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A Composition Formula of the Pathway Integral Transform Operator

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Abstract. In the present paper, we aim at presenting composition formula of integral transform operator due to Nair, which is expressed in terms of the generalized Wright hypergeometric function, by inserting the generalized Bessel function of the first kind $\mathrm{w}_\nu(z)$. Furthermore the special cases for the product of trigonometric functions are also consider.

Keywords: Pathway fractional integral operator, generalized (Wright) hypergeometric functions, generalized Bessel function of the first kind, trigonometric functions.

MSC 2000 classification: 26A33, 33C70, 33E20, 33E12

1 Introduction

The history of fractional calculus is about 300 years old. It is a field of applied mathematics that deals with derivatives and integrals of arbitrary order. The fractional integral operator, which involves various special functions in it, has a significance importance and its applications are used in various subfield of applicable mathematical analysis. Such as, in turbulence and fluid dynamics, stochastic dynamical system, plasma physics and controlled thermonuclear fusion, nonlinear control theory, image processing, non-linear biological systems, astrophysics. During the last four decades, a number of researchers have done deep study i.e., the properties, applications and different extensions of various hypergeometric operators of fractional integration (see, for example, [2, 4, 5, 6, 11, 12, 13, 14, 15, 17, 18, 19, 24, 29] and [30]).

Bessel functions play an important role in the study of solutions of differential equations, and they are associated with a wide range of problems in important areas of mathematical physics, like problems of acoustics, radio physics, hydrodynamics, and atomic and nuclear physics. These considerations have to led various workers in the field of special functions to explore the possible extensions and applications for the Bessel functions. A useful generalization of the Bessel function $\mathrm{w}_\nu(z)$ has been introduced and studied in [7, 8, 9] and [10].

In recent years, a remarkably large number of fractional integral formulas involving a variety of special functions have been developed by so many authors (see, for example, [15]; for a very recent work, see also [1, 3] and [27]). Many fractional integral formulas have also been presented (see, for example, [7, 8, 9, 10, 20] and [26]), which contain the generalized Bessel function $\mathrm{w}_\nu(z)$ (1.9). Here, we aim at presenting composition formula of the pathway fractional integral operator, which are expressed in terms of the generalized Wright hypergeometric function, by inserting the generalized Bessel function of the first kind (1.9) with suitable arguments into the pathway fractional integration operator of (1.1). Furthermore some interesting special cases of our main result in terms of the product of cosine, hyperbolic cosine, sine and hyperbolic sine functions, respectively are also discussed.

Here, we start by recalling some known functions and earlier works. Recently, Nair ([27], p. 239) introduce a pathway fractional integral operator by using the pathway idea of Mathai [21] and developed further by Mathai and Haubold [22, 23], is defined as follows:
Let $f(x) \in L(a,b),\ \eta \in \mathbb{C}, \Re(\eta) > 0,\ a > 0$ and let us take a pathway parameter $\alpha < 1$, then:

$$\left(P_{0^+}^{(\eta,\alpha)} f\right)(x) = x^\eta \int_0^{\left[\frac{x}{a(1-\alpha)}\right]} [1 - \frac{a\,(1-\alpha)\,t}{x}]^{\frac{\eta}{(1-\alpha)}} f(t)dt. \tag{1.1}$$

For a real scalar α, the pathway model for scalar random variables is represented by the following probability density function ($p.d.f.$):

$$f(x) = c\,|x|^{\gamma-1}\left[1 - a(1-\alpha)\,|x|^\delta\right]^{\frac{\beta}{(1-\alpha)}}, \tag{1.2}$$

provided that $-\infty < x < \infty, \delta > 0, \beta \geq 0, [1 - a(1-\alpha)\,|x|^\delta] > 0, \gamma > 0$, where c is the normalizing constant and α is called the pathway parameter.
For real α, the normalizing constant is as follows:

$$c = \frac{1}{2}\frac{\delta\,[a(1-\alpha)]^{\frac{\gamma}{\delta}}\,\Gamma\left(\frac{\gamma}{\delta} + \frac{\beta}{1-\alpha} + 1\right)}{\Gamma\left(\frac{\gamma}{\delta}\right)\Gamma\left(\frac{\beta}{1-\alpha} + 1\right)}, \quad \text{for} \quad \alpha < 1, \tag{1.3}$$

