THE MACKEY DUAL OF A BANACH SPACE (*)

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Dedicated to the memory of Professor Gottfried Köthe

1. INTRODUCTION

If $X$ is a Banach space, then the Mackey topology on the dual space $X^*$ is the topology $\tau(X^*, X)$ of uniform convergence on weakly compact subsets of $X$. The classical Mackey-Arens Theorem tells us that $\tau$ is the finest locally convex topology on $X^*$ whose dual space is $X$.

The topology $\tau$ is finer than the weak* topology $\sigma(X^*, X)$ and coarser than the norm topology on $X^*$, with equality holding only in special cases. The weak topology $\sigma(X^*, X^{**})$ also lies between the weak* and norm topologies; $\tau$ and weak are usually not comparable, but their compact sets often are, as shown by Grothendieck [11] in his study of the Dunford-Pettis and reciprocal Dunford-Pettis properties.

In general, the Mackey topology on $X^*$ has received much less attention from functional analysts and topologists than its more famous relatives: norm, weak, and weak*. Recently, however, the authors have introduced and studied a new class of Banach spaces in which $\tau$ arises in a natural way [27]. A Banach space $X$ is strongly WCG(SWCW) if and only if there is a weakly compact subset $K$ of $X$ such that for every weakly compact subset $L$ of $X$ and $\epsilon > 0$, there is a positive integer $n$ with $L \subseteq nK + \epsilon B (B = \text{closed unit ball of } X)$. This is a properly stronger notion than the familiar WCG property, since every SWCG space is weakly sequentially complete.

The relevant fact is that $X$ is SWCG if and only if $(B^*, \tau)$, the dual unit ball with the relative Mackey topology, is (completely) metrizable [27, Th. 2.1]. Taking this as our starting point, we pursue here the topological study of $(X^*, \tau)$ and $(B^*, \tau)$. Two topological properties emerge as central. One is Michael's notion of an $R_0$-space [18]: remarkably, if any of $(B$, weak), $(X$, weak), $(B^*, \tau)$, or $(X^*, \tau)$ has the $R_0$-property, then so do all the others. This allows us to employ the powerful machinery of duality theory in a novel way. For example, it becomes easy to show that if $X$ is a separable SWCG space, then $(X$, weak) must be an $R_0$-space. The Banach space results in [18] can then be placed in a general context.

The other key topological notion is that of a $k$-space: a space in which the topology is determined by the compact subsets [6]. It is folklore that $(B$, weak) is a $k$-space iff $X$ contains no isomorphic copy of $l_1$; we prove more than this (Theorem 5.1). If $(B^*, \tau)$ is a

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k-space, then \( X \) is weakly sequentially complete. If \((X^*, \tau)\) is a k-space, then either \( X \) is reflexive or \( X \) is hereditarily \( l_1 \).

In general topology, the \( k \)-and \( \aleph_0 \)-spaces are precisely the quotients of separable metric spaces [18]. In the Banach space setting, more is true: \((B, \text{weak})\) is a \( k \)-and \( \aleph_0 \)-space iff it is separably metrizable (iff \( X^* \) is norm separable). Also, if \( X \) is isomorphic to its square, then \((B^*, \tau)\) is a \( k \)-and \( \aleph_0 \)-space iff it is separably metrizable (iff \( X \) is separable and SWCG). The point is that the \( k \)-and \( \aleph_0 \)-properties are often a «factorization» of the separable metric property in this context.

The Batt-Hierneyer space [3; 27, 2.6] is a separable, weakly sequentially complete dual space for which \((B^*, \tau)\) is neither a \( k \)-space nor an \( \aleph_0 \)-space. The space \( l_0(l_1) \) is a separable, weakly sequentially complete dual space for which \((B^*, \tau)\) is an \( \aleph_0 \)-spaces, but not a \( k \)-space. The permanence properties of \( k \)-spaces and \( \aleph_0 \)-spaces in this setting are also studied.

2. SOME FACTS ABOUT \( X^*, \tau \)

We use [5, 26, 32] as references for Banach spaces and locally convex spaces. Section 1 of [27] presents some special results about \((X^*, \tau)\). This space is considered explicitly in [11, 13, 15, 16, 31].

The term «operator» means continuous linear operator from one Banach space into another. An operator \( T : X \to Y \) is said to be a Dunford-Pettis operator (or completely continuous) iff it maps weakly compact sets to norm compact sets. Then \( X \) is said to have the Dunford-Pettis property (DP) iff every weakly compact operator \( T : X \to Y \) is a Dunford-Pettis operator. Similarly, \( X \) is said to have the reciprocal Dunford-Pettis property (RDP) iff every Dunford-Pettis operator \( T : X \to Y \) is weakly compact. If \( K \) is a compact Hausdorff space, then \( X = C(K) \) enjoys both DP and RDP. Basic references for this area are [4, 11].

**Proposition 2.1.** [11, p. 135, p. 152]. (a) \( X \) has DP iff every \( \sigma(X^*, X^{**}) \)-compact subset of \( X^* \) is \( \tau(X^*, X) \)-compact; (b) \( X \) has RDP iff every \( \tau(X^*, X) \)-compact subset of \( X^* \) is \( \sigma(X^*, X^{**}) \)-compact.

If (a) holds, then the weak and Mackey topologies agree on any weakly compact subset of \( X^* \), since both topologies are compact and finer than the Hausdorff topology \( \sigma(X^*, X) \). A similar result holds for \( \tau \)-compact sets in (b).

**Proposition 2.2.** [11, p. 134; 15]. A subset \( H \) of \( X^* \) is relatively \( \tau \)-compact iff every weakly convergent sequence in \( X \) converges uniformly on \( H \).

