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On left univocal factorizations of semigroups

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Abstract. We should like to construct all the semigroups that have a left univocal factorization with factors to pair (A, B) of prescribed semigroups such their intersection consists of only one element, that is right identity of A and left identity of B.

Keywords: Factorizable semigroup, Left univocal factorization.

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1 Introduction

Following Tolo [10], a semigroup S is said to be *factorizable* if it can be written as the set product AB of proper subsemigroups A and B. In this case, we call the pair (A, B) a *factorization* of S, with factors A and B. As in Catino [1], a factorization (A, B) of a semigroup S is called *left univocal* if $ab = a'b'$ implies $a = a'$, for all $a, a' \in A$ and $b, b' \in B$. There are several papers on this subject, for istance see [1, 2, 4, 5, 6, 7, 8, 9].

The main aim of the paper is to construct a semigroup S from a given semigroups A and B in such a way that (A, B) is a left univocal factorization of S where $A \cap B$ consists of only one element, that is right identity of A and left identity of B. The construction obtained generalizes those of Köhler [Theorem 2.1, [4]], of Krishnan [Theorem 3, [5]] on monoids and of Catino [Theorem 5, [1]] on univocal factorizations.

The notations and terminology of Howie [3] will be used, with a few minor exceptions.

2 Results and proofs

A well–known result about left univocal factorizations of semigroups is the following lemma (Lemma 1 of [1]).

Lemma 1. *Let* (A, B) *a left univocal factorization of a semigroup* S*. Then* $A \cap B$ *is a left zero semigroup formed by right identities of* A*.*

Then we can obtain an easy but useful result.

Proposition 1. *Let* (A, B) *a left univocal factorization of a semigroup* S*. If* E *is a non-empty subset of* $A \cap B$*, then* $\bar{B} := EB$ *is a subsemigroup of* S *and* $(A, \bar{B})$ *is a left univocal factorization of* S *with* $A \cap \bar{B} = E$*.*

Proof. Let E be is a non-empty subset of $A \cap B$. Put $\bar{B} := EB$, it easy to show that $(A, \bar{B})$ is a left univocal factorization of S and $E \subseteq A \cap \bar{B}$. Moreover, let z be an element of $A \cap \bar{B}$ and let $e \in A$ and $b \in B$ be such that $z = eb$. Then $zb = ebb$ and thus $z = e$, since (A, B) is a left univocal factorization of S. Therefore $A \cap \bar{B} = E$. QED

Corollary 1. *Let* (A, B) *a left univocal factorization of a semigroup* S *and* e *an element of* $A \cap B$*. Then* (A, eB) *is a left univocal factorization of* S *and* $A \cap eB = \{e\}$*. In this case e is a right identity of* A *and a left identity of* eB*.*

Note that we may consider left univocal factorizations (A, B) with $A \cap B = \{e\}$ and e non left identity of B (see [1], Example of p. 168). But, of course, also in this case we may consider a left univocal factorization $(A, \bar{B})$ in which the only element of the intersection of two factors is a left identity of $\bar{B}$.

We should now like to construct all semigroups that have a left univocal factorization with factors isomorphic to a pair (A, B) of prescribed semigroups where $A \cap B$ consists of only one element, that is a right identity of A and a left identity of B.

Let (A, B) a left univocal factorization of a semigroup S and let e be a right identity element of A and a left identity of B such that $A \cap B = \{e\}$.

