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On Killing vector fields on a tangent bundle with g-natural metric Part I

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Abstract. The tangent bundle of a Riemannian manifold (M, g) with a non-degenerate $g-$ natural metric G that admits a Killing vector field is investigated. Using Taylor's formula (TM, G) is decomposed into four classes. The equivalence of the existence of Killing vector field on M and TM is proved. More detailed investigaton of the classes is done in the part II of the paper.

Keywords: Key words: Riemannian manifold, tangent bundle, g - natural metric, Killing vector field, non-degenerate metric

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1 Introduction

Geometry of a tangent bundle goes back to 1958 when Sasaki published ([16]). Having given Riemannian metric g on a differentiable manifold M, he constructed a Riemannian metric G on the tangent bundle TM of M, known today as the Sasaki metric. Since then different topics of geometry of the tangent bundle were studied by many geometers. Other metrics on the tangent bundle, obtained from the base metric g (as lifts), had been considered and studied. Actually, all these metrics belong to a large class of metrics on TM, known as $g-$ natural ones, constructed in ([13]), see also ([5]). $g-$ natural metrics can be regarded as jets of a Riemannian metric g on a manifold M ([2]).

In this paper we are interested in the classification of Killing vector field on the tangent bundle TM endowed with an arbitrary $g-$ natural metric G. The same subject had been studied in ([3]), ([17]) and ([18]) in the particular cases where G is the Cheeger-Gromoll metric g^{CG}, the complete lift g^c and the Sasaki metric g^S, respectively. In all the cases, a classification of Killing vector fields on the tangent bundle had been obtained. Similar results were obtained independently in ([15]).

We start by developing the method by Tanno ([18]) to investigate Killing vector fields on TM with arbitrary, non-degenerate $g-$ natural metrics. The method applies Taylor's formula to components of the vector field that is supposed to be an infinitesimal affine transformation, in particular an infinitesimal isometry. The infinitesimal affine transformation is determined by the values of its components and their first partial derivatives at a point ([12], p. 232). It appears by applying the Taylor's formula there are at most four "generators" of the infinitesimal isometry: two vectors and two tensors of type $(1,1)$.

The paper is organized as follows. In Chapter 2 we describe the conventions and give basic formulas we shall need. We also give a short resumé on a tangent bundle of a Riemannian manifold. In Chapter 3 we calculate the Lie derivative of a $g-$natural metric G on TM in terms of horizontal and vertical lifts of vector fields from M to TM. Furthermore, we obtain the Lie derivative of G with respect to an arbitrary vector field in terms of an adapted frame. By applying the Taylor's formula to the Killing vector field on a neighbourhood of the set $M \times \{0\}$ we get a series of conditions relating components and their covariant derivatives. Finally we prove some lemmas of a general character. It is worth mentioning that at this level there is a restriction on one of the generators to be non-zero. The further restrictions of this kind will appear later on.

In Chapter 4, making use of these conditions and lemmas, we split the non-degenerate $g-$natural metrics on TM into four classes (Theorem 2).

As a consequence of the splitting theorem and Theorem 3 as well, we obtain the main

Theorem 1. *If the tangent bundle of a Riemannian manifold (M,g), $dimM >$ 2, with a $g-$ natural, non-degenerate metric G admits a Killing vector field, then there exists a Killing vector field on M.*

Conversely, any Killing vector field X on a Riemannian manifold (M,g) gives rise to a Killing vector field Z on its tangent bundle endowed with a non-degenerate $g-$ natural metric. Precisely, Z is the complete lift of X.

Finally, in the Appendix we collect some known facts and theorems that we use throughout the paper and also prove lemmas of a general character.

In part II of this work ([9], see also [10]) further properties of the classes indicated in Theorem 2 are investigated separately. Moreover, a complete structure of the Lie algebra of Killing vector fields on TM for some subclasses is given. Some classical lifts of some tensor fields from (M,g) to (TM,G) are also discussed.

Throughout the paper all manifolds under consideration are smooth and Hausdorff ones. The metric g of the base manifold M is always assumed to be

Riemannian one.

The computations in local coordinates were partially carried out and checked using MathTensor™ and Mathematica® software.

2 Preliminaries

2.1 Conventions and basic formulas

Let (M, g) be a pseudo-Riemannian manifold of dimension n with metric g. The Riemann curvature tensor R is defined by

$$R(X, Y) = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]}.$$

In a local coordinate neighbourhood $(U, (x^1, \dots, x^n))$ its components are given by

$$R(\partial_i, \partial_j)\partial_k = R(\partial_i, \partial_j, \partial_k) = R^r_{kji}\partial_r = \left(\partial_i \Gamma^r_{jk} - \partial_j \Gamma^r_{ik} + \Gamma^r_{is}\Gamma^s_{jk} - \Gamma^r_{js}\Gamma^s_{ik}\right)\partial_r,$$

where $\partial_k = \frac{\partial}{\partial x^k}$ and Γ^r_{jk} are the Christoffel symbols of the Levi-Civita connection ∇. We have

$$\partial_l g_{hk} = g_{hk;l} = \Gamma^r_{hl} g_{rk} + \Gamma^r_{kl} g_{rk}. \tag{1}$$

The Ricci identity is

$$\nabla_i \nabla_j X_k - \nabla_j \nabla_i X_k = X_{k,ji} - X_{k,ij} = -X^s R_{skji}. \tag{2}$$

The Lie derivative of a metric tensor g is given by

$$(L_X g)(Y, Z) = g(\nabla_Y X, Z) + g(Y, \nabla_Z X) \tag{3}$$

for all vector fields X, Y, Z on M. In local coordinates $(U, (x^1, \dots, x^n))$ we get

$$\left(L_{X^r \partial_r} g\right)_{ij} = \nabla_i X_j + \nabla_j X_i,$$

where $X_k = g_{kr} X^r$.

We shall need the following properties of the Lie derivative

$$L_X \Gamma^h_{ji} = \nabla_j \nabla_i X^h + X^r R_{rjis} g^{sh} = \frac{1}{2} g^{hr}\left[\nabla_j (L_X g_{ir}) + \nabla_i (L_X g_{jr}) - \nabla_r (L_X g_{ji})\right]. \tag{4}$$

If $L_X \Gamma^h_{ji} = 0$, then X is said to be an infinitesimal affine transformation.

The vector field X is said to be the Killing vector field or infinitesimal isometry if

$$L_X g = 0.$$

For a Killing vector field X we have

$$L_X \nabla = 0, \quad L_X R = 0, \quad L_X (\nabla R) = 0, \ldots .$$

([19], p. 23 and 24).