A proof of the next result can be found in [8].

**Proposition 2.3.** The following conditions on \( X \) are equivalent: (a) every Dunford-Pettis operator \( T : X \to Y \) is compact; (b) every \( \tau(X^*, X) \)-compact set is norm-compact; (c) \( X \)
contains no isomorphic copy of $l_1$.

**Corollary 2.4.** $X$ has the hereditary RDP iff $X$ contains no isomorphic copy of $l_1$.

**Proposition 2.5.** The following conditions on $X$ are equivalent: (a) every weakly compact operator $T : X \to Y$ is compact; (b) every $\sigma(X^*, X^{**})$-compact set is norm-compact (i.e., $X^*$ is a Schur space); (c) $X$ has DP, and $X$ contains no isomorphic copy of $l_1$.

**Proof.** The equivalence of (a) and (b) is left to the reader. For (b) $\iff$ (c), see [23; 4, pp. 23-24].

The final result of this section should be compared with [5, p. 223, Ex. 2]. Let $S(X^*)$ be the set of vectors in $X^*$ with norm 1.

**Proposition 2.6.** (a) If $X$ is reflexive, then $S(X^*)$ is $\tau$-closed in $X^*$; (b) if $X$ is not reflexive, then $S(X^*)$ is $\tau$-dense in $B(X^*)$; (c) if $X$ contains no isomorphic copy of $l_1$, then $S(X^*)$ is $\tau$-sequentially closed in $X^*$.

**Proof.** (a) Since $\tau$ is the norm topology. (b) Let $x^* \in X^*$, $\|x^*\| < 1$, and let $K$ be a weakly compact subset of $X$. For each $\epsilon > 0$, there is a member $y^*$ of $S(X^*)$ such that $\sup \{|y^*(x)| : x \in K\} < \epsilon$; otherwise, the closed absolutely convex hull of $K$ would contain $\epsilon \cdot B(X)$, and $X^*$ would be reflexive. A simple argument shows that either $\|x^* + y^*\| \geq 1$ or $\|x^* - y^*\| \geq 1$. Hence there is a scalar $t$, $|t| \leq 1$, such that $z^* = x^* + ty^* \in S(X^*)$, and $\sup \{|z^*(x) - x^*(x)| : x \in K\} < \epsilon$. (c) This follows from 2.3.

### 3. COMPACTNESS, METRIZABILITY, AND SEPARABILITY

It is well-known that $X$ with its weak topology is an angelic space [24], so that the various notions of compactness coincide. This need not be true for $(X^*, \tau)$. If $H \subseteq X^*$ is $\tau$-sequentially compact, then the $\tau$-closure of $H$ is $\tau$-totally bounded and $\tau$-complete [27, 1.1], hence $\tau$-compact. If $X = L_2[0, 1]$, then $H = B^*$ is $\tau$-compact, but not $\tau$-sequentially compact [13]. If $X$ is a WCG space, then $(B^*, \text{weak}^*)$ is an Eberlein compact [17, Th. 3.3]. Thus $(X^*, \text{weak}^*)$ is an angelic space, and so is $(X^*, \tau)$ [24, 0.5].

**Proposition 3.1.** The following are equivalent: (a) $(B^*, \tau)$ is compact; (b) $(B^*, \tau)$ is locally compact; (c) $(B^*, \tau)$ is $\sigma$-compact; (d) $X$ is a Schur space.

**Proof.** (a) $\implies$ (b) and (a) $\implies$ (c) are obvious.

(c) $\implies$ (a): $B^*$ is a countable union of $\tau$-compact sets, which are norm-closed. Thus one of these sets has a norm-interior point, and it follows by standard arguments that $(B^*, \tau)$ is compact.

(b) $\implies$ (d): Choose a weakly compact absolutely convex subset $K$ of $X$ such that $K^0 \cap B^*$ is $\tau$-compact. Let $x_n \to 0$ in $(X, \text{weak})$. Then $x_n \to 0$ uniformly on $K^0 \cap B^*$, by
2.2. Since \((K^0 \cap B^*)^0 = (K \cup B)^{00}\), for any \(\epsilon > 0\) there is a positive integer \(n_0\) such that \(x_n \in \epsilon(K \cup B)^{00}\) for all \(n \geq n_0\). Since \(K \subset mB\) for some \(m \geq 1\), we have \(x_n \in \epsilon mB\) for all \(n \geq n_0\). Thus \(\|x_n\| \to 0\), and so \(X\) is a Schur space.

(d) \(\Rightarrow\) (a): the Mackey and weak* topologies coincide on \(B^*\) [26, p. 85].

The space \((X, weak)\) is metrizable iff \(X\) is finite-dimensional; the space \((X^*, \tau)\) is metrizable iff \(X\) is reflexive (so that \(\tau = \text{norm topology on } X^*\)). It is well-known that \((B, weak)\) is metrizable iff \(X^*\) is norm separable. The question of complete metrizability of \((B, weak)\) is investigated in [7]. The Banach spaces for which \((B^*, \tau)\) is (completely) metrizable are precisely the strongly WCG space [27]. Examples include reflexive spaces, separable Schur spaces, and spaces \(L_1(\mu)\) for \(\mu\) a \(\sigma\)-finite measure [27, 2.3].