$$= \frac{1}{2} \frac{\delta \left[a(1-\alpha)\right]^{\frac{\gamma}{\delta}} \Gamma\left(\frac{\beta}{\alpha-1}\right)}{\Gamma\left(\frac{\gamma}{\delta}\right) \Gamma\left(\frac{\beta}{\alpha-1} - \frac{\gamma}{\delta}\right)}, \quad \text{for} \quad \frac{1}{\alpha-1} - \frac{\gamma}{\delta} > 0, \ \alpha > 1, \tag{1.4}$$

$$= \frac{1}{2} \frac{\delta \left(a\beta\right)^{\frac{\gamma}{\delta}}}{\Gamma\left(\frac{\gamma}{\delta}\right)}, \alpha \to 1. \tag{1.5}$$

We observe that for $\alpha < 1$, it is a finite range density with $[1-a(1-\alpha)\,|x|^{\delta}] > 0$ and (1.2) remains in the extended generalized type-1 beta family. The pathway density in (1.2), for $\alpha < 1$, includes the extended type-1 beta density, the triangular density, the uniform density and many other $p.d.f'.s$.

For $\alpha > 1$, we have:

$$f(x) = c\,|x|^{\gamma-1} \left[1 + a(\alpha-1)\,|x|^{\delta}\right]^{-\frac{\beta}{(\alpha-1)}}, \tag{1.6}$$

provided that $-\infty < x < \infty, \delta > 0, \beta \geq 0, \alpha > 1$, which is the extended generalized type-2 beta model for real x. It includes the type-2 beta density, the F density, the Student-t density, the Cauchy density and many more.

Here, we consider only the case of pathway parameter $\alpha < 1$. For $\alpha \to 1$ both (1.2) and (1.6) take the exponential from, since

$$\lim_{\alpha \to 1} \; c\,|x|^{\gamma-1} \left[1 - a(1-\alpha)\,|x|^{\delta}\right]^{\frac{\eta}{(1-\alpha)}} = \lim_{\alpha \to 1} \; c\,|x|^{\gamma-1} \left[1 + a(\alpha-1)\,|x|^{\delta}\right]^{-\frac{\eta}{(\alpha-1)}}$$

$$= c\,|x|^{\gamma-1} \exp(-a\eta\,|x|^{\delta}). \tag{1.7}$$

This include the generalized Gamma, the Weibull, the chi-square, the Laplace, Maxwell-Boltzmann and other related densities.

When $\alpha \to 1_{-}$,
$\left[1 - \frac{a(1-\alpha)}{x}\right]^{\frac{\eta}{(1-\alpha)}} \to e^{-\frac{a\eta}{x}t}$.
Then, operator (1.1) reduces to the Laplace integral transform of f with parameter $\frac{a\eta}{x}$:

$$\left(P_{0^+}^{(\eta,1)} f\right)(x) = x^{\eta} \int_0^{\infty} e^{-\frac{a\eta}{x}} f(t)dt = x^{\eta} L_f\left(\frac{a\eta}{x}\right). \tag{1.8}$$

If, we set $\alpha = 0, a = 1$, and replacing η by $\eta - 1$ in (1.1) the integral operator reduces to the Riemann-Liouville fractional integral operator (see, for example, [16], [17], [28], [29] and [31]).

Here, we investigate the composition of the integral transform operator (1.1) with the product of generalized Bessel function of the first kind $\mathrm{w}_\nu(z)$, which is defined for $z \in \mathbb{C} \setminus \{0\}$ and $b, c, \nu \in \mathbb{C}$ with $\Re(\nu) > -1$ by the following series (see, for example,[7, p. 10, Eq. (1.15)]; for a very recent work, see also [8, 9, 10], [26, p. 182, Eq. (2.2)], and [20, p. 2, Eq. (8)]):

$$\mathrm{w}_\nu(z) = \sum_{k=0}^{\infty} \frac{(-1)^k c^k \left(\frac{z}{2}\right)^{\nu+2k}}{k!\,\Gamma(\nu + k + \frac{1+b}{2})}, \tag{1.9}$$

where $\mathbb{C}$ denotes the set of complex numbers and $\Gamma(z)$ is the familiar Gamma function (see, [32, Section 1.1]).