2.2 Tangent bundle

Let x be a point of a Riemannian manifold (M, g), $\dim M = n$, covered by coordinate neighbourhoods $(U, (x^j, \ j = 1, \ldots, n))$, . Let TM be the tangent bundle of M and $\pi : TM \longrightarrow M$ be the natural projection on M. If $x \in U$ and $u = u^r \frac{\partial}{\partial x^r}_{|x} \in T_x M$ then $(\pi^{-1}(U), ((x^r), (u^r), \ r = 1, \ldots, n))$, is a coordinate neighbourhood on TM.

For all $(x, u) \in TM$ we denote by $V_{(x,u)}TM$ the kernel of the differential at (x, u) of the projection $\pi : TM \longrightarrow M$, i.e.,

$$V_{(x,u)}TM = Ker \left(d\pi_{|(x,u)} \right),$$

which is called the vertical subspace of $T_{(x,u)}TM$ at (x, u).

To define the horizontal subspace of $T_{(x,u)}TM$ at (x, u), let $V \subset M$ and $W \subset T_x M$ be open neighbourhoods of x and 0 respectively, diffeomorphic under exponential mapping $exp_x : T_x M \longrightarrow M$. Furthermore, let $S : \pi^{-1}(V) \longrightarrow T_x M$ be a smooth mapping that translates every vector $Z \in \pi^{-1}(V)$ from the point y to the point x in a parallel manner along the unique geodesic connecting y and x. Finally, for a given $u \in T_x M$, let $R_{-u} : T_x M \longrightarrow T_x M$ be a translation by u, i.e. $R_{-u}(X_x) = X_x - u$. The connection map

$$K_{(x,u)} : T_{(x,u)}TM \longrightarrow T_x M$$

of the Levi-Civita connection ∇ is given by

$$K_{(x,u)}(Z) = d(exp_p \circ R_{-u} \circ S)(Z)$$

for any $Z \in T_{(x,u)}TM$.

For any smooth vector field $Z : M \longrightarrow TM$ and $X_x \in T_x M$ we have

$$K(dZ_x(X_x)) = (\nabla_X Z)_x .$$

Then $H_{(x,u)}TM = Ker(K_{(x,u)})$ is called the horizontal subspace of $T_{(x,u)}TM$ at (x, u).

The space $T_{(x,u)}TM$ tangent to TM at (x,u) splits into direct sum

$$T_{(x,u)}TM = H_{(x,u)}TM \oplus V_{(x,u)}TM.$$

We have isomorphisms

$$H_{(x,u)}TM \sim T_xM \sim V_{(x,u)}TM.$$

For any vector $X \in T_xM$ there exist unique vectors in $T_{(x,u)}TM$, X^h and X^v, given respectively by $d\pi(X^h) = X$ and $X^v(df) = Xf$, for any function f on M. X^h and X^v are called the horizontal and the vertical lifts of X to the point $(x,u) \in TM$.

The vertical lift of a vector field X on M is the unique vector field X^v on TM such that at each point $(x,u) \in TM$ its value is the vertical lift of X_x to the point (x,u). The horizontal lift of a vector field is defined similarly.

If $((x^j),\ (u^j), i = 1,\dots,n)$ is a local coordinate system around the point $(x,u) \in TM$ where $u \in T_xM$ and $X = X^j \frac{\partial}{\partial x^j}$, then

$$X^h = X^j \frac{\partial}{\partial x^j} - u^r X^s \Gamma^j_{rs} \frac{\partial}{\partial u^j}, \quad X^v = X^j \frac{\partial}{\partial u^j},$$

where Γ^j_{rs} are the Christoffel symbols of the Levi-Civita connection ∇ on (M,g). We shall write $\partial_k = \frac{\partial}{\partial x^k}$ and $\delta_k = \frac{\partial}{\partial u^k}$ (cf. [8] or [11], see also [20]).

In the paper we shall frequently use the frame $(\partial^h_k, \partial^v_l) = \left(\left(\frac{\partial}{\partial x^k}\right)^h, \left(\frac{\partial}{\partial x^l}\right)^v \right)$ known as the adapted frame.

Lemma 1. *The Lie brackets of vector fields on the tangent bundle of a pseudo-Riemannian manifold M are given by*

$$\begin{aligned}
\left[X^h, Y^h\right]_{(x,u)} &= [X,Y]^h_{(x,u)} - v\left\{R\left(X_x, Y_x\right)u\right\}, \\
\left[X^h, Y^v\right]_{(x,u)} &= (\nabla_X Y)^v_{(x,u)} = (\nabla_Y X)^v_{(x,u)} + [X,Y]^v_{(x,u)}, \\
\left[X^v, Y^v\right]_{(x,u)} &= 0
\end{aligned}$$

for all vector fields X, Y on M.

Every metric g on M gives rise to the class of so called $g-$ natural metrics. The well-known Cheeger-Gromoll and Sasaki metrics are special cases of $g-$ natural metrics ([13]). $g-$ natural metrics are characterized by the following

Lemma 2. *([5], [6]) Let (M,g) be a Riemannian manifold and G be a $g-$natural metric on TM. There exist functions a_j, b_j $:< 0,\infty) \longrightarrow R$, $j =$*

$1, 2, 3$, *such that for every* X, Y, $u \in T_xM$

$$\begin{aligned} G_{(x,u)}(X^h, Y^h) &= (a_1 + a_3)(r^2)g_x(X, Y) + (b_1 + b_3)(r^2)g_x(X, u)g_x(Y, u), \\ G_{(x,u)}(X^h, Y^v) &= a_2(r^2)g_x(X, Y) + b_2(r^2)g_x(X, u)g_x(Y, u), \\ G_{(x,u)}(X^v, Y^h) &= a_2(r^2)g_x(X, Y) + b_2(r^2)g_x(X, u)g_x(Y, u), \\ G_{(x,u)}(X^v, Y^v) &= a_1(r^2)g_x(X, Y) + b_1(r^2)g_x(X, u)g_x(Y, u), \end{aligned} \tag{5}$$

where $r^2 = g_x(u, u)$. *For* $\dim M = 1$ *the same holds for* $b_j = 0$, $j = 1, 2, 3$.

Setting $a_1 = 1$, $a_2 = a_3 = b_j = 0$ we obtain the Sasaki metric, while setting $a_1 = b_1 = \frac{1}{1+r^2}$, $a_2 = b_2 = 0 = 0$, $a_1 + a_3 = 1$, $b_1 + b_3 = 1$ we get the Cheeger-Gromoll one.

Following ([5]) we put

(1) $a(t) = a_1(t)\,(a_1(t) + a_3(t)) - a_2^2(t)$,

(2) $F_j(t) = a_j(t) + tb_j(t)$,

(3) $F(t) = F_1(t)\,[F_1(t) + F_3(t)] - F_2^2(t)$
for all $t \in < 0, \infty)$.

We shall often abbreviate: $A = a_1 + a_3$, $B = b_1 + b_3$.