If \(X\) is separable, then \((X^*, weak^*)\) is a countable union of compact metric spaces, hence hereditarily separable. In this case, it follows from the Hahn-Banach Theorem that any absolutely convex subset of \(X^*\) is separable for the Mackey topology. However, \((B^*, \tau)\) need not be hereditarily separable. Kirk [16] studied \((X^*, \tau)\) for \(X = C(K)\), \(K\) a compact Hausdorff space, and showed that when \(K\) is first countable the natural image of \(K\) in \(B^*\) is \(\tau\)-closed and discrete. Thus \(X = C[0, 1]\) is a separable space for which \((B^*, \tau)\) is not hereditarily separable, and not Lindelöf. Indeed, since \((B^*, \tau)\) is separable with an uncountable closed discrete subset, it is not even normal.

If \(X = l_\infty\), then \((X^*, weak^*)\) contains \(l_1\) as a dense subspace, and so \((X^*, \tau)\) is separable, although \(X\) is not separable.

4. \(\mathcal{R}_0\)-SPACES

Topologists have studied a number of properties which fall under the heading of «generalized metric spaces» [12]. Among the most interesting of these is the notion of an \(\mathcal{R}_0\)-space introduced by Michael [18]. A collection \(\mathcal{P}\) of (not necessarily open) subsets of a topological space \(T\) is called a pseudobase for \(T\) if, whenever \(C \subset U\) with \(C\) compact and \(U\) open, then \(C \subset P \subset U\) for some \(P \in \mathcal{P}\). An \(\mathcal{R}_0\)-space is then a regular space with a countable pseudobase. A collection \(\mathcal{P}\) of subsets of \(T\) is called a \(k\)-network for \(T\) if, whenever \(C \subset U\) with \(C\) compact and \(U\) open, then \(C \subset P_1 \cup \ldots \cup P_n \subset U\) for some finite collection \(\{P_i\}\) of members of \(\mathcal{P}\). Clearly \(T\) is an \(\mathcal{R}_0\)-space if and only if it is regular and has a countable \(k\)-network.

For the convenience of the reader, we summarize key results from [18].

**Theorem 4.1 (Michael).** (a) All separable metric spaces and their regular quotient spaces are \(\mathcal{R}_0\)-spaces; (b) first countable or locally compact \(\mathcal{R}_0\)-spaces are separably metrizable; (c) every \(\mathcal{R}_0\)-space is separable, has the Lindelöf property, and every closed subset is a \(G_\delta\) set; (d) the class of \(\mathcal{R}_0\)-spaces is preserved by all subsets and by countable products; (e) if \((T, t)\) is an \(\mathcal{R}_0\)-space, and \(t'\) is a regular topology on \(T\) having the same compact sets as \(t\), then
(T, t') is an \( R_0 \)-space; (f) if \( S \) and \( T \) are \( R_0 \)-spaces, so is the continuous function space \( C(S, T) \) with the compact-open topology; (g) to show that \( T \) is an \( R_0 \)-space, it suffices (in the definition of pseudobase) to consider open sets \( U \) selected from a sub-base for the topology; (h) if a regular space \( T \) is covered by closed \( R_0 \)-subspaces \( (A_n)_{n \in \mathbb{N}} \), and if each compact subset of \( T \) is contained in some \( A_n \), then \( T \) is an \( R_0 \)-space; (i) a regular space is both an \( R_0 \)-space and a \( k \)-space if and only if it is a quotient of a separable metric space.

Our goal here is to incorporate these results into a Banach space setting. Already in [18] it was shown that

1) if \( X \) is a Banach space with separable dual, or if \( X = l_1 \), then \((X, \text{weak})\) is an \( R_0 \)-space;

and

2) if \( X = C(K), K \) a compact Hausdorff space, then \((X, \text{weak})\) is an \( R_0 \)-space if and only if \( K \) is countable.

According to 4.1 (e), we need only consider separable Banach spaces.

**Theorem 4.2.** If any of the spaces \((X, \text{weak}), (B, \text{weak}), (X^*, \tau)\) and \((B^*, \tau)\) has the \( R_0 \)-property, so do all the others.

**Proof.** The results 4.1 (d) and (h) show that \( X \) and \( B \) satisfy or fail the \( R_0 \)-condition together, with a similar outcome for \( X^* \) and \( B^* \). If \( S = (X, \text{weak}) \) is an \( R_0 \)-space and \( T = \mathbb{R} \), then \((X^*, \tau)\) is a subspace of \( C(S, T) \), endowed with the compact-open topology. Hence \((X^*, \tau)\) is an \( R_0 \)-space, by 4.1 (d) and (f).

Finally, if \( S = (X^*, \tau) \) is an \( R_0 \)-space, and \( T = \mathbb{R} \), let \( \gamma \) denote the topology on \( X \) of uniform convergence on compact subsets of \( S \). Then \((X, \gamma)\) is a subspace of \( C(S, T) \) with the compact-open topology and thus an \( R_0 \)-space. The topologies \( \gamma \) and \( \sigma(X, X^*) \) have exactly the same compact sets. This follows from a standard result in duality theory [26, p. 85], with \( E \) the complete locally convex space \((X^*, \tau), F = \mathbb{R}, H \) a weakly compact subset of \( L(E, F) = X \). Now 4.1 (e) shows that \((X, \text{weak})\) is an \( R_0 \)-space.

This remarkable duality has no analogue for the property of metrizability. If \( X = c_0 \), then \((B, \text{weak})\) is metrizable, since \( X \) has separable dual; but \((B^*, \tau)\) is not metrizable, since \( c_0 \) is not SWCG. If \( X = l_1 \), then \((B^*, \tau)\) is metrizable, since \( X \) is SWCG, but \((B, \text{weak})\) is not, since \( X^* \) is not separable. We will also see later (5.2) that there is no analogue for the property of being a \( k \)-space.

