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ON SOME CLASS OF EQUATIONS WITH POTENTIAL OPERATORS

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Dedicated to the memory of Professor Gottfried Köthe

New conditions of solvability of quasi-linear integral Hammerstein type equations with symmetrical kernels are suggested. The conditions were obtained with the help of special integral-functional inequalities (see [1]). They make possible to enforce practically all the known before theorems on Hammerstein type equations which were obtained with variational methods.

1. NOTATIONS

In the main part of the paper the scale of spaces $L_p = L_p(\Omega, \mathbb{R}^1)$ $(1 < p < \infty)$ is used where ω is a closed bounded domain in $\mathbb{R}^N$.

Let $G(t,s) : \Omega \times \Omega \to \mathbb{R}^1$ denote a positive semidefinite symmetrical kernel which determines the linear integral operator

$$(1) \qquad Ax(t) = \int_\Omega G(t,s)x(t)\,d\mu(s).$$

Suppose the operator A is completely continuous in the space L_2 (with a standard scalar product $[\cdot,\cdot]$ and norm $\|\cdot\|$). In this case the eigenspace $E_0 \subset L_2$ which coincides with eigenvalue $x = \| A \|_{L_2 \to L_2}$ is finite-dimensional. The distribution functions

$$(2) \qquad \chi(\delta; e) = mes\{t; t \in \Omega,\ |e(t)| \le \delta\}$$

of elements $e(t) \in E_0$ are used. The positive semidefinite selfadjoint in L_2 square root of the operator A is denoted by K.

Let $f(t,x) : \Omega \times \mathbb{R}^1 \to \mathbb{R}^1$ denote a function measurable in t and continuous in x. Let $f(t,x)$ satisfy the double-side estimate

$$(3) \qquad |f(t,x)| \le c|x|^\alpha + b \qquad (t \in \Omega,\ x \in \mathbb{R}^1)$$

where $\alpha \ge 1$, $b, c > 0$ and the one-side estimate

$$(4) \qquad \int_0^x f(t,u)\,du \le \frac{1}{2}kx^2 - \Phi(t,|x|) \qquad (t \in \Omega,\ x \in \mathbb{R}^1)$$

where $\Phi(t,u)$ $(t \in \Omega, u \geq 0)$ is some scalar function belonging to a class $\mathscr{R}(u_0)$. Class $\mathscr{R}(u_0)$ consists of bounded superpositionally measurable (see [2]) functions $\Phi(t,u)$, which for $u \geq u_0$ satisfy the following conditions: $\Phi(t,u)$ do not increase in u; $\Phi(t,u)$ are nonnegative; $\Phi(t,u)$ are continuous in u; for each function $\Phi(t,u) \in \mathscr{R}(u_0)$ a subset $\Omega_0 \subset \Omega(mes\,\Omega_0 > 0)$ exists such that $\Phi(t,u) > 0$ for $t \in \Omega_0$, $u \geq u_0$.

If the condition (3) holds then the superposition operator

$$fx = f[t, x(t)] \tag{5}$$

acts from each space $L_p(p \geq \alpha)$ to space $L_{p/\alpha}$ (α is the number from (3)). In particular the operator (5) acts from space $L_{1+\alpha}$ to space $(L_{1+\alpha})^* = L_{1+\alpha^{-1}}$.

The estimate (3) and the operator (1) are called *connected* if the operator A acts from $(L_{1+\alpha})^*$ to $L_{1+\alpha}$ being completely continuous. In this case the operator K acts from L_2 in $L_{1+\alpha}$ being completely continuous. Adjoint to the operator $K : L_2 \to L_{1+\alpha}$ the operator K^* acts from $(L_{1+\alpha})^*$ to L_2 being completely continuous also.