Lemma 3. *([5], Proposition 2.7) The necessary and sufficient conditions for a* $g-$ *natural metric* G *on the tangent bundle of a Riemannian manifold* (M, g) *to be non-degenerate are* $a(t) \neq 0$ *and* $F(t) \neq 0$ *for all* $t \in < 0, \infty)$. *If* $\dim M = 1$ *this is equivalent to* $a(t) \neq 0$ *for all* $t \in < 0, \infty)$.

2.3 The Levi-Civita connection

The Levi-Civita connection $\widetilde{\nabla}$ of a Riemannian g - natural metric G on TM was calculated and presented in ([4], [5], [6]), with some misprints (see, for instance, ([1]) for correct expressions without misprints).

The same expressions remain valid for non-degenerate $g-$ natural metric (cf. ([7])).

Let T be a tensor field of type $(1, s)$ on M. For any $X_1, \ldots, X_s \in T_xM$, $x \in M$, we define horizontal and vertical vectors at a point $(x, u) \in TTM$ setting respectively

$$h\,\{T(X_1, \ldots, u, \ldots, X_{s-1}\} = \sum_{r=1}^{dimM} u^r [T(X_1, \ldots, \partial_r, .., X_{s-1})]^h,$$

$$v\left\{T(X_1,\ldots,u,\ldots,X_{s-1}\right\}=\sum_{r=1}^{dimM}u^r[T(X_1,\ldots,\partial_r,..,X_{s-1})]^v.$$

By the similar formulas we define

$$h\left\{T(X_1,\ldots,u,\ldots,u,\ldots,X_{s-1}\right\}\ \text{and}\ h\left\{T(X_1,\ldots,u,\ldots,u,\ldots,X_{s-1}\right\}.$$

Moreover, we put $h\left\{T(X_1,\ldots,X_s\right\}=(T(X_1,\ldots,X_s))^h$ and $v\left\{T(X_1,\ldots,X_s\right\}=(T(X_1,\ldots,X_s))^v$. Therefore $h\{X\}=X^h$ and $v\{X\}=X^v$ ([4], pp. 22-23).

Finally, we write

$$R(X,Y,Z)=R(X,Y)Z\ \text{and}\ R(X,Y,Z,V)=g(R(X,Y,Z),V)$$

for all $X,Y,Z,V\in T_xM$.

Proposition 1. *([1], [7]) Let (M,g) be a Riemannian manifold, ∇ its Levi-Civita connection and R its Riemann curvature tensor. If G is a non-degenerate $g-$natural metric on TM, then the Levi-Civita connection $\widetilde{\nabla}$ of (TM,G) at a point $(x,u)\in TM$ is given by*

$$\begin{aligned}
\left(\widetilde{\nabla}_{X^h}Y^h\right)_{(x,u)} &= (\nabla_XY)^h_{(x,u)}+h\left\{A(u,X_x,Y_x)\right\}+v\left\{B(u,X_x,Y_x)\right\},\\
\left(\widetilde{\nabla}_{X^h}Y^v\right)_{(x,u)} &= (\nabla_XY)^v_{(x,u)}+h\left\{C(u,X_x,Y_x)\right\}+v\left\{D(u,X_x,Y_x)\right\},\\
\left(\widetilde{\nabla}_{X^v}Y^h\right)_{(x,u)} &= h\left\{C(u,Y_x,X_x)\right\}+v\left\{D(u,Y_x,X_x)\right\},\\
\left(\widetilde{\nabla}_{X^v}Y^v\right)_{(x,u)} &= h\left\{E(u,X_x,Y_x)\right\}+v\left\{F(u,X_x,Y_x)\right\},
\end{aligned}$$

for all vector fields X, Y on M, where $P=a_2'-\frac{b_2}{2}$, $Q=a_2'+\frac{b_2}{2}$ and

$$\begin{aligned}
A(u,X,Y)=-\frac{a_1a_2}{2a}\left[R(X,u,Y)+R(Y,u,X)\right]+&\\
\frac{a_2B}{2a}\left[g(Y,u)X+g(X,u)Y\right]+&\\
\frac{1}{aF}\Big\{\ a_2\left[a_1\left(F_1B-F_2b_2\right)+a_2\left(b_1a_2-b_2a_1\right)\right]R(X,u,Y,u)+&\\
\left[aF_2B'+B\left[a_2(F_2b_2-F_1B)+A(a_1b_2-a_2b_1)\right]\right]g(X,u)g(Y,u)+&\\
aF_2A'g(X,Y)\ \Big\}u,&
\end{aligned}$$

$$B(u,X,Y) = \frac{a_2^2}{a}R(X,u,Y) - \frac{a_1 A}{2a}R(X,Y,u) - \frac{AB}{2a}\left[g(Y,u)X + g(X,u)Y\right] + \frac{1}{aF}\Big\{ a_2[a_2(F_2 b_2 - F_1 B) + A(b_2 a_1 - b_1 a_2)]R(X,u,Y,u) + \Big[-a(F_1+F_3)B' + B\left[A\left((F_1+F_3)b_1 - F_2 b_2\right) + a_2\left(a_2 B - b_2 A\right)\right] \Big] g(X,u)g(Y,u) - a(F_1+F_3)A' g(X,Y) \Big\} u,$$

$$C(u,X,Y) = -\frac{a_1^2}{2a}R(Y,u,X) + \frac{a_1 B}{2a}g(X,u)Y + \frac{1}{a}\left(a_1 A' - a_2 P\right)g(Y,u)X + \frac{1}{aF}\Big\{ \frac{a_1}{2}\left[a_2\left(a_2 b_1 - a_1 b_2\right) + a_1\left(F_1 B - F_2 b_2\right)\right]R(X,u,Y,u) + a\left(\frac{F_1}{2}B + F_2 P\right)g(X,Y) + \Big[aF_1 B' + \left(A' + \frac{B}{2}\right)\left[a_2\left(a_1 b_2 - a_2 b_1\right) + a_1\left(F_2 b_2 - BF_1\right)\right] + P\left[a_2\left(b_1\left(F_1+F_3\right) - b_2 F_2\right) - a_1\left(b_2 A - a_2 B\right)\right] \Big] g(X,u)g(Y,u) \Big\} u,$$

$$D(u,X,Y) = \frac{1}{a}\Big\{ \frac{a_1 a_2}{2}R(Y,u,X) - \frac{a_2 B}{2}g(X,u)Y + \left(AP - a_2 A'\right)g(Y,u)X \Big\} + \frac{1}{aF}\Big\{ \frac{a_1}{2}\left[A(a_1 b_2 - a_2 b_1) + a_2(F_2 b_2 - F_1 B)\right]R(X,u,Y,u) - a\left[\frac{F_2}{2}B + (F_1+F_3)P\right]g(X,Y) + \Big[-aF_2 B' + \left(A' + \frac{B}{2}\right)\left[A(a_2 b_1 - a_1 b_2) + a_2(F_1 B - F_2 b_2)\right] + P\left[A(b_2 F_2 - b_1(F_1+F_3)) + a_2(b_2 A - a_2 B)\right] \Big] g(X,u)g(Y,u) \Big\} u,$$