**Theorem 4.3.** If \( X \) is separable and SWCG, then \((X, \text{weak})\) is an \( R_0 \)-space.

**Dual Proof.** \((B^*, \tau)\) is separable and metrizable, hence an \( R_0 \)-space, by 4.1(a). The result follows from 4.2.
**Dual Proof.** (We include this to show how a pseudobase can be constructed explicitly). Let \((K_n)\) be a strongly generating sequence of weakly compact subsets of \(X\), and let \((f_m)\) be a countable dense subset of \((B^*, \tau)\). Consider the sub-base for the weak topology on \(X\) consisting of all sets \(f^{-1}(-\infty, q)\), where \(f \in X^*, \| f \| = 1\), and \(q\) is rational. With a view to applying 4.1(g), let \(L\) be a weakly compact subset of such a half-space \(f^{-1}(-\infty, q)\). Choose a rational \(\epsilon\) such that \(\max f(L) < q - \epsilon\), choose \(n\) such that \(L \subseteq K_n + (\epsilon/4)B\), and choose \(m\) such that \(\max \{|f(t) - f_m(t)| : t \in L \cup K_n\} < \frac{\epsilon}{4}\).

We claim that \(L \subseteq (f_m^{-1}(-\infty, q - \epsilon/2) \cap K_n) + (\epsilon/4)B \subseteq f^{-1}(-\infty, q)\). For the first inclusion, let \(x \in L\), and choose \(y \in K_n\), \(z \in B\) with \(x = y + (\epsilon/4)z\). Then \(f_m(y) < q - \epsilon/2\); for if not, then \(f(y) > f_m(y) - \epsilon/4 \geq q - 3\epsilon/4\), so that \(f(x) = f(y) + (\epsilon/4) f(z) \geq f(y) - \epsilon/4 > q - \epsilon\), a contradiction. A short calculation now verifies the second inclusion. According to [18, p. 986], the family of all finite unions of finite intersections of sets \((f_m^{-1}(-\infty, q_1) \cap K_n) + q_2 B\), for \(q_1\) and \(q_2\) rational, is a countable pseudobase for \((X, weak)\).

The next result unifies (and goes a bit further than) the Banach space results in [18].

**Proposition 4.4.** The class of Banach space \(X\) such that \((X, weak)\) is an \(\aleph_0\)-space includes all spaces with separable dual, separable Schur spaces, and separable \(L_1(\mu)\) spaces. It is preserved by closed subspaces, but not by quotients.

**Proof.** If \(X\) has separable dual, then \((B, weak)\) is separably metrizable, so 4.1(a) applies. The result for separable Schur spaces and separable \(L_1(\mu)\) spaces follows from 4.3 and [27, 2.3].

The property passes to any subset, by 4.1(d). The space \(l_1\) belongs to this class, but its quotient space \(C[0, 1]\) does not: \((B^*, \tau)\) is not Lindelöf, as noted in Section 3, so it cannot be an \(\aleph_0\)-space, by 4.1(c). The quotient map is also a quotient map for the weak topologies on \(l_1\) and \(C[0, 1]\) [26, p. 135]. Now a quotient space of a quotient space is a quotient space (direct verification), yet this example does not violate 4.1(a). The reason (which will follow from 4.1(i) and 5.4) is that \((l_1, weak)\) cannot be a quotient of a separable metric space.

We remark that \((**)\) can be easily established with the techniques used here. Let \(X = C(K)\). If \((X, weak)\) is an \(\aleph_0\)-space, then so is \((X^*, \tau)\), and so is \((X^*, weak)\), using 2.1 and 4.1(e). Thus \(X^*\) is weakly (hence norm) separable, by 4.1(c). Since the point masses at points of \(K\) have distance 2 apart in \(X^*\), this forces \(K\) to be countable. Conversely, if \(K\) is countable, then \(X^* = l_1(K)\) is separable, and so \((X, weak)\) is an \(\aleph_0\)-space, by 4.4.

**Proposition 4.5.** The class of Banach spaces \(X\) such that \((X, weak)\) (or \((X^*, \tau)\)) is an \(\aleph_0\)-space is preserved by countable \(\ell^p\)-sums, \(1 \leq p < \infty\), and by countable \(c_0\)-sums.
Proof. Let \( \{X_n\}^\infty_{n=1} \) be Banach spaces such that \((X_n, \text{weak})\) is an \(\aleph_0\)-space for all \(n\), and let \(X = l_p(X_n) = \{(x_n): x_n \in X_n, \sum^\infty_{n=1} ||x_n||^p < \infty\}\). For \(p = 1\), \((B(X^*), \tau(X^*, X))\) is the topological product of the \(\aleph_0\)-space \((B(X^*_n), \tau(X^*_n, X_n))\) [27, 1.2], so 4.1(d) can be applied.

Now let \(1 < p < \infty\). For each \(n\), let \(\mathcal{P}_n\) be a countable pseudobase for \((B(X_n), \text{weak})\). Let \(\mathcal{P}\) be the (countable) collection of all sets \(\{P_1 + P_2 + \ldots + P_m + Z_m\} \cap B(X)\), where \(P_i \in \mathcal{P}_i, 1 \leq i \leq m,\) and \(Z_m = \{x \in X: x_i = 0, 1 \leq i \leq m\}\).