2. MAIN RESULTS

Consider the integral equation

$$x(t) = \int_\Omega G(t,s) f[s, x(s)] \,\mathrm{d}\mu(s). \tag{6}$$

If the estimate (3) holds and the linear operator (1) is connected with it then the estimate (4) with $k < x$ and $-\Phi(t,u) = const > 0$ implies the existence of at least one solution of (6). This well-known result can be proved with the help of rather simple constructions. The ideas of these constructions belong to Hammerstein and Golomb ([3], [4]). The ideas are also used in this paper being supplemented with the use of a priori estimates of norms of solutions of new type inequalities.

Theorem 1. *Let the estimate (3) hold and the linear operator (1) be connected with this estimate. Let $\Phi(t,u) \in \mathscr{R}(u_0)$ for all $R > 0$ and $u_* \geq u_0$ satisfy the equality*

$$\lim_{\delta \to 0} \sup_{e(t) \in E_0; \|e\|=1} \frac{\chi(\delta; e)}{\int_\Omega \Phi\left[t, u_* + R\delta^{-1}|e(t)|\right] \mathrm{d}\mu} = 0. \tag{7}$$

Then a number $\varepsilon > 0$ exists such that the condition (4), where the coefficient k satisfies the inequality

$$kx < 1 + \varepsilon, \tag{8}$$

implies the existence of at least one solution $x(t) \in L_{1+\alpha}$ of (6).

The condition (7) seems cumbrous on the face of it. But in real situations it becomes simpler. Let us give an example for a function $\Phi(t,u) \equiv \Phi(u)$ $(u \geq u_0)$ and a one-dimensional subspace E_0. In this case the condition (7) can be rewritten as

$$\lim_{\delta\to 0} \frac{\chi(\delta;e)}{\int_0^\infty \Phi\left(u_0 + R\delta^{-1}\xi\right) \mathrm{d}\chi(\xi;e)} = 0. \tag{9}$$

If $\chi(\delta;e)$ satisfies the estimates

$$c_2\delta^\gamma \leq \chi(\delta;e) \leq c_1\delta^\gamma \qquad \left(\gamma > 0;\ 0 \leq \delta \leq \delta_0\right),$$

then (9) is equivalent to the equality

$$\int_{u_0}^\infty u^{\gamma-1}\Phi(u)\mathrm{d}u = 0.$$

If $\chi(\delta_0;e) = 0$ for some $\delta_0 > 0$ then (7) holds for any function $\Phi(t,u) \in \mathscr{R}(u_0)$.

Note that (7) implies the equalities $mes\,\{t : e(t) = 0\} = 0$ for $e(t) \in E_0$, $\|e\| \neq 0$.

The restrictions of Theorem 1 do not guarantee a priori estimate of norms of solutions of (1).

Theorem 1 is proved in the next section.

Analogous to the proof of Theorem 1 constructions can be used for proving different similar to Theorem 1 statements. For example some finite number of negative eigenvalues of the operator A is possible (see [5], [6]); systems of nonlinear integral equations can be considered; the restrictions of Theorem 1 can be weakened by attracting of other topologies (using Schäfer's ideas [7], developed by Petry, Zabreiko and other authors) and functional spaces different from L_p (see for example [2]).

Let us give one of the possible modifications of Theorem 1 in the complete form.

Theorem 2. *Let the kernel $G(t,s)$ and the function $f(t,x)$ be continuous with respect to the set of variables. Let $\Phi(t,u) \in \mathscr{R}(u_0)$ for all $R > 0$ and $u_* \geq u_0$ satisfy the equality (7). Then a $\varepsilon > 0$ exists such that the condition (4), where the coefficient k satisfies the inequality (4), implies the existence of at least one continuous solution $x(t)$ of (6).*

The estimate (3) is not used in Theorem 2.

3. PROOF OF THEOREM 1

The important role in the proof is played by a special statement on a priori estimates of norms of solutions of the integral-functional inequalities.