$$E(u,X,Y) = \frac{1}{a}\left(a_1 Q - a_2 a_1'\right)[g(X,u)Y + g(Y,u)X] +$$
$$\frac{1}{aF}\Big\{ a\left[F_1 b_2 - F_2(b_1 - a_1')\right] g(X,Y) +$$
$$\Big[a(2F_1 b_2' - F_2 b_1') + 2a_1'\left[a_1(a_2 B - b_2 A) + a_2(b_1(F_1+F_3) - b_2 F_2)\right] +$$
$$2Q\left[a_1(F_2 b_2 - F_1 B) + a_2(a_1 b_2 - a_2 b_1)\right] \Big] g(X,u)g(Y,u) \Big\} u,$$

$$F(u,X,Y) = \frac{1}{a}\left(A a_1' - a_2 Q\right)[g(X,u)Y + g(Y,u)X] +$$
$$\frac{1}{aF}\Big\{ a\left[(F_1+F_3)(b_1 - a_1') - F_2 b_2\right] g(X,Y) +$$
$$\Big[a((F_1+F_3)b_1' - 2F_2 b_2') + 2a_1'\left[a_2(b_2 A - a_2 B) + A(b_2 F_2 - b_1(F_1+F_3))\right] +$$
$$2Q\left[a_2(F_1 B - F_2 b_2) + A(a_2 b_1 - a_1 b_2)\right] \Big] g(X,u)g(Y,u) \Big\} u.$$

3 Killing vector field

3.1 Lie derivative

Applying the formula (3) to the non-degenerate $g-$natural metric G on TM and vertical and horizontal lifts of vector fields X, Y, Z on M, using Proposition 1, we get

$$(L_{X^v}G)(Y^v, Z^v) =$$
$$b_1 g(X,Z)g(Y,u) + b_1 g(X,Y)g(Z,u) +$$
$$2a_1' g(Y,Z)g(X,u) + 2b_1' g(X,u)g(Y,u)g(Z,u),$$

$$(L_{X^h}G)(Y^v, Z^v) = 0,$$

whence

$$\left(L_{H^a\partial_a^h + V^a\partial_a^v}G\right)(\partial_k^v, \partial_l^v) = V^a\left(L_{\partial_a^v}G\right)(\partial_k^v, \partial_l^v) + \partial_k^v H^a G\left(\partial_a^h, \partial_l^v\right) +$$
$$\partial_k^v V^a G\left(\partial_a^v, \partial_l^v\right) + \partial_l^v H^a G\left(\partial_k^v, \partial_a^h\right) + \partial_l^v V^a G\left(\partial_k^v, \partial_a^v\right).$$

Next we find

$$(L_{X^h}G)\left(Y^v, Z^h\right) =$$
$$- a_1 R(Y,u,X,Z) + a_2 g(\nabla_Z X, Y) + b_2 g(\nabla_Z X, u)g(Y,u),$$

$$\begin{aligned}(L_{X^v}G)\left(Y^v, Z^h\right) &= a_1 g(\nabla_Z X, Y) + b_1 g(\nabla_Z X, u) g(Y,u) + \\ &b_2\left[g(X,Z)g(Y,u) + g(X,Y)g(Z,u)\right] + \\ &2a_2' g(Y,Z)g(X,u) + 2b_2' g(X,u)g(Y,u)g(Z,u),\end{aligned}$$

whence

$$\begin{aligned}&\left(L_{H^a\partial_a^h + V^a\partial_a^v}G\right)\left(\partial_k^v, \partial_l^h\right) = \\ &H^a\left(L_{\partial_a^h}G\right)\left(\partial_k^v, \partial_l^h\right) + V^a\left(L_{\partial_a^v}G\right)\left(\partial_k^v, \partial_l^h\right) + \partial_k^v H^a G\left(\partial_a^h, \partial_l^h\right) + \\ &\partial_k^v V^a G\left(\partial_a^v, \partial_l^h\right) + \partial_l^h H^a G\left(\partial_k^v, \partial_a^h\right) + \partial_l^h V^a G\left(\partial_k^v, \partial_a^v\right).\end{aligned}$$

Finally, we have

$$\begin{aligned}(L_{X^h}G)\left(Y^h, Z^h\right) &= A\left[g(\nabla_Z X, Y) + g(\nabla_Y X, Z)\right] + \\ &B\left[g(\nabla_Z X, u)g(Y,u) + g(\nabla_Y X, u)g(Z,u)\right] - \\ &a_2\left[R(Y,u,X,Z) + R(Z,u,X,Y)\right],\end{aligned}$$

$$\begin{aligned}(L_{X^v}G)\left(Y^h, Z^h\right) &= a_2\left[g(\nabla_Z X, Y) + g(\nabla_Y X, Z)\right] + \\ &b_2\left[g(\nabla_Z X, u)g(Y,u) + g(\nabla_Y X, u)g(Z,u)\right] + \\ &B\left[g(X,Y)g(Z,u) + g(X,Z)g(Y,u)\right] + \\ &2A' g(Y,Z)g(X,u) + 2B' g(X,u)g(Y,u)g(Z,u),\end{aligned}$$

whence

$$\begin{aligned}&\left(L_{H^a\partial_a^h + V^a\partial_a^v}G\right)\left(\partial_k^h, \partial_l^h\right) = \\ &H^a\left(L_{\partial_a^h}G\right)\left(\partial_k^h, \partial_l^h\right) + V^a\left(L_{\partial_a^v}G\right)\left(\partial_k^h, \partial_l^h\right) + \partial_k^h H^a G\left(\partial_a^h, \partial_l^h\right) + \\ &\partial_k^h V^a G\left(\partial_a^v, \partial_l^h\right) + \partial_l^h H^a G\left(\partial_k^h, \partial_a^h\right) + \partial_l^h V^a G\left(\partial_k^h, \partial_a^v\right).\end{aligned}$$

Suppose now that

$$Z = Z^a\partial_a + \widetilde{Z}^\alpha\delta_\alpha = Z^a\partial_a^h + (\widetilde{Z}^\alpha + Z^a u^r \Gamma_{ar}^\alpha)\partial_\alpha^v = H^a\partial_a^h + V^\alpha\partial_\alpha^v$$

is a vector field on TM, H^a, V^a being the horizontal and vertical components of the vector field Z on TM respectively.