Firstly, for each $a \in A$ define a relation θ_a on B by

$$b \, \theta_a \, b' :\Longleftrightarrow ab = ab'$$

for all elements b and b' of B. Then θ_a is an equivalence relation on B, even a right congruence. Moreover, if $|_l$ denotes left divisibility on A, i.e. $a \mid_l a'$ if there exists $c \in A$ such that $ca = a'$, then we have

$$\forall a, a' \in A \quad a \mid_l a' \Longrightarrow \theta_a \subseteq \theta_{a'} \tag{1}$$

$$\theta_e = id_B \tag{2}$$

For all $a \in A$ and $b \in B$, let $b \triangleright a$ be the element of A such that $ba = (b \triangleright a)b'$, for some $b' \in B$. Then, by the Axiom of Choise, there exists a mapping $\triangleleft$: $A \times B \longrightarrow B$ such that $ba = (b \triangleright a)((a, b)\triangleleft)$. If we denote $(a, b)\triangleleft$ by $b \triangleleft a$, then

$$ba = (b \triangleright a)(b \triangleleft a).$$

Without loss of generality we may assume that

$$e \triangleleft a = e \quad \text{and} \quad b \triangleleft e = be \tag{3}$$

for all $a \in A$ and $b \in B$. Moreover, we have

$$e \triangleright a = ea \quad \text{and} \quad b \triangleright e = e \tag{4}$$

Now, let $a_1, a_2 \in A$ and let $b_1, b_2 \in B$. Then

$$\begin{aligned}((b_1b_2) \triangleright a_1)((b_1b_2) \triangleleft a_1) &= b_1b_2a_1 = b_1(b_2 \triangleright a_1)(b_2 \triangleleft a_1) \\ &= (b_1 \triangleright (b_2 \triangleright a_1))(b_1 \triangleleft (b_2 \triangleright a_1))(b_2 \triangleleft a_1)\end{aligned}$$

and

$$\begin{aligned}(b_1 \triangleright (a_1a_2))(b_1 \triangleleft (a_1a_2)) &= b_1a_1a_2 = (b_1 \triangleright a_1)(b_1 \triangleleft a_1)a_2 \\ &= (b_1 \triangleright a_1)((b_1 \triangleleft a_1) \triangleright a_2)((b_1 \triangleleft a_1) \triangleleft a_2))\end{aligned}$$

It follows that

$$(b_1b_2) \triangleright a_1 = b_1 \triangleright (b_2 \triangleright a_1) \tag{5}$$

$$b_1 \triangleright (a_1a_2) = (b_1 \triangleright a_1)((b_1 \triangleleft a_1) \triangleright a_2) \tag{6}$$

$$(b_1 \triangleleft (a_1a_2))\, \theta_{b_1 \triangleright (a_1a_2)}\, ((b_1 \triangleleft a_1) \triangleleft a_2) \tag{7}$$

$$((b_1b_2) \triangleleft a_1)\, \theta_{(b_1b_2) \triangleright a_1}\, ((b_1 \triangleleft (b_2 \triangleright a_1))(b_2 \triangleleft a_1) \tag{8}$$

for all $a_1, a_2 \in A$ and $b_1, b_2 \in B$.
Finally, let $a_1, a_2 \in A$ and $b_1, b_1', b_2, b_2' \in B$ such that $a_1b_1 = a_1b_1'$ and $a_2b_2 = a_2b_2'$. Then

$$\begin{aligned}a_1(b_1 \triangleright a_2)(b_1 \triangleleft a_2)b_2 &= a_1b_1a_2b_2 = a_1b_1'a_2b_2' \\ &= a_1(b_1' \triangleright a_2)(b_1' \triangleleft a_2)b_2'\end{aligned}$$

Thus, we have that if $b_1\, \theta_{a_1}\, b_1'$ and $b_2\, \theta_{a_2}\, b_2'$, then

$$a_1(b_1 \triangleright a_2) = a_1(b_1' \triangleright a_2) \tag{9}$$

$$((b_1 \triangleleft a_2)b_2)\, \theta_{a_1(b_1 \triangleright a_2)}\, ((b_1' \triangleleft a_2)b_2') \tag{10}$$