Consider a compact set $\mathscr{S} \subset L_2$ of functions $e(t) : \Omega \to \mathbf{R}^n$. With each function $e(t) \subset \mathscr{S}$ its distribution function (2) is considered.

Let us call the set $\mathscr{S}$ and the function $\Phi(t, u)$ *corresponding* to each other if for any $\beta > 0$ such a positive decreasing function $\alpha(u)$ $(u \geq 0)$ and such a number $c = c(\beta) > 0$ exist that the inequality

$$(10) \qquad \| h(t) \|^2 \leq -\beta \int_\Omega \Phi[t, |\xi e(t) + h(t)|] \mathrm{d}\mu + \beta \cdot \alpha(\| \xi e(t) + h(t) \|)$$

has not solutions $h(t) \in L_2$ for $e(t) \in \mathscr{S}$ and $|\xi| > c$.

In other words the set $\mathscr{S}$ corresponds to function $\Phi(t, u)$ if a positive decreasing function $\alpha(u)$ exists such that a priori estimate $|\xi| \leq c$ of the first component of all the solutions $\{\xi, h(t)\}$ of (10) for $e(t) \in \mathscr{S}$ is fulfilled.

Lemma 1. [1]. *Let $\Phi(t, u) \in \mathscr{R}(u_0)$ $(u_0 > 0)$ and for any $R > 0$*

$$(11) \qquad \lim_{\delta \to 0} \sup_{e(t) \in \mathscr{S}} \frac{\chi(\delta; e)}{\int_\Omega \Phi\left[t, u_0 + R\delta^{-1}|e(u)|\right] \mathrm{d}\mu} = 0.$$

Then the set $\mathscr{S}$ corresponds to the function $\Phi(t, u)$.

Let us pass now to the direct proof of Theorem 1. Consider in L_2 the operator equation

$$(12) \qquad x = K^* \mathfrak{f} K x.$$

Each solution $x(t)$ of (12) determines the solution $y(t) = Kx(t) \in L_{1+\alpha}$ of (6). Therefore to prove Theorem 1 it is sufficient to establish the existence of at least one solution of (12).

Consider the nonlinear functional (Golomb's functional)

$$\Gamma[x(t)] = \int_\Omega F[t, , Kx(t)] \mathrm{d}\mu$$

where

$$F(t, x) = \int_0^x f(t, z) \mathrm{d}z.$$

This functional is defined and weakly continuous in L_2. Consider the functional

$$V[x(t)] \stackrel{\text{def}}{=} \frac{1}{2}[x, x] - \Gamma[x(t)],$$

it is lower semicontinuous. If the functional $V[x(t)]$ is positive on some sphere $\{\| x \|_{L_2} = const\}$ then (as $V[0] = 0$) some point $x_* \in L_2$ is a point of its local *minimum*. In the point x_* the gradient

$$grad\, V[x(t)] = x - K^* \mathfrak{f}\, K x$$

of the functional $V[x(t)]$ is equal to 0 i.e. the point x_* is a solution of the equation (12). Such a scheme of the type (12) equations analysis was used in several works.

In the proof the projectors P and Q are used accordingly to the subspace E_0 and to the orthogonal addition E_1 $(E_0 \oplus E_1 = L_2)$. Let $q \in [0, 1)$ be a value $\frac{1}{\sqrt{x}} \| K \|_{E_1 \to E_1}$. The following relations are evident:

$$(13) \qquad \| KPx \| = \sqrt{x} \| Px \|, \quad \| KQx \| \leq q\sqrt{x} \| Qx \| \quad (x \in L_2).$$