Lemma 4. *Let G be a non-degenerate $g-$ natural metric (i.e., of the form (5)) defined on the tangent bundle TM of a manifold (M,g). With respect to the base $\left(\partial_k^v,\partial_l^h\right)$ we have*

$$
\begin{aligned}
\left(L_{H^a\partial_a^h+V^\alpha\partial_\alpha^v}G\right)\left(\partial_k^h,\partial_l^h\right) = &\\
&- a_2\left[R_{aklr}+R_{alkr}\right]H^a u^r + \\
&A\left[\left(\partial_k^h H^a + H^r\Gamma_{rk}^a\right)g_{al}+\left(\partial_l^h H^a + H^r\Gamma_{rl}^a\right)g_{ak}\right]+\\
&B\left[\left(\partial_k^h H^a + H^r\Gamma_{rk}^a\right)u_a u_l+\left(\partial_l^h H^a + H^r\Gamma_{rl}^a\right)u_a u_k\right]+\\
&a_2\left[\left(\partial_k^h V^a + V^r\Gamma_{rk}^a\right)g_{al}+\left(\partial_l^h V^a + V^r\Gamma_{rl}^a\right)g_{ak}\right]+\\
&b_2\left[\left(\partial_k^h V^a + V^r\Gamma_{rk}^a\right)u_a u_l+\left(\partial_l^h V^a + V^r\Gamma_{rl}^a\right)u_a u_k\right]+\\
&2A' g_{kl}V^b u_b + 2B' V^a u_a u_k u_l + B\left(V_k u_l + V_l u_k\right),
\end{aligned}
\tag{6}
$$

$$
\begin{aligned}
\left(L_{H^a\partial_a^h+V^\alpha\partial_\alpha^v}G\right)\left(\partial_k^v,\partial_l^h\right) = &\\
&- a_1 R_{alkr}u^r H^a + \partial_k^v H^a\left(A g_{al} + B u_a u_l\right)+\\
&a_2\left(\partial_l^h H^a + H^r\Gamma_{rl}^a\right)g_{ak} + b_2\left(\partial_l^h H^a + H^r\Gamma_{rl}^a\right)u_a u_k+\\
&\partial_k^v V^a\left(a_2 g_{al} + b_2 u_a u_l\right)+\\
&a_1\left(\partial_l^h V^a + V^r\Gamma_{rl}^a\right)g_{ak} + b_1\left(\partial_l^h V^a + V^r\Gamma_{rl}^a\right)u_a u_k+\\
&2a_2' g_{kl}V^b u_b + 2b_2' V^a u_a u_k u_l + b_2\left(V_k u_l + V_l u_k\right),
\end{aligned}
\tag{7}
$$

$$
\begin{aligned}
\left(L_{H^a\partial_a^h+V^\alpha\partial_\alpha^v}G\right)\left(\partial_k^v,\partial_l^v\right) = &\\
&a_2\left(\partial_k^v H^a g_{al} + \partial_l^v H^a g_{ak}\right) + b_2\left(\partial_k^v H^a u_a u_l + \partial_l^v H^a u_a u_k\right)+\\
&b_1\left(V_k u_l + V_l u_k\right) + 2a_1' g_{kl}V^b u_b + 2b_1' V^b u_b u_k u_l+\\
&a_1\left(\partial_k^v V^a g_{al} + \partial_l^v V^a g_{ak}\right) + b_1\left(\partial_k^v V^a u_a u_l + \partial_l^v V^a u_a u_k\right).
\end{aligned}
\tag{8}
$$

3.2 Taylor's formula and coefficients

Throughout the paper the following hypothesis will be used:

(M,g) is a Riemannian manifold of dimension n with metric g, (9)
covered by the coordinate system $(U,\ (x^r))$.
(TM,G) is the tangent bundle of M with $g-$natural non-

degenerate metric G, covered by a coordinate system
$(\pi^{-1}(U),\ (x^r, u^s))$, r, s run through the range $\{1, \dots, n\}$.
Z is a Killing vector field on TM with local components $(Z^r, \widetilde{Z}^s)$
with respect to the local base (∂_r, δ_s) .

Let

$$H^a = Z^a = Z^a(x,u) = \\ X^a + K^a_p u^p + \frac{1}{2}E^a_{pq}u^p u^q + \frac{1}{3!}F^a_{pqr}u^p u^q u^r + \frac{1}{4!}G^a_{pqrs}u^p u^q u^r u^s + \cdots , \quad (10)$$

$$\widetilde{Z}^a = \widetilde{Z}^a(x,u) = \\ Y^a + \widetilde{P}^a_p u^p + \frac{1}{2}Q^a_{pq}u^p u^q + \frac{1}{3!}S^a_{pqr}u^p u^q u^r + \frac{1}{4!}V^a_{pqrs}u^p u^q u^r u^s + \cdots \quad (11)$$

be expansions of the components Z^a and $\widetilde{Z}^a$ by Taylor's formula in a neighbourhood in T_xM of a point $(x,0) \in TM$. For each index a the coefficients are values of partial derivatives of Z^a, $\widetilde{Z}^a$ respectively, taken at a point $(x,0)$ and therefore are symmetric in all lower indices. For simplicity we have omitted the remainders.

Lemma 5. *([18]) The quantities*

$$X = (X^a(x)) = (Z^a(x,0)),$$
$$Y = (Y^a(x)) = \left(\widetilde{Z}^a(x,0)\right),$$
$$K = \left(K^a_p(x)\right) = (\delta_p Z^a(x,0)),$$
$$E = \left(E^a_{pq}(x)\right) = (\delta_p\delta_q Z^a(x,0)),$$
$$P = \left(P^a_p(x)\right) = \left(\left(\delta_p \widetilde{Z}^a\right)(x,0) - \partial_p(Z^a(x,0))\right)$$

are tensor fields on M.

Applying the operators ∂^v_k and ∂^h_k to the horizontal components we get

$$\partial^v_k H^a = K^a_k + E^a_{kq}u^q + \frac{1}{2}F^a_{kpq}u^p u^q + \frac{1}{3!}G^a_{kpqr}u^p u^q u^r + \cdots ,$$

$$\partial^h_k H^a = \Theta_k X^a + \Theta_k K^a_p u^p + \\ \frac{1}{2}\Theta_k E^a_{pq}u^p u^q + \frac{1}{3!}\Theta_k F^a_{pqr}u^p u^q u^r + \frac{1}{4!}\Theta_k G^a_{pqrs}u^p u^q u^r u^s + \cdots .$$

on a neighbourhood of a point $(x,0)\in TM$, where for any $(1,z)-$ tensor T we have put

$$\Theta_k T^a_{hij\dots} = \nabla_k T^a_{hij\dots} - \Gamma^a_{rk} T^r_{hij\dots}.$$