Here, it is important to mentioning that, for $c = 1$ and $b = 1$ the generalized Bessel function of the first kind (1.9), reduces in to the Bessel function of the first kind $J_\nu(z)$ and for $c = -1$ and $b = 1$ the function $\mathrm{w}_\nu(z)$ reduces in to the terms of incomplete Bessel function of the first kind $I_\nu(z)$. Similarly, for $c = 1$ and $b = 2$ the function $\mathrm{w}_\nu(z)$ reduces in to $\frac{2j_\nu}{\sqrt{\pi}}$, while if $c = -1$ and $b = 2$, then $\mathrm{w}_\nu(z)$ becomes $\frac{2i_\nu}{\sqrt{\pi}}$. In the sequel, from (1.9), we also have $\mathrm{w}_\nu(0) = 0$. Therefore, the generalized Bessel function of the first kind presented in this paper are easily converted in terms of the various kind of Bessel functions after some suitable parametric replacement.

Then, we can show that the composition formula is expressed in terms of the generalized (Wright) hypergeometric functions (see, for example, [33, p. 21]):

$${}_p\Psi_q \left[\begin{array}{l} (\alpha_1, A_1), \ldots, (\alpha_p, A_p); \\ (\beta_1, B_1), \ldots, (\beta_q, B_q); \end{array} z \right] = \sum_{k=0}^{\infty} \frac{\prod_{j=1}^{p} \Gamma(\alpha_j + A_j k)}{\prod_{j=1}^{q} \Gamma(\beta_j + B_j k)} \frac{z^k}{k!}. \tag{1.10}$$

Wright [34] introduced the generalized Wright function (1.10) and proved several theorems on the asymptotic expansion of ${}_p\Psi_q$(z) (for instance, see, [34, 35, 36]) for all values of the argument z, under the condition:

$$1 + \sum_{j=1}^{q} B_j - \sum_{j=1}^{p} A_j \geq 0. \tag{1.11}$$

A generalized hypergeometric function ${}_pF_q$ is defined and represented as follows (see, [32, Section 1.5]):

$${}_pF_q \left[\begin{array}{l} (\alpha_p)\,; \\ (\beta_q)\,; \end{array} z \right] = \sum_{n=0}^{\infty} \frac{\prod_{j=1}^{p} (\alpha_j)_n}{\prod_{j=1}^{q} (\beta_j)_n} \frac{z^n}{n!}, \quad \text{provided} \quad p \leq q; p = q+1 \, and \, |z| < 1, \tag{1.12}$$

where $(\lambda)_n$ is the Pochhammer symbol defined (for $\lambda \in \mathbb{C}$) by (see, [32, p. 2 and pp. 4-6]):

$$\begin{aligned}(\lambda)_n : &= \begin{cases} 1 & (n = 0) \\ \lambda(\lambda+1)\ldots(\lambda+n-1) & (n \in \mathbb{N} := \{1, 2, 3, \ldots\}) \end{cases} \\ &= \frac{\Gamma(\lambda+n)}{\Gamma(\lambda)} \quad (\lambda \in \mathbb{C} \setminus \mathbb{Z}_0^-).\end{aligned} \tag{1.13}$$

and $\mathbb{Z}_0^-$ denotes the set of nonpositive integers.

The function (1.12) is a special case of the generalized Wright function (1.10) for $\alpha_1 = \ldots = \alpha_p = \beta_1 = \ldots = \beta_q$:

$${}_p\Psi_q \left[\begin{matrix} (\alpha_1, 1), \ldots, (\alpha_p, 1); \\ (\beta_1, 1), \ldots, (\beta_q, 1); \end{matrix} z \right] = \frac{\prod_{j=1}^{p} \Gamma(\alpha_j)}{\prod_{j=1}^{q} \Gamma(\beta_j)} \, {}_pF_q \left[\begin{matrix} \alpha_1, \ldots, \alpha_p; \\ \beta_1, \ldots, \beta_q; \end{matrix} z \right]. \tag{1.14}$$

In this paper, we apply the integral operators (1.1) to the generalized Bessel function of the first kind $\mathrm{w}_\nu(\mathrm{z})$ (1.9) and express the image in terms of generalized Wright hypergeometric functions.

2 Pathway Fractional Integration of Generalized Bessel functions

Our results in this section are based on the preliminary assertions giving by composition formula of pathway fractional integral (1.1) with a power function.