We apply 4.1(g): let \(K\) be weakly compact in \(B(X)\), and consider \(x^* \in X^*\) and \(\alpha \in R\) such that \(K \subset U = \{x \in B(X): x^*(x) < \alpha\}\), a subbasic open set for the weak topology. Now \(X^* = l_q(x^*_n)\), where \(p\) and \(q\) are conjugate exponents. Hence there exist \(m\) and a rational number \(\beta\) such that if \(y^* = (x^*_1, x^*_2, \ldots, x^*_m, 0, 0, \ldots)\) is the \(m\)th truncate of \(x^*\), then \(K \subset V = \{x \in B(X): y^*(x) < \beta\}\) \(\subset U\).

Let \(\mathcal{B}\) be the (countable) collection of all \(m\)-tuples of rational numbers whose sum is \(\beta\). For each \(i, 1 \leq i \leq m,\) and rational \(q_i, \) let \(S(i, q_i) = \{x_i \in B(X_i): x^*_i(x_i) < q_i\}\) and \(T(i, q_i) = \{x \in B(X): x^*_i(x) < q_i\}\). Thus if \(\pi_i\) is the natural projection of \(B(X)\) onto \(B(X_i)\), \(T(i, q_i) = \pi_i^{-1}(S(i, q_i))\). For each \(C = (q_1^i, q_2^i, \ldots, q_m^i) \in \mathcal{B}\), let \(W_C = \cap_{i=1}^m T(i, q_i^i) \subset B(X)\). Then \(K \subset \cup_{C} W_C : C \in \mathcal{B} = V\), so \(\exists C_1, \ldots, C_p \in \mathcal{B}\) with \(K \subset \bigcup_{j=1}^{k} W_{C_j} \subset V \subset U\). It is a standard result that there exist weakly compact sets \(K_j\) with \(\bigcup_{j=1}^{k} K_j = K\) and \(K_j \subset W_{C_j}\) for all \(j\).

For each \(j, 1 \leq j \leq k,\) and \(i, 1 \leq i \leq m,\) choose \(P_{ij} \in \mathcal{P}_i\) with \(\pi_i(K_j) \subset P_{ij} \subset S(i, q_i^j)\). Then \(K_j \subset (P_{1,j} + P_{2,j} + \ldots + P_{m,j} + Z_m) \cap B(X) \subset W_{C_j}\), and so \(K \subset \bigcup_{j=1}^{k} (P_{1,j} + P_{2,j} + \ldots + P_{m,j} + Z_m) \cap B(X) \subset V \subset U\). The result follows.

The same argument works for countable \(c_0\)-sums, but not for countable \(l_\infty\)-sums, since the truncates of \(x^*\) do not converge to \(x^*\) in norm. Note that \((l_\infty, \text{weak})\) is not an \(\aleph_0\)-space, since it is not separable.

The Batt-Hiermeyer space 4.6. Let \(X = BH\) be the separable, weakly sequentially complete dual space introduced in [3]. The space \(BH\) is a tree space which behaves like \(l_1\) on totally ordered subsets of the binary tree, but like \(l_2\) on subsets of the binary tree whose members are pairwise incomparable. In [27, 2.6] it is shown that \(X\) is not an SWCG space. Here we present the deeper result that \((X, \text{weak})\) (equivalently, \((B^*, \tau)\) is not an \(\aleph_0\)-space (cf. 4.3).

It suffices to find uncountable collections \((K_A)_{A \in \Gamma}\) and \((U_A)_{A \in \Gamma}\) of compact and open subsets of \((X, \text{weak})\) such that \(K_A \subset U_A\) for all \(A\), but \(K_D \not\subset U_A\) for \(A \neq D\). For then if \((P_n)\) were a countable pseudobase for \((X, \text{weak})\), we would have \(K_A \subset P_{m(A)} \subset U_A\), so that there would be a sequence \((U_n)\) with each \(K_A\) contained in some \(U_n\), a contradiction.
Following the notation of [27, 2.6], let $\Gamma$ be the set of all infinite, convex, totally ordered subsets $A$ of $C$, the binary tree, which begin at $(0)$. Fix such an $A = \{t_n\}$, and let $x_n(A) = (1/2^n) e_{t_n} + e_{(t_n,i_n)}$, where $i_n = 0$ or $1$ is chosen so that $(t_n, i_n) \neq t_{n+1}$. Let $K_A = \{x_n(A)\} \cup \{0\}$, and let $U_A = \{x \in X : \exists m$ such that $|\langle x_m(A) - x, x_A^* \rangle| < \frac{1}{4}\}$. We show that $x_n(A) \to 0$ weakly. It will follow that $K_A$ is weakly compact, $U_A$ is weakly open, and $K_A \subset U_A$.

Let $P_A$ denote the natural projection $x \to x|A$ on $BH$. Then the sequence $(P_A(x_n(A)))$ converges to $0$ in norm, so it suffices to show that $(x_n(A) - P_A(x_n(A)))$ is equivalent to the unit vector basis of $l_2$. This is an easy consequence of the fact that the $(t_n, i_n)$ are pairwise incomparable: if $(c_j)$ is a sequence of scalars, then

$$
\| \sum_{j=1}^{m} c_j(x_j(A) - P_A(x_j(A))) \| = \left( \sum_{j=1}^{m} |c_j(x_j(A) - P_A(x_j(A)))(t_j, i_j)|^2 \right)^{1/2} = \left( \sum_{j=1}^{m} |c_j|^2 \right)^{1/2}.
$$

Fix $A, D \in \Gamma, A \neq D$, and let $s = \min \{ t \in C : t \in A, t \notin D \}$. Let $s_0$ be the immediate predecessor of $s$, and suppose $s_0$ is the $n$th member of $A$ (and $D$). Then $\langle x_A^*, x_n(D) \rangle = 1/2 + 1$, while $\langle x_A^*, x_m(A) \rangle \leq 1/2$ for all $m$. Thus $K_D \notin U_A$, to complete the argument.