Moreover, if we put:

$$S^a_k = \widetilde{P}^a_k + X^b\Gamma^a_{bk} = \widetilde{P}^a_k - \partial_k X^a + \nabla_k X^a = P^a_k + \nabla_k X^a,$$

$$T^a_{kp} = Q^a_{kq} + K^b_k\Gamma^a_{bp} + K^b_p\Gamma^a_{bk},$$

$$\begin{aligned} F_{lkpq} &= \delta_k\delta_p\delta_q Z^a(x,0)g_{al}, \\ W_{lkpq} &= \left(\delta_k\delta_p\delta_q\overline{Z}^a(x,0) + E^c_{pk}\Gamma^a_{cq} + E^c_{qk}\Gamma^a_{cp} + E^c_{pq}\Gamma^a_{ck}\right)g_{al} = \\ &\quad \left(S^a_{kpq} + E^c_{pk}\Gamma^a_{cq} + E^c_{qk}\Gamma^a_{cp} + E^c_{pq}\Gamma^a_{ck}\right)g_{al}, \end{aligned}$$

$$Z^a_{kpqr} = V^a_{kpqr} + F^c_{kpq}\Gamma^a_{cr} + F^c_{kqr}\Gamma^a_{cp} + F^c_{krp}\Gamma^a_{cq} + F^c_{pqr}\Gamma^a_{ck},$$

then the vertical component writes

$$V^a = Y^a + S^a_p u^p + \frac{1}{2!}T^a_{pq}u^p u^q + \frac{1}{3!}W^a_{pqr}u^p u^q u^r + \frac{1}{4!}Z^a_{pqrs}u^p u^q u^r u^s + \cdots$$

and

$$\partial^v_k V^a = S^a_k + T^a_{kp}u^p + \frac{1}{2}W^a_{kpq}u^p u^q + \frac{1}{3!}Z^a_{kpqr}u^p u^q u^r + \dots,$$

$$\begin{aligned} \partial^h_k V^a = {} & \Theta_k Y^a + \Theta_k S^a_p u^p + \\ & \frac{1}{2!}\Theta_k T^a_{pq}u^p u^q + \frac{1}{3!}\Theta_k W^a_{pqr}u^p u^q u^r + \frac{1}{4!}\Theta_k Z^a_{pqrs}u^p u^q u^r u^s + \cdots \end{aligned} \tag{12}$$

on a neighbourhood of a point $(x,0)\in TM$.

We shall often use the following definitions and abbreviations:

$$S^a_p = P^a_p + \nabla_p X^a, \quad S_{kp} = S^a_p g_{ak}, \quad P_{lk} = P^a_k g_{al},$$

$$K_{lp} = K^a_p g_{al}, \quad E_{kpq} = E_{kqp} = E^a_{pq} g_{ak}, \quad T_{lkp} = T^a_{kp} g_{al}.$$

Substituting (10) - (12) into the right hand sides of (6)-(8) we obtain on some neighbourhood of $(x,0)$ expressions that are sums of polynomials in variables u^r with coefficients depending on x^t multiplied by functions depending on $r^2 = g_{rs}u^r u^s$ plus terms that contain remainders. Suppose that $Z = Z^r\partial_r + \widetilde{Z}^r\delta_r$ is a Killing vector field on TM. Then the left hand sides vanish and substituting $u = (u^j) = 0$ we obtain on M

$$A\left(\nabla_k X_l + \nabla_l X_k\right) + a_2\left(\nabla_k Y_l + \nabla_l Y_k\right) = 0, \tag{I_1}$$

$$AK_{lk} + a_2 \left(P_{lk} + \nabla_k X_l + \nabla_l X_k\right) + a_1 \nabla_l Y_k = 0, \qquad (II_1)$$

$$a_2 \left(K_{lk} + K_{kl}\right) + a_1 \left(S_{lk} + S_{kl}\right) = 0, \qquad (III_1)$$

where $A = A(0)$, $a_j = a_j(0)$. Differentiating with respect to δ_k, making use of the property

$$\delta_k f(r^2) = 2f'(r^2) g_{ks} u^s$$

and substituting $u^j = 0$ we find

$$A \left(\nabla_k K_{lp} + \nabla_l K_{kp}\right) + a_2 \left[\nabla_k S_{lp} + \nabla_l S_{kp} - X^a \left(R_{aklp} + R_{alkp}\right)\right] + 2A' g_{kl} Y_p + B \left(Y_k g_{lp} + Y_l g_{kp}\right) = 0, \qquad (I_2)$$

$$AE_{lkp} + a_1 \left(\nabla_l S_{kp} - X^a R_{alkp}\right) + a_2 \left(\nabla_l K_{kp} + T_{lkp}\right) + 2a_2' g_{kl} Y_p + b_2 \left(Y_k g_{lp} + Y_l g_{kp}\right) = 0, \qquad (II_2)$$

$$a_1 \left(T_{lkp} + T_{klp}\right) + a_2 \left(E_{lkp} + E_{klp}\right) + b_1 \left(Y_k g_{lp} + Y_l g_{kp}\right) + 2a_1' g_{kl} Y_p = 0, \qquad (III_2)$$

on M, where $A' = A'(0)$, $a_j' = a_j'(0)$ etc.

For any $(0,2)-$ tensor T we put

$$\overline{T}_{ab} = T_{ab} + T_{ba}, \quad \widehat{T}_{ab} = T_{ab} - T_{ba}$$

It is easily seen, that the quantities F and W are symmetric in the last three indices. Proceeding in the same way as before we easily obtain expressions of the second order:

$$\left(L_{H^a \partial_a^h + V^\alpha \partial_\alpha^v} G\right) \left(\partial_k^h, \partial_l^h\right)_{pq} |_{(x,0)} =$$
$$A \left(\nabla_k E_{lpq} + \nabla_l E_{kpq}\right) + a_2 \left(\nabla_k T_{lpq} + \nabla_l T_{kpq}\right) + 2A' g_{kl} \overline{S}_{pq} +$$
$$B \left[\left(\nabla_k X_p + S_{kp}\right) g_{ql} + \left(\nabla_k X_q + S_{kq}\right) g_{pl} + \left(\nabla_l X_p + S_{lp}\right) g_{qk} + \left(\nabla_l X_q + S_{lq}\right) g_{pk}\right] +$$
$$b_2 \left(\nabla_k Y_p g_{ql} + \nabla_k Y_q g_{pl} + \nabla_l Y_p g_{qk} + \nabla_l Y_q g_{pk}\right) -$$
$$a_2 \left[K_p^a \left(R_{alkq} + R_{aklq}\right) + K_q^a \left(R_{alkp} + R_{aklp}\right)\right] = 0, \qquad (I_3)$$