2.1 Lemma 1. ([Nair [27], Lemma 1])

Let $\eta \in \mathbb{C}, \Re(\eta) > 0$, $\beta \in \mathbb{C}$ and $\alpha < 1$, If $\Re(\beta) > 0$, and $\Re\left(\frac{\eta}{1-\alpha}\right) > -1$, then:

$$\left\{ P_{0^+}^{(\eta,\alpha)} \left[t^{\beta-1} \right] \right\}(x) = \frac{x^{\eta+\beta}}{[a(1-\alpha)]^{\beta}} \frac{\Gamma(\beta)\Gamma(1+\frac{\eta}{1-\alpha})}{\Gamma(1+\frac{\eta}{1-\alpha}+\beta)}. \tag{2.1}$$

The pathway fractional integration (1.1) of product of the generalized Bessel function of the first kind (1.9) is given by the following result.

Theorem 1. *Let $\eta, \sigma, \nu, b, c \in \mathbb{C}$ and $\alpha < 1$, be such that:*

$$\Re(\eta) > 0, \Re(\sigma) > 0, \Re(\nu + \frac{b+1}{2}) > -1,\ \Re(\sigma+\nu) > 0 \, and \, \Re\left(\frac{\eta}{1-\alpha}\right) > -1, \tag{2.2}$$

Then there holds the following formula:

$$\begin{aligned}\left(P_{0^+}^{(\eta,\alpha)}\left[t^{\sigma-1}w_\nu(t)\right]\right)(x) &= x^{\sigma+\eta}\frac{\Gamma(1+\frac{\eta}{1-\alpha})}{[a(1-\alpha)]^{\sigma+\nu}}\cdot\left(\frac{x}{2}\right)^\nu \\ \cdot {}_1\Psi_2&\left[\begin{matrix}\left(\sigma+\nu,2\right),- \\ \left(1+\sigma+\nu+\frac{\eta}{1-\alpha},2\right),\left(\nu+\frac{b+1}{2},1\right)\end{matrix};-\frac{cx^2}{4[a(1-\alpha)]^2}\right].\end{aligned} \tag{2.3}$$

Proof: For convenience, let the left-hand side of the (PFIF)(2.3), be denoted by $\mathcal{I}$. Applying (1.9), using (1.1) and (1.10) and changing the order of integration and summation, we get

$$\begin{aligned}\mathcal{I} &= \left(P_{0^+}^{(\eta,\alpha)}\left[t^{\sigma-1}\sum_{k=0}^{\infty}\frac{(-c)^k\left(\frac{t}{2}\right)^{\nu+2k}}{k!\Gamma(\nu+\frac{b+1}{2}+k)}\right]\right)(x), \\ &= \sum_{k=0}^{\infty}\frac{(-c)^k\left(\frac{1}{2}\right)^{\nu+2k}}{k!\Gamma(\nu+\frac{b+1}{2}+k)}\left(P_{0^+}^{(\eta,\alpha)}\left\{t^{\sigma+\nu+2k-1}\right\}\right)(x).\end{aligned}$$

By (2.2), for any $k \in \mathbb{N}_0$, $\Re(\sigma+\nu+2k) \geq \Re(\sigma+\nu) > 0$ and $\Re\left(\frac{\eta}{1-\alpha}\right) > -1$. Applying Lemma 1 and using (2.1) with β replaced by $(\sigma+\nu+2k)$ after a little simplification, we get

$$\begin{aligned}\mathcal{I} &= \sum_{k=0}^{\infty}\frac{(-c)^k\left(\frac{1}{2}\right)^{\nu+2k}}{k!\Gamma(\nu+\frac{b+1}{2}+k)}\times\frac{\Gamma(\sigma+\nu+2k)\,\Gamma\left(1+\frac{\eta}{1-\alpha}\right)}{\Gamma\left(1+\sigma+\frac{\eta}{1-\alpha}+\nu+2k\right)}\frac{x^{\eta+\sigma+\nu+2k}}{[a(1-\alpha)]^{\sigma+\nu+2k}}, \\ &= x^{\sigma+\eta}\frac{\Gamma(1+\frac{\eta}{1-\alpha})}{[a(1-\alpha)]^{\sigma+\nu}}\cdot\left(\frac{x}{2}\right)^\nu \\ &\sum_{k=0}^{\infty}\frac{\Gamma(\sigma+\nu+2k)}{\Gamma\left(\nu+\frac{b+1}{2}+k\right)\Gamma\left(1+\sigma+\frac{\eta}{1-\alpha}+\nu+2k\right)}\frac{(-cx^2)^k}{k![2a(1-\alpha)]^{2k}}.\end{aligned} \tag{2.4}$$

This, in accordance with (1.10), gives the required result (2.3). This is complete the proof of the Theorem 1.

