) \in (S_1^*)_o$ by Lemma 3.3 in [14]. Now we choose a set I such that each Banach space E_k is isomorphic to a subspace of $\ell_\infty(I)$, where E_k is the completion of $(E, \| \|_k)$. We identify the tensor product $\ell_\infty(I)\widehat{\otimes}_\pi\lambda(B)$ with the space of all $y = (y_j)$, such that each $y_j \in \ell_\infty(I)$ and for every $k \in \mathbb{N}$ we have

$$\|y\|_k = \sup_j \|y_j\|_\infty b_{j;k} < +\infty$$

where $\| \|_\infty$ denotes the norm of $\ell_\infty(I)$. An obvious modification of Proposition 3.5 in [14] yields $(E,\ell_\infty(I)\widehat{\otimes}_\pi\lambda(B)) \in (S_1)$. Taking tensor product with $\ell_\infty(I)$ we get an exact sequence

$$0 \to \ell_\infty(I)\widehat{\otimes}_\pi\lambda(B) \to \ell_\infty(I)\widehat{\otimes}_\pi\lambda(A) \to \ell_\infty(I)^{\mathbb{N}} \to 0.$$

Since $\operatorname{Ext}(E,\ell_\infty(I)\widehat{\otimes}_\pi\lambda(B)) = 0$, the natural imbedding T of E into $\ell_\infty(I)^{\mathbb{N}}$ can be lifted to a continuous linear operator $\widehat{T} : E \to \ell_\infty(I)\widehat{\otimes}_\pi\lambda(A)$. It is easily checked that $\widehat{T}$ is one-to-one and has a closed range.

We would like to point out that the Köthe space $\lambda(A)$ we have constructed actually depends only on φ. This is so, because for each strictly increasing function $\varphi : (0,\infty) \to$ $\to (0,\infty)$, an examination of the proof of Lemma 5.4 in [13] reveals that there is a nuclear Köthe space $\lambda(B)$ such that $(E,\lambda(B)) \in (S_1^*)$ for any Fréchet space E which satisfies (DN_φ).

Finally, we would like to note that L. Holmström [5] proved that every nuclear countably normable (= nuclear asymptotically normable) Fréchet space is isomorphic to a subspace of some nuclear Köthe space which admits a continuous norm.

4. Finally we use the results of section 3 for another description of the class of asymptotically normable Fréchet spaces and for a comparison with the class quasi-normable Fréchet spaces. The first Lemma is obvious.

Lemma 4.1. *If E is asymptotically normable and $H \subset E$ a subspace then H is asymptotically normable.*

Lemma 4.2. *If E and F are asymptotically normable then so is $L_b(E'_b, F)$.*

Proof. The proof is straightforward if one notices that E being asymptotically normable is equivalent to: there is k_o such that for every k there is a p so that for every $\varepsilon > 0$ we can choose $M > 0$ with

$$U_k^o \subset M U_{k_o}^o + \varepsilon U_p^o$$

(cf. [12], Lemma 1.4).

From these Lemmas and Theorem 3.3 one obtains:

Theorem 4.3. *The class of asymptotically normable Fréchet spaces is the smallest class of Fréchet spaces containing the nuclear Köthe spaces with continuous norm (or the countably normable Schwartz spaces) the Banach spaces and is closed under ε-tensor products and subspaces.*

For comparison we recall that the class of quasi-normable Fréchet spaces is the smallest class of Fréchet spaces containing the nuclear Köthe spaces (or nuclear Köthe spaces with continuous norms, or Schwartz spaces, or countably normable Schwartz spaces) and the Banach spaces and is closed under π-tensor products and quotient spaces.

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