$$\left(L_{H^a \partial_a^h + V^\alpha \partial_\alpha^v} G\right) \left(\partial_k^v, \partial_l^h\right)_{pq} |_{(x,0)} =$$
$$AF_{lkpq} + a_2 W_{lkpq} + a_1 \nabla_l T_{kpq} + a_2 \nabla_l E_{kpq} + 2a_2' g_{kl} \overline{S}_{pq} -$$
$$a_1 \left(K_p^a R_{alkq} + K_q^a R_{alkp}\right) + B \left(K_{pk} g_{ql} + K_{qk} g_{pl}\right) +$$
$$b_2 \left(\overline{S}_{pk} g_{ql} + \overline{S}_{qk} g_{pl} + S_{lp} g_{qk} + S_{lq} g_{pk} + \nabla_l X_p g_{kq} + \nabla_l X_q g_{kp}\right) +$$
$$b_1 \left(\nabla_l Y_p g_{kq} + \nabla_l Y_q g_{kp}\right) = 0, \qquad (II_3)$$

$$\left(L_{H^a\partial_a^h+V^\alpha\partial_\alpha^v}G\right)(\partial_k^v,\partial_l^v)_{pq}|_{(x,0)}=$$
$$a_2\left(F_{lkpq}+F_{klpq}\right)+a_1\left(W_{lkpq}+W_{klpq}\right)+2a_1'g_{kl}\overline{S}_{pq}+$$
$$b_1\left(\overline{S}_{kp}g_{ql}+\overline{S}_{kq}g_{pl}+\overline{S}_{lp}g_{qk}+\overline{S}_{lq}g_{pk}\right)+$$
$$b_2\left(K_{pk}g_{ql}+K_{qk}g_{pl}+K_{pl}g_{qk}+K_{ql}g_{pk}\right)=0.\quad (III_3)$$

Finally, expressions of the third order are:

$$\left(L_{H^a\partial_a^h+V^\alpha\partial_\alpha^v}G\right)\left(\partial_k^h,\partial_l^h\right)_{pqr}|_{(x,0)}=$$
$$A\left[\nabla_kF_{lpqr}+\nabla_lF_{kpqr}\right]+a_2\left[\nabla_kW_{lpqr}+\nabla_lW_{kpqr}\right]-$$
$$a_2\left[E^a_{pq}\left(R_{alkr}+R_{aklr}\right)+E^a_{qr}\left(R_{alkp}+R_{aklp}\right)+E^a_{rp}\left(R_{alkq}+R_{aklq}\right)\right]+$$
$$B\left[\nabla_k\overline{K}_{qp}g_{lr}+\nabla_k\overline{K}_{rq}g_{lp}+\nabla_k\overline{K}_{pr}g_{lq}+\nabla_l\overline{K}_{qp}g_{kr}+\nabla_l\overline{K}_{rq}g_{kp}+\nabla_l\overline{K}_{pr}g_{kq}\right]+$$
$$b_2\left[\nabla_k\overline{S}_{qp}g_{lr}+\nabla_k\overline{S}_{rq}g_{lp}+\nabla_k\overline{S}_{pr}g_{lq}+\nabla_l\overline{S}_{qp}g_{kr}+\nabla_l\overline{S}_{rq}g_{kp}+\nabla_l\overline{S}_{pr}g_{kq}\right]+$$
$$B\left[g_{lp}T_{kqr}+g_{lq}T_{krp}+g_{lr}T_{kpq}+g_{kp}T_{lqr}+g_{kq}T_{lrp}+g_{kr}T_{lpq}\right]+$$
$$2B'\left[\left(g_{pk}g_{ql}+g_{qk}g_{pl}\right)Y_r+\left(g_{qk}g_{rl}+g_{rk}g_{ql}\right)Y_p+\left(g_{rk}g_{pl}+g_{pk}g_{rl}\right)Y_q\right]+$$
$$2A'g_{kl}M_{pqr}=0,\quad (I_4)$$

$$\left(L_{H^a\partial_a^h+V^\alpha\partial_\alpha^v}G\right)\left(\partial_k^v,\partial_l^h\right)_{pqr}|_{(x,0)}=$$
$$AG_{lkpqr}+a_2Z_{lkpqr}+a_2\nabla_lF_{kpqr}+a_1\nabla_lW_{kpqr}-$$
$$a_1\left[E^a_{pq}R_{alkr}+E^a_{qr}R_{alkp}+E^a_{rp}R_{alkq}\right]+$$
$$b_2\left[\nabla_l\overline{K}_{qp}g_{kr}+\nabla_l\overline{K}_{rq}g_{kp}+\nabla_l\overline{K}_{pr}g_{kq}\right]+$$
$$B\left[g_{lr}\left(E_{qkp}+E_{pkq}\right)+g_{lp}\left(E_{rkq}+E_{qkr}\right)+g_{lq}\left(E_{pkr}+E_{rkp}\right)\right]+$$
$$b_1\left[\nabla_l\overline{S}_{qp}g_{kr}+\nabla_l\overline{S}_{rq}g_{kp}+\nabla_l\overline{S}_{pr}g_{kq}\right]+$$
$$b_2\left[g_{kp}T_{lqr}+g_{kq}T_{lrp}+g_{kr}T_{lpq}\right]+b_2\left[g_{lp}M_{kqr}+g_{lq}M_{krp}+g_{lr}M_{kpq}\right]+$$
$$2b_2'\left[\left(g_{pk}g_{ql}+g_{qk}g_{pl}\right)Y_r+\left(g_{qk}g_{rl}+g_{rk}g_{ql}\right)Y_p+\left(g_{rk}g_{pl}+g_{pk}g_{rl}\right)Y_q\right]+$$
$$2a_2'g_{kl}M_{pqr}=0,\quad (II_4)$$

where $M_{pqr}=T_{pqr}+T_{qrp}+T_{rpq}$ and

$$Z_{lkpqr}=\left(V^a_{kpqr}+F^c_{kpq}\Gamma^a_{cr}+F^c_{kqr}\Gamma^a_{cp}+F^c_{krp}\Gamma^a_{cq}+F^c_{pqr}\Gamma^a_{ck}\right)g_{al}$$

which is symmetric in the last four lower indices.

Moreover, we have

$$\left(L_{H^a\partial_a^h+V^\alpha\partial_\alpha^v}G\right)(\partial_k^v,\partial_l^v)_{pqr}|_{(x,0)} =$$
$$a_2\left(G_{lkpqr}+G_{klpqr}\right)+a_1\left(Z_{lkpqr}+Z_{klpqr}\right)+$$
$$b_2\left[g_{lr}\left(E_{qkp}+E_{pkq}\right)+g_{lp}\left(E_{rkq}+E_{qkr}\right)+g_{lq}\left(E_{pkr}+E_{rkp}\right)\right]+$$
$$b_2\left[g_{kr}\left(E_{qlp}+E_{plq}\right)+g_{kp}\left(E_{rlq}+E_{qlr}\right)+g_{kq}\left(E_{plr}+E_{rlp}\right)\right]+$$
$$b_1\left[g_{kp}M_{lqr}+g_{kq}M_{lrp}+g_{kr}M_{lpq}\right]+b_1\left[g_{lp}M_{kqr}+g_{lq}M_{krp}+g_{lr}M_{kpq}\right]+$$
$$2b_1'\left[\left(g_{pk}g_{ql}+g_{qk}g_{pl}\right)Y_r+\left(g_{qk}g_{rl}+g_{rk}g_{ql}\right)Y_p+\left(g_{rk}g_{pl}+g_{pk}g_{rl}\right)Y_q\right]+$$
$$2a_1'g_{kl}M_{pqr}=0. \quad (III_4)$$

Important remark: Hereafter, and unless otherwise specified, all the coefficients a_j, b_j, a_j', b_j', A, A', B, B', $\dots$ are considered to be constants, equal to the values at 0 of the corresponding functions.