Now, let A be a semigroup with right identity e_1 and let B be a semigroup with left identity e_2 such that $A \cap B = \{e\}$, and $e := e_1 = e_2$. Denote by $\mathcal{E}(B)$ the lattice of equivalence relation on B, let $\theta : A \longrightarrow \mathcal{E}(B)$, $\triangleright : A \times B \longrightarrow B$ and $\triangleleft : B \times A \longrightarrow B$. If $a \in A$ and $b \in B$, then denote $(a, b)\triangleright$ by $a \triangleright b$, $(b, a)\triangleleft$ by $b \triangleleft a$ and $a\theta$ by θ_a. The quintet $(A, B, \theta, \triangleright, \triangleleft)$ is called *admissible* if conditions (1)–(10) hold.
So far we have seen that any left univocal fattorization (A, B) of a semigroup S as above gives an admissible quintet. In the following we will show that these data suffice to reconstruct S from A and B.

Theorem 1. *Given a admissible quintet* $(A, B, \theta, \triangleright, \triangleleft)$ *there is a semigroup* S *with a left univocal factorization* $(\bar{A}, \bar{B})$ *such that* $\bar{A}$ *and* $\bar{B}$ *are isomorphic to* A *and* B*, respectively. Moreover,* $\bar{A} \cap \bar{B}$ *has only element* $\bar{e}$ *that is a right identity of* $\bar{A}$ *and a left identity of* $\bar{B}$*.*

Proof. Define on $A \times B$ the following equivalence relation

$$(a_1, b_1) \sim (a_2, b_2) :\Longleftrightarrow a_1 = a_2 \quad \text{and} \quad b_1 \theta_{a_1} b_2$$

for all $a_1, a_2 \in A,\ b_1, b_2 \in B$ and denote by $\langle a, b\rangle$ the equivalence class of (a, b), for all $a \in A$ and $b \in B$.
We define on $S := (A \times B)/\sim$ the following multiplication

$$\langle a_1, b_1\rangle\langle a_2, b_2\rangle := \langle a_1(b_1 \triangleright a_2), (b_1 \triangleleft a_2)b_2\rangle$$

for all $a_1, a_2 \in A,\ b_1, b_2 \in B$. By conditions (9) and (10) this in fact is well-defined.

To prove associativity, let $a, a_1, a_2 \in A,\ b, b_1, b_2 \in B$. Then

$$\begin{aligned}(\langle a, b\rangle\langle a_1, b_1\rangle)\langle a_2, b_2\rangle &= \langle a(b \triangleright a_1)(b \triangleleft a1)b_1\langle a_2, b_2\rangle \\ &= \langle a(b \triangleright a_1)(((b \triangleleft a_1)b_1) \triangleright a_2), (((b \triangleleft a_1)b_1) \triangleright a_2)b_2\rangle\end{aligned}$$

and

$$\begin{aligned}\langle a, b\rangle(\langle a_1, b_1\rangle\langle a_2, b_2\rangle) &= \langle a_2, b_2\rangle\langle a(b_1 \triangleright a_2)(b_1 \triangleleft a_2)b_2 \\ &= \langle a(b \triangleright (a_1(b_1 \triangleright a_2))), (b \triangleleft (a_1(b_1 \triangleright a_2)))(b_1 \triangleleft a_2)b_2\rangle\end{aligned}$$

By (5) and (6) we have

$$a(b \triangleright a_1)(((b \triangleleft a_1)b_1) \triangleright a_2) = a(b \triangleright (a_1(b_1 \triangleright a_2))).$$

We remark that by (10), applied to $a_2 := e$, and by (3),(4) we have that θ_{a_1} is a right congruence, for all $a_1 \in A$.
By this and by (8),(1) we have

$$(((b \triangleleft a_1)b_1) \triangleright a_2)b_2 \ \ \theta_{a(b \triangleright a_1)(((b \triangleleft a_1)b_1) \triangleright a_2)} \ (((b \triangleleft a_1) \triangleleft (b_1 \triangleright a_2))(b_1 \triangleleft a_2))b_2.$$