5. $k$-SPACES AND TOPOLOGICAL FACTORIZATION

A Hausdorff space $T$ is a $k$-space iff a subset which intersects each compact set in a closed set must be closed. Equivalently, the topology on $T$ is the finest yielding the same collection of compact sets as itself. The class of $k$-spaces is extensive— it includes both locally compact and first countable spaces [6]. Its major defect is its failure to be preserved under finite products; this plays a role in 5.7.

The space $(X, \text{weak})$ cannot be a $k$-space unless $X$ is finite-dimensional. Indeed if the $k$-space property holds, then the topology $\gamma$ denied in the proof of 4.2 coincides with the weak topology. Thus every norm-convergent sequence in $X^*$, being $\tau$-compact, must have finite-dimensional linear span. Hence $X^*$ and $X$ are finite-dimensional.

The situation for $(B, \text{weak})$ is far more interesting. A Hausdorff space $T$ is said to be a Fréchet-Urysohn space iff whenever $p \in A \subset T$, then some sequence in $A$ converges to $p$. A space is Fréchet-Urysohn iff every subset is a $k$-space in the relative topology [2]. Also, $T$ is said to be a sequential space iff every sequentially closed subset is closed.

The following result seems to be well-known, but we have not found it recorded in this form. An early version appears in [9].
Theorem 5.1. The following conditions on a Banach space \( X \) are equivalent: (a) \((B, \text{weak})\) is a Fréchet-Urysohn space; (b) \((B, \text{weak})\) is a sequential space; (c) \((B, \text{weak})\) is a \( k \)-space; (d) \( X \) contains no isomorphic copy of \( l_1 \).

Proof. (a) \( \Rightarrow \) (b) \( \Rightarrow \) (c) is easily seen to hold for any Hausdorff topological space.

(c) \( \Rightarrow \) (d): Suppose \( l_1 \) is isomorphic to a subspace of \( X \). Then \((B(l_1), \text{weak})\) is homeomorphic to a closed subset of \((nB(X), \text{weak})\) for some \( n \), hence is a \( k \)-space. But this is clearly not true, since the weak and norm topologies on \( B(l_1) \) have the same compact sets.

(d) \( \Rightarrow \) (a): Let \( p \) belongs to the weak closure of \( A \) in \( B(X) \). Every Banach space in its weak topology has the property of countable tightness: there is a countable subset \( C \) of \( A \) whose weak closure contains \( p \) [32, p. 229]. Then \( Y \), the closed linear span of \( \{p\} \cup C \) in \( X \), is a separable space containing no isomorphic copy of \( l_1 \). By the Odell-Rosenthal Theorem [22], \((B(Y^*), \omega^*)\) is a pointwise compact set of Baire-1 functions on the compact metric space \((B(Y^*), \omega^*)\). Thus \((B(Y^*), \omega^*)\) is a Rosenthal compact in the sense of Godefroy [10], so it is a compact angelic space, and therefore has the Fréchet-Urysohn property. Since \((B(Y), \text{weak})\) is a subspace of \((B(Y^*), \omega^*)\), some sequence in \( C \) must converge weakly to \( p \).

In contrast to 4.2, the property of being a \( k \)-space is very far from being a «dual property».

Corollary 5.2. \((B, \text{weak})\) is a \( k \)-space if and only if every \( \tau \)-compact subset of \( X^* \) is norm-compact. Thus \((B, \text{weak})\) and \((B^*, \tau)\) are both \( k \)-spaces if and only if \( X \) is reflexive.

Proof. The first assertion is an immediate consequence of 2.3 and 5.1. The condition «every \( \tau \)-compact set is norm-compact» says that the \( k \)-space associated with \((B^*, \tau)\) is \((B^*, \text{norm})\). The second assertion now follows.

Corollary 5.3. The class of Banach spaces such that \((B, \text{weak})\) is a \( k \)-space is preserved by closed subspaces, quotients, and finite products.

Proof. This is immediate from 5.1 and the fact [4, p. 42] that the property «\( X \) contains no isomorphic copy of \( l_1 \)» is a three-space property. The families of \( k \)-spaces and Fréchet-Urysohn spaces are not preserved by two-fold products in general topology.

Factorization theorem for \((B, \text{weak})\). The following are equivalent: (a) \((B, \text{weak})\) is (separably) metrizable; (b) \((B, \text{weak})\) is a \( k \)-and-\( \aleph_0 \)-space; (c) \( X^* \) is separable.

Proof. The equivalence of (c) with either version of (a) is well-known, and the separable version of (a) clearly implies (b).

(b) \( \Rightarrow \) (c): By 5.2, the Mackey and norm topologies on \( X^* \) admit the same compact sets. By 4.1(e) and (4.2), \((X^*, \text{norm})\) is an \( \aleph_0 \)-space, so it is separable.
This is much stronger than the best available result in general topology (4.1(i)).

**Example 5.5.** Let $X$ be the James tree space [14], a separable Banach space which contains no isomorphic copy of $l_1$, yet has non-separable dual. Then since $X$ is itself a dual space, $(B, weak)$ is a $k$-space, and admits a coarser compact metric topology, but is not an $\aleph_0$-space.

We turn now to the $k$-space question for $(X^*, \tau)$ and $(B^*, \tau)$.

**Theorem 5.6.** If $X$ is either reflexive or a Schur space, then $(X^*, \tau)$ is a $k$-space.

**Proof.** If $X$ is a Schur space, then $\tau$ is the topology of uniform convergence on norm-compact subsets of $X$. By the Banach-Dieudonné Theorem [26, p. 151], $\tau$ is the finest topology (locally convex or not) which agrees with the weak* topology on weak*-compact sets. Thus $(X^*, \tau)$ is a $k$-space.