Remark 1. If, we set $\alpha = 0, a = 1$, and then replacing η by $\eta - 1$ in (2.3), we can arrive at the known result due to Malik *et al.* [20].

If, we set $b = c = 1$ in (2.3) and give some suitable parametric replacement in the resulting result, we can arrive at the Equation (30) in Nair ([27], p. 247).

3 Pathway fractional integration of cosine,hyperbolic cosine, sine and hyperbolic sine functions

In this section, we derive certain new integral formulas for the cosine, hyperbolic cosine, sine and hyperbolic sine functions involving in the PFIF of (2.3).

To do this, we set $\nu = -\frac{b}{2}$ in (1.9), then the generalized Bessel function of the first kind $\mathrm{w}_\nu(\mathrm{z})$ in (1.9) have following relation with cosine function when c is replaced by c^2 (see, for example, [20]):

$$\mathrm{w}_{-b/2,c^2}(z) = \left(\frac{2}{z}\right)^{\frac{b}{2}} \frac{\cos cz}{\sqrt{\pi}}, \tag{3.1}$$

while, if $\nu = -\frac{b}{2}$ and $c = -c^2$, $\mathrm{w}_\nu(\mathrm{z})$ in (1.9) have the following relation with the hyperbolic cosine function as:

$$\mathrm{w}_{-b/2,-c^2}(z) = \left(\frac{2}{z}\right)^{\frac{b}{2}} \frac{\cosh cz}{\sqrt{\pi}}. \tag{3.2}$$

Replacing c by c^2 and then setting $\nu = -\frac{b}{2}$ in (2.3), and applying the expression in (3.1) to the resulting identities, we obtain the pathway fractional integral formula stated in Corollary 1 below.

Corollary 1. *Let $\eta, \sigma, b, c \in \mathbb{C}, \alpha < 1,$ be such that:*

$$\Re(\eta) > 0, \Re(\sigma) > 0, \Re\left(\sigma - \frac{b}{2}\right) > 0 \, and \, \Re\left(\frac{\eta}{1-\alpha}\right) > -1,$$

Then there holds the following formula:

$$\begin{aligned}\left(P_{0^+}^{(\eta,\alpha)}\left[t^{\sigma-1}cos(ct)\right]\right)(x) &= x^{\sigma+\eta}\sqrt{\pi}\frac{\Gamma(1+\frac{\eta}{1-\alpha})}{[a(1-\alpha)]^{\sigma-\frac{b}{2}}}\cdot \\ &\cdot {}_1\Psi_2\left[\begin{matrix}\left(\sigma-\dfrac{b}{2},2\right),- \\ \left(1+\sigma-\dfrac{b}{2}+\dfrac{\eta}{1-\alpha},2\right),\left(\dfrac{1}{2},1\right)\end{matrix};-\frac{(cx)^2}{4[a(1-\alpha)]^2}\right].\end{aligned} \tag{3.3}$$

Remark 2. By setting $\alpha = 0, a = 1$ and then replacing η by $\eta - 1$ in (3.3), we can arrive at known result given by Malik *et al.*([20],p. 5).
If we set $c = 1$ in (3.3) and give some suitable parametric replacement in the resulting result, we can arrive at the Equation (33) in Nair ([27], p. 248).

Similarly by setting $\nu = -\frac{b}{2}$ and replacing c by $-c^2$ in (2.3), and applying the expression in (3.2) to the resulting identities, we get the pathway fractional integral formula asserted in Corollary 2.

Corollary 2. *Let $\eta, \sigma, b, c \in \mathbb{C}, \alpha < 1$, be such that:*

$$\Re(\eta) > 0, \Re(\sigma) > 0, \Re\left(\sigma - \frac{b}{2}\right) > 0 \, and \, \Re\left(\frac{\eta}{1-\alpha}\right) > -1,$$

Then there holds the following formula:

$$\begin{aligned}\left(P_{0^+}^{(\eta,\alpha)}\left[t^{\sigma-1}cosh(ct)\right]\right)(x) = x^{\sigma+\eta}\sqrt{\pi}\frac{\Gamma(1+\frac{\eta}{1-\alpha})}{[a(1-\alpha)]^{\sigma-\frac{b}{2}}}\cdot \\ \cdot {}_1\Psi_2\left[\begin{matrix}\left(\sigma-\frac{b}{2},2\right), - \\ \left(1+\sigma-\frac{b}{2}+\frac{\eta}{1-\alpha},2\right), \left(\frac{1}{2},1\right)\end{matrix}; \frac{(cx)^2}{4[a(1-\alpha)]^2}\right].\end{aligned} \tag{3.4}$$

Remark 3. If, we set $\alpha = 0, a = 1$ and then replacing η by $\eta - 1$ in (3.4), we can arrive at the known result given by Malik *et al.*([20], p. 5).