Similary, by (7) and (1) we have

$$(b \triangleleft (a_1(b_1 \triangleright a_2)))(b_1 \triangleleft a_2)b_2 \ \ \theta_{a(b \triangleright (a_1(b_1 \triangleright a_2)))} \ (((b \triangleleft a_1) \triangleleft (b_1 \triangleright a_2))(b_1 \triangleleft a_2))b_2.$$

This finally shows

$$(b \triangleleft (a_1(b_1 \triangleright a_2)))(b_1 \triangleleft a_2)b_2 \ \ \theta_{a(b \triangleright (a_1(b_1 \triangleright a_2)))} \ (b \triangleleft (a_1(b_1 \triangleright a_2)))(b_1 \triangleleft a_2)b_2.$$

Thus the above multiplication is associative.
It is easily seen by (2),(3) and (4) that the mappings

$$\alpha : A \longrightarrow S,\ a \longmapsto \langle a, e\rangle \quad \text{and} \quad \beta : B \longrightarrow S,\ b \longmapsto \langle e, b\rangle$$

are monomorphisms. Put $\bar{A} = A\alpha$, $\bar{B} = B\beta$ and $\bar{e} = \langle e, e\rangle$, using (3) and (4) it easy to shows that $(\bar{A}, \bar{B})$ is a left univocal factorization of S such that $\bar{A}$ and $\bar{B}$ are isomorphic to A and B, respectively, $\bar{A} \cap \bar{B} = \{\bar{e}\}$ and $\bar{e}$ is a right identity of $\bar{A}$ and a left identity of $\bar{B}$. QED

Finally, following Köhler [4], a left univocal factorization (A, B) of a semigroup S is a *quasi-decomposition* of S if $Ba \subseteq aB$, for all $a \in A$. If $A \cap B = \{e\}$ with e be a right identity element of A and a left identity of B such that $A \cap B = \{e\}$, then e is an identity of A. We note the e is not necessarily an identity of B. In this regard, consider $S := \{a, b, c, d, e\}$ with the following Cayley moltiplication table

	a	b	c	d	e
a	b	b	a	a	a
b	b	b	b	b	b
c	a	b	c	d	e
d	a	b	c	d	e
e	a	b	e	e	e

Then S is a semigroup that admits the left univocal factorization (A, B) with $A := \{a, b, c\}$ and $B := \{c, d, e\}$ in which c is a left identity of B but not a right identity.

References

[1] F. Catino: *Factorizable semigroups*, Semigroup Forum **36** (1987), no. 2, 167–174.

[2] S.Y. Chen, S.C. Hsieh: *Factorizable inverse semigroups*, Semigroup Forum **8** (1974), 283–297.

[3] J.M. Howie: *An Introduction to Semigroup Theory*, Academic Press, London, 1976.

[4] P. Köhler: *Quasi-decompositions of semigroups*, Houston J. Math., **5** (1979), no. 4, 525 – 542.

[5] V.S. Krishnan: *Quasi resolved semigroups and quintets*, Semigroup Forum **17** (1979), no. 3, 267–277.

[6] T.T. Peng, X.J. Guo, K.P. Shum: *Factorizable rpp monoids*, International Mathematical Forum **2** (2007), 143–153.

[7] T.T. Peng, X.J. Guo: *Factorizable quasi-adequate semigroups*, Chin. Quart. J. of Math. **26**(1) (2011), 93–94.

[8] S. Qui, M. Hartman, X.J. Guo, K.P. Shum: *Almost factorizable IC quasi-adequat semigroups*, Comm. Algebra **39** (2011), no.5, 1796–1808.

[9] J. Schmidt: *Quasi-decompositions, exact sequences and triple sums of semigroups, I. General theory, II Applications*, Colloquia Math. Soc. János Bolyai, Contributions to Universal Algebra, Szeged, **17** (1975), 365–397 (I), 399–428 (II).

[10] K. Tolo: *Factorizable semigroups*, Pacific J. Math. **31** (1969), 523–535.