Towards a partial converse of this result, let $A_n$ be a space homeomorphic to $\{1/m\}_{m=1}^\infty \cup \{0\}$, for each positive integer $n$, such that $A_n \cap A_p = \{0\}$ for $n \neq p$. The hedgehog space $H$ is the quotient space of $\cup_{n=1}^\infty A_n$ obtained by identifying all the 0 points in the $A_n$. The space $H$ is a $k$-space, but is not first countable at 0. A result of Michael [19] shows, in particular, that for a metrizable space $T$, $T \times H$ is a $k$-space iff $T$ is locally compact. Hence if $T$ is a non-locally compact metric space, and $Y$ is a space containing a closed copy of $H$, then $T \times Y$ is not a $k$-space.

Now observe that for $X = l_1$, $(X^*, \tau)$ contains a closed copy of $H$. Indeed let $S = \{n e_m\}_{m=1}^\infty \cup \{0\} \subset l_\infty$, where $e_m$ denotes the $m$th unit vector. For fixed $n$, the sequence $\{n e_m\}_{m=1}^\infty$ is $\tau$-convergent to 0; hence there is a natural 1-1 correspondence between $S$ and $H$. Moreover, the correspondence preserves compact sets (in both directions). Now $H$ is a $k$-space, and so is $S$ (it is a closed subset of $(l_\infty, \tau)$, which is a $k$-space by 5.6). Hence the correspondence is a homeomorphism.

**Theorem 5.7.** If $(X^*, \tau)$ is a $k$-space, then either $X$ is reflexive or $X$ is hereditarily $l_1$ (i.e., every infinite-dimensional closed subspace contains an isomorphic copy of $l_1$).

**Proof.** First we show that if $Z$ is an infinite-dimensional closed subspace of $X$, then either (a) $Z$ is reflexive or (b) $Z$ contains an isomorphic copy of $l_1$. Now $(Z^*, \tau)$ is a quotient space of $(X^*, \tau)$ [26, p. 135]. Since $k$-spaces are exactly the quotients of locally compact spaces [6, p. 248], and a quotient of a quotient is a quotient, we have that $(Z^*, \tau)$ is a $k$-space. If $Z$ contains no isomorphic copy of $l_1$, then every $\tau$-compact set in $Z^*$ is norm-compact, by 2.3. Hence $\tau$ and norm must coincide on $Z^*$, so $Z$ is reflexive.

Now suppose that both alternatives (a) and (b) occur. Thus $X$ contains infinite-dimensional subspaces $Z_1$ and $Z_2$ which are reflexive and isomorphic to $l_1$, respectively. Then $Z_1$ and $Z_2$ are totally incomparable Banach spaces, so by [25, Th. 1], $Z_1 + Z_2$ is closed in $X$.
and isomorphic to $Z_1 \times Z_2$. Now $T = (Z_1^*, \text{norm})$ is a non-locally compact metric space, and $Y = (Z_2^*, \tau) = (l_\infty, \tau)$ contains a closed copy of $H$. Hence $T \times Y = ((Z_1 \times Z_2)^*, \tau)$ is not a $k$-space. This space is a quotient of $(X^*, \tau)$, so that space is not a $k$-space either, a contradiction. This completes the proof.

Theorem 5.8. If $(B^*, \tau)$ is a $k$-space, then $X$ is weakly sequentially complete.

Proof. Let $\gamma$ again denote the topology on $X$ of uniform convergence on $\tau$-compact subsets of $X^*$. We show that $(X, \gamma)$ is a complete locally convex space. Applying Grothendieck's Completeness Theorem, let $f$ be a linear functional on the topological dual space $(X, \gamma)' = X^*$ such that $f[H]$ is weak*–continuous for each $\tau$-compact $H \subset X^*$. Then $f$ is $\tau$–continuous on each compact subset of the $k$-space $(B^*, \tau)$ and so $f[B^*]$ is $\tau$–continuous. Then $f^{-1}(0) \cap B^*$ is $\tau$–closed and convex, hence weak*–closed, and so $f \in X$ [26, p. 149].

Now [31, 1.3, 1.4] shows that $\gamma$ is the finest locally convex topology on $X$ which has the same convergent (or Cauchy) sequences as the weak topology. The weak sequential completeness of $X$ follows immediately.

Since a metric space is a $k$-space, this is a stronger result than the fact that an SWCG space is weakly sequentially complete [27, 2.5]. Note that for $X = L_1[0,1]$, $(B^*, \tau)$ is a $k$-space, but $(X^*, \tau)$ is not ([27, 2.3] and 5.7). Also it can be shown that the Banach space $X_0$ constructed in [1], a separable, hereditarily $l_1$ SWCG space, fails to have $(X_0^*, \tau)$ a $k$-space. Thus the converse to 5.7 is false. We do not know if the converse to 5.6 is true.

If $X$ is infinite-dimensional and enjoys both the Dunford-Pettis and reciprocal Dunford-Pettis properties, then $(B^*, \tau)$ cannot be a $k$-space. Indeed the topologies $\tau(X^*, X)$ and $\sigma(X^*, X^{**})$ have the same compact sets, by 2.1. Thus $\sigma(X^*, X^{**})$ would be coarser than $\tau(X^*, X)$ on $B^*$, so $X$ would be reflexive. A reflexive space with $DP$ is finite dimensional.