Below we consider the set

$$(14) \qquad \mathcal{F} = \{e(t) : e(t) \in E_0, \| e \| = 1\}.$$

By Lemma 1 the set (14) corresponds to the function $\Phi(t, u)$. Therefore for any $\beta > 0$ such a positive decreasing function $\alpha(u)$ $(u \geq 0)$ and such a number $c = c(\beta) > 0$ exist that for all solutions $y(t) = \xi e(t) + h(t)$ $(e(t) \in \mathcal{F})$ of the inequality

$$(15) \qquad \| h(t) \|^2 \leq -\beta \int_\Omega \Phi[t, |y(t)|] d\mu + \beta \cdot \alpha(\| y(t) \|)$$

the following estimate holds:

$$(16) \qquad \| y(t) \| \leq c(\beta).$$

Below the number

$$(17) \qquad \beta_1 = x\varepsilon_1^{-1}$$

is used where $\varepsilon_1 = \frac{1}{2}(1 - q^2)$, let $c_1 = c(\beta_1)$. Suppose

$$M = \frac{2}{\varepsilon_1} \mu\Omega \cdot \sup_{t \in \Omega, u \geq 0} |\Phi(t, u)|,$$

$$\rho = \sqrt{2 c_1^2 + Mx} + 1.$$

Lemma 2. *Let the coefficient k in (4) satisfy (8) where*

$$\varepsilon = min\left\{\varepsilon_1, 2x \cdot \frac{\alpha(\rho)}{\rho^2}\right\}. \tag{18}$$

Then for all $x(t) \in L_2$ satisfying

$$\rho - 1 < \sqrt{x} \parallel x \parallel < \rho \tag{19}$$

the functional $V[x(t)]$ is positive.

The statement of Theorem 1 follows from Lemma 2: in the point $x(t) \equiv 0$ the functional $V[x(t)]$ is equal to 0, in the spherical layer (19) it is positive, hence in the ball $\{\sqrt{x} \parallel x \parallel < \rho - 1\}$ it takes its least value at a point $x_* \in L_2$ which is a solution of (12). We are to prove Lemma 2 for to complete the proof of Theorem 1.

Proof of Lemma 2. The chain of relations

$$V[x(t)] \geq \frac{1}{2} \parallel x \parallel^2 - \int_\Omega \left\{\frac{1}{2}k|Kx(t)|^2 - \Phi[t, |Kx(t)|]\right\} d\mu \geq$$
$$\geq \frac{1}{2} \parallel x \parallel^2 - \frac{1}{2}k \parallel Kx \parallel^2 + \int_\Omega \Phi[t, |Kx(t)|] d\mu \geq \frac{1}{2} \parallel Qx \parallel^2 +$$
$$+ \frac{1}{2} \parallel Px \parallel^2 - \frac{1}{2}\left(\frac{1}{x} + \frac{\varepsilon}{x}\right) \parallel Kx \parallel^2 + \int_\Omega \Phi[t, |Kx(t)|] d\mu \geq$$
$$\geq \frac{1}{2} \parallel Qx \parallel^2 + \frac{1}{2} \parallel Px \parallel^2 - \frac{1}{2} \parallel Px \parallel^2 - \frac{1}{2}q^2 \parallel Qx \parallel^2 -$$
$$- \frac{1}{2}\frac{\varepsilon}{x} \parallel Kx \parallel^2 + \int_\Omega \Phi[t, |Kx(t)|] d\mu \geq \frac{1}{2}(1 - q^2) \parallel Qx \parallel^2 -$$
$$- \frac{1}{2}\frac{\varepsilon}{x} \parallel Kx \parallel^2 + \int_\Omega \Phi[t, |Kx(t)|] d\mu \geq \varepsilon_1 \parallel Qx \parallel^2 -$$
$$- \frac{1}{2}\frac{\varepsilon}{x} \parallel Kx \parallel^2 + \int_\Omega \Phi[t, |Kx(t)|] d\mu$$

implies the inequality

$$V[x(t)] \geq \varepsilon_1 \parallel Qx \parallel^2 - \frac{1}{2}\frac{\varepsilon}{x} \parallel Kx \parallel^2 + \int_\Omega \Phi[t, |Kx(t)|] d\mu. \tag{20}$$

In this relations we used the determinations of ε and ε_1, the inequality $k \leq x^{-1}(1 + \varepsilon)$ for k and the estimates (13).