3.3 Lemmas

Lemma 6. *Under hypothesis (9) ,we have, on M:*

$$a_1T_{lkp}+a_2E_{lkp}=a_1'\left(Y_lg_{kp}-Y_kg_{lp}-Y_pg_{kl}\right)-b_1Y_lg_{kp}, \quad (13)$$

$$AE_{lkp}+a_2T_{lkp}+a_2'(g_{kl}Y_p+g_{pl}Y_k)+\frac{1}{2}b_2(2g_{kp}Y_l+g_{lp}Y_k+g_{kl}Y_p)=0, \quad (14)$$

$$aE_{lkm}=(a_2b_1-a_1b_2-a_2a_1')g_{km}Y_l-$$
$$\frac{1}{2}(a_1b_2-2a_2a_1'+2a_1a_2')(g_{lm}Y_k+g_{lk}Y_m), \quad (15)$$

$$aT_{lkm}=(Aa_1'+a_2b_2-Ab_1)g_{km}Y_l+\frac{1}{2}(a_2b_2-2Aa_1'+2a_2a_2')(g_{lm}Y_k+g_{lk}Y_m), \quad (16)$$

$$aM_{lkm}=[2a_2(b_2+a_2')-A(b_1+a_1')](g_{km}Y_l+g_{lk}Y_m+g_{ml}Y_k). \quad (17)$$

Moreover,

$$a_2\left[\nabla_k\left(\nabla_lX_p+\nabla_pX_l\right)+\nabla_l\left(\nabla_kX_p+\nabla_pX_k\right)-\nabla_p\left(\nabla_lX_k+\nabla_kX_l\right)\right]+$$
$$a_1\left(\nabla_k\nabla_lY_p+\nabla_l\nabla_kY_p\right)=2A'g_{kl}Y_p+B\left(Y_kg_{lp}+Y_lg_{kp}\right), \quad (18)$$

$$a\left(\nabla_kK_{lp}+\nabla_lK_{kp}\right)+(a_2b_2+2a_1A'-2a_2a_2')Y_pg_{kl}+$$
$$\frac{1}{2}(-a_2b_2+2a_1B+2a_2a_2')(Y_kg_{lp}+Y_lg_{kp})=0. \quad (19)$$

Proof. Alternating (III_2) in (l,p), then interchanging the indices (p,k) and adding the resulting equation to (III_2), we obtain (13).

Differentiating covariantly (III_1) we get

$$a_2\left(\nabla_k K_{lp} + \nabla_k K_{pl}\right) + a_1\left(\nabla_k S_{lp} + \nabla_k S_{pl}\right) = 0.$$

Symmetrizing (II_2) in (k,p) and subtracting the resulting equation from the above one we find (14).

Now (15) and (16) result immediately from (13) and (14).

From (II_1) we easily get

$$A\nabla_k K_{lp} + a_2\left(\nabla_k P_{lp} + \nabla_k\nabla_p X_l + \nabla_k\nabla_l X_p\right) + a_1\nabla_k\nabla_l Y_p = 0,$$

whence, symmetrizing in (k,l), subtracting from (I_2), by the use of the Ricci identity, we obtain (18).

To prove (19) first we symmetrize (II_2) in (k,l) and combine it with (I_2) to obtain

$$\begin{aligned}&a\left(\nabla_k K_{lm} + \nabla_l K_{km}\right) - a_2\left[A\left(E_{lkm} + E_{klm}\right) + a_2\left(T_{lkm} + T_{klm}\right)\right] + \\ &\qquad 2\left(a_1A' - 2a_2a_2'\right)g_{kl}Y_m + \left(a_1B - 2a_2b_2\right)\left(g_{lm}Y_k + g_{km}Y_l\right) = 0.\end{aligned}$$

On the other hand, symmetrizing (14) in (k,l) and subtracting from the above we obtain (19). This completes the proof. QED

Lemma 7. *Under hypothesis (9) we have, on M*

$$\begin{aligned}&2a\nabla_l K_{km} = a_1^2 Y^r R_{rmkl} - a_1 B g_{km} Y_l + \\ &\qquad (-a_1B + a_2b_2 - 2a_2a_2')g_{lm}Y_k + (-a_2b_2 - 2a_1A' + 2a_2a_2')g_{kl}Y_m, \qquad (20)\end{aligned}$$

$$\begin{aligned}&2a\left(\nabla_l S_{km} - X^r R_{rlkm}\right) + a_1a_2Y^r R_{rmkl} - a_2Bg_{km}Y_l + \\ &\qquad \left[-a_2B + A\left(b_2 - 2a_2'\right)\right]g_{lm}Y_k + \left[-2a_2A' - A\left(b_2 - 2a_2'\right)\right]g_{kl}Y_m = 0. \qquad (21)\end{aligned}$$

Proof. From (II_2) we subtract (14) to obtain

$$a_2\nabla_l K_{km} + a_1\left(\nabla_l S_{km} - X^r R_{rlkm}\right) + \left(a_2' - \frac{b_2}{2}\right)\left(g_{kl}Y_m - g_{ml}Y_k\right) = 0. \qquad (22)$$

On the other hand, interchanging in (II_1) k and m, differentiating covariantly with respect to ∂_k, alternating in (k,l) and applying the Ricci identity, we find

$$A\left(\nabla_k K_{lm} - \nabla_l K_{km}\right) + a_2\left(\nabla_k S_{lm} - \nabla_l S_{km}\right) + a_2X^r R_{rmkl} + a_1Y^r R_{rmkl} = 0.$$

Subtracting from (I_2), in virtue of the Bianchi identity, we get

$$2A\nabla_l K_{km} + 2a_2 \left(\nabla_l S_{km} - X^r R_{rlkm}\right) - a_1 Y^r R_{rmkl} + 2A' g_{kl} Y_m + B\left(g_{lm} Y_k + g_{km} Y_l\right) = 0.$$

The last equation together with (22) yields the result. QED

Lemma 8. *Under hypothesis (9) suppose* $\dim M > 2$. *Then, on* M, *we have*

$$T_{kl} = T_{lk} = 2