Example 5.9. A weakly sequentially complete space $X$ such that $(B^*, \tau)$ is not a $k$-space. Let $X = C[0,1]^*$, the space of bounded regular Borel measures on $[0,1]$. Then $X$ can be expressed as an uncountable $l_1$-sum of separable $L_1(\mu)$ spaces. By [27, 1.2, 2.3], $(B^*, \tau)$ is homeomorphic to an uncountable product of complete separable metric spaces. Such a product is never a $k$-space unless all but countably many of the factors are compact [21], which is not the case here.

The examples $l_2(l_1)$ and $BH$ (4.5 and 4.6) are separable, weakly sequentially complete dual spaces for which $(B^*, \tau)$ is not a $k$-space (it is an $K_0$–space for the first of these, but not the second). For $l_2(l_1)$ this will follow from the results presented below (5.10, 5.11). The rather lengthy proof for $BH$ is omitted.

Theorem 5.10. Let $\{X_n\}_{n=1}^\infty$ be SWCG Banach spaces, and let $1 < p < \infty$. Then if $X = l_p(X_n), X$ is SWCG iff all but finitely many of the $X_n$ are reflexive.
Proof. The sufficiency is straightforward. For the necessity, we may suppose that $X_n$ is non-reflexive for all $n$. Let $K_0$ be a strongly generating, weakly compact subset of $X$, and let $K_n = \pi_n(K_0)$ for each $n$. Clearly $K_n$ is strongly generating for $X_n$.

For each $n$, we can choose $x_n \in B(X_n)$ such that $x_n \notin nK_n + \frac{1}{2} B(X_n)$. Otherwise, we would have $B(X_n) \subset nK_n + \frac{1}{2} B(X_n)$, so $\frac{1}{2} B(X_n) \subset \frac{n}{2} K_n + \frac{1}{2} B(X_n)$, and thus $B(X_n) \subset (n + \frac{n}{2}) K_n + \frac{1}{2} B(X_n)$, etc. An application of Grothendieck's Criterion [5, p. 227] shows that $B(X_n)$ is weakly compact, so $X_n$ is reflexive, a contradiction.

Now let $L = \{(0, 0, \ldots, x_n, 0, \ldots)\}_{n=1}^{\infty} \cup \{0\}$, a weakly convergent sequence in $X$. Since $K_0$ is strongly generating, there is a positive integer $m$ with $L \subset mK_0 + \frac{1}{2} B(X)$. But then $x_n \in mK_n + \frac{1}{2} B(X_n)$ for all $n$, a contradiction. \[\square\]

This should be compared with [27, 2.9].

It is natural to inquire if an analogue of 5.4 holds for $(B^*, \tau)$. In other words, if $(B^*, \tau)$ is a $k$-and-$\mathcal{R}_0$-space, must it be metrizable, so that $X$ is SWCG? Since $(X, \text{weak})$ will also be an $\mathcal{R}_0$-space under these conditions, the problem is restricted to separable $X$.

Although we have not solved this question completely, we are able to present some partial results, based on the penetrating topological work of Tanaka [28, 29, 30].

**Theorem 5.11.** Let $X$ be a separable Banach space such that $X$ is isomorphic to $X \times X$. Then $X$ is SWCG if and only if $(B^*, \tau)$ is a $k$-and-$\mathcal{R}_0$-space.

**Proof.** The necessity is clear, since $(B^*, \tau)$ is a separable metric space. If $D = (B^*, \tau)$ is a $k$-and-$\mathcal{R}_0$-space, and $T : X \to X \oplus_1 X$ is a linear homeomorphism, then $T^*$ is a homeomorphism of $D \times D$ onto a closed subset of a suitable multiple of $D$, so $D \times D$ is a $k$-and-$\mathcal{R}_0$-space. According to [29, Th. 1.1], either $D$ is a separable metric space or $D$ is $\sigma$-compact. In the latter case, $X$ is a separable Schur space, by 3.1, so $D$ is still metrizable. \[\square\]

Since $X = l_2(l_1)$ is isomorphic to its square, it follows from 4.5, 5.10, and 5.11 that $(B^*, \tau)$ is not a $k$-space in this instance.

**Theorem 5.12.** Let $X$ be a separable Banach space, and let $Y = l_1(X)$, the $l_1$-sum of countably many copies of $X$. Then $X$ is SWCG if and only if $(B(Y^*), \tau)$ is a $k$-and-$\mathcal{R}_0$-space.

**Proof.** The necessity follows from [27, 2.9]. Conversely, $(B(Y^*), \tau)$ is homeomorphic to a countable product of copies of $(B(Y^*), \tau)$, by [27, 1.2]. A $k$-and-$\mathcal{R}_0$-space is a sequential space [20, Th. 7.3]. A direct application of [28, Th. 1.3(i)] shows that $(B(Y^*), \tau)$ is metrizable, so $X$ is SWCG. \[\square\]

The remarkable result [30, Th. 4.4] reveals exactly how a $k$-and-$\mathcal{R}_0$-space $(B^*, \tau)$ could...
fail to be metrizable, if indeed this can occur at all. Let $H$ again denote the hedgehog space. Let $S_2 = (N \times N) \cup (N \times \{0\}) \cup \{(0,0)\}$, where each point of $N \times N$ is isolated; a base of neighbourhoods of $(n,0)$ consists of sets of the form $\{(n,0)\} \cup \{(n,m) : m \geq m_0\}$; and $U$ is a neighbourhood of $(0,0)$ if $(0,0) \in U$, and $U$ is a neighbourhood of all but finitely many $(n,0)$. Then $(B^*, \tau)$ is metrizable if and only if it contains no (closed) copy of $H$ or $S_2$. 
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