In the sequel, we recall the following well known formulas (see, for example, [20]):

$$\mathrm{w}_{1-b/2,c^2}(z) = \left(\frac{2}{z}\right)^{\frac{b}{2}} \frac{sincz}{\sqrt{\pi}}. \tag{3.5}$$

$$\mathrm{w}_{1-b/2,-c^2}(z) = \left(\frac{2}{z}\right)^{\frac{b}{2}} \frac{sinhcz}{\sqrt{\pi}}. \tag{3.6}$$

Considering (3.5) and (3.6), and using (2.3), we get the following pathway fractional integral formulas asserted in Corollaries 3 and 4 in terms of sine and hyperbolic sine functions, respectively.

Corollary 3. *Let $\eta, \sigma, b, c \in \mathbb{C}, \alpha < 1$, be such that:*

$$\Re(\eta) > 0, \Re(\sigma) > 0, \Re\left(\sigma - \frac{b}{2}\right) > -1 \, and \, \Re\left(\frac{\eta}{1-\alpha}\right) > -1,$$

Then there holds the following formula:

$$\left(P_{0^+}^{(\eta,\alpha)}\left[t^{\sigma-1}\sin(ct)\right]\right)(x) = x^{\sigma+\eta+1}\sqrt{\pi}\,\frac{c}{2}\frac{\Gamma(1+\frac{\eta}{1-\alpha})}{[a(1-\alpha)]^{\sigma-\frac{b}{2}+1}}\cdot$$
$$\cdot{}_1\Psi_2\left[\begin{matrix}\left(\sigma+1-\frac{b}{2},2\right),- \\ \left(2+\sigma-\frac{b}{2}+\frac{\eta}{1-\alpha},2\right),\left(\frac{3}{2},1\right)\end{matrix};-\frac{(cx)^2}{4[a(1-\alpha)]^2}\right]. \tag{3.7}$$

Remark 4. By setting $\alpha = 0, a = 1$ and replacing η by $\eta - 1$ in (3.7), we can arrive at the known result given by Malik *et al.*([20], p. 8).

Corollary 4. *Let* $\eta, \sigma, b, c \in \mathbb{C}, \alpha < 1,$ *be such that:*

$$\Re(\eta) > 0, \Re(\sigma) > 0,\ \Re\left(\sigma-\frac{b}{2}\right) > -1\,and\,\Re\left(\frac{\eta}{1-\alpha}\right) > -1,$$

Then there holds the following formula:

$$\left(P_{0^+}^{(\eta,\alpha)}\left[t^{\sigma-1}\sinh(ct)\right]\right)(x) = x^{\sigma+\eta+1}\sqrt{\pi}\frac{c}{2}\frac{\Gamma(1+\frac{\eta}{1-\alpha})}{[a(1-\alpha)]^{\sigma-\frac{b}{2}}}\cdot$$
$$\cdot{}_1\Psi_2\left[\begin{matrix}\left(\sigma+1-\frac{b}{2},2\right),- \\ \left(2+\sigma-\frac{b}{2}+\frac{\eta}{1-\alpha},2\right),\left(\frac{3}{2},1\right)\end{matrix};\frac{(cx)^2}{4[a(1-\alpha)]^2}\right]. \tag{3.8}$$

Remark 5. By setting $\alpha = 0, a = 1$ and replacing η by $\eta - 1$ in (3.8), directly we can arrive at the known result given by Malik *et al.*([20], p. 8).

4 Concluding Remarks

We conclude this investigation by remarking that the results obtained here are general in character and useful in deriving various integral formulas in the theory of the pathway fractional integration operator. Most of the results obtained here, besides being of a very general character, have been put in a compact form avoiding the occurrence of infinite series and thus making them useful from the point of view of applications. The result obtained in the present paper provides an extension of the results given by Agarwal and Purohit [1] and Nair [27] as mentioned earlier.

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