Let $V[x(t)] \leq 0$. Then (20) implies the inequality

$$\varepsilon_1 \parallel Qx \parallel^2 \leq \frac{1}{2}\frac{\varepsilon}{x} \parallel Kx \parallel^2 + \mu\Omega \sup_{t\in\Omega, u\geq 0} |\Phi(t,u)|$$

therefore

$$\varepsilon_1 \parallel Qx \parallel^2 \leq \frac{1}{2}\varepsilon \parallel x \parallel^2 + \frac{1}{2}\varepsilon_1 M$$

and

$$\varepsilon_1 \parallel Qx \parallel^2 \leq \frac{1}{2}\varepsilon_1 \parallel x \parallel^2 + \frac{1}{2}\varepsilon_1 M.$$

Thus

$$2 \parallel Qx \parallel^2 \leq \parallel x \parallel^2 + M$$

i.e. all $x(t)$ for which $V[x(t)] \leq 0$ satisfy the estimate

$$\parallel Qx \parallel^2 \leq \parallel Px \parallel^2 + M. \tag{21}$$

Let $V[x(t)] \leq 0$ and $\parallel x \parallel \leq \frac{1}{\sqrt{x}}\rho$. Then by (20) the inequality

$$\varepsilon_1 \parallel Qx \parallel^2 \leq \frac{1}{2}\frac{\varepsilon}{x} \parallel Kx \parallel^2 - \int_\Omega \Phi[t, |Kx(t)|] \mathrm{d}\mu$$

holds which implies the relation

$$\beta^{-1} \parallel KQx \parallel^2 \leq \frac{1}{2}\frac{\varepsilon}{x} \parallel Kx \parallel^2 - \int_\Omega \Phi[t, |Kx(t)|] \mathrm{d}\mu \tag{22}$$

where β is the number (17). But for $\parallel x \parallel \leq \frac{1}{\sqrt{x}}\rho$ the estimate $\parallel Kx \parallel \leq \rho$ holds therefore (22), by (18), implies

$$\beta^{-1} \parallel KQx \parallel^2 \leq \alpha(\rho) - \int_\Omega \Phi[t, |Kx(t)|] \mathrm{d}\mu$$

and

$$\parallel KQx \parallel^2 \leq -\beta \int_\Omega \Phi[t, |Kx(t)|] \mathrm{d}\mu + \beta\alpha(\parallel Kx \parallel). \tag{23}$$

Let us use the denominations: $y(t) = Kx(t)$, $e(t) = \sqrt{\varkappa} \cdot \frac{Px}{\|Px\|}$ for $\| Px \| \neq 0$, $e(t)$ is an arbitrary element from $\mathcal{F}$ for $\| Px \| = 0$, $h(t) = KQy(t)$, $\xi = \| Px \|$. With this denominations the inequalities (10) and (23) are equivalent i.e. for all x satisfying $V[x(t)] \leq 0$ and $\| x \| \leq \frac{1}{\sqrt{\varkappa}}\rho$ the estimate $\| y(t) \| \leq c_1$ ($\| Kx \| \leq c_1$) holds. But if $\| Kx \| \leq c_1$ then $\| Px \|^2 \leq c_1^2 \frac{1}{\sqrt{\varkappa}}$ and (by (20)) $\| x \|^2 \leq 2c_1^2 \frac{1}{\sqrt{\varkappa}} + M$. The last relation means that $\sqrt{\varkappa} \| x \| \leq \rho - 1$. So the inequalities $V[x(t)] \leq 0$ and $\| x \| \leq \frac{1}{\sqrt{\varkappa}}\rho$ imply the estimate $\| x \| \leq \frac{1}{\sqrt{\varkappa}}\rho - 1$.

Both Lemma 2 and Theorem 1 are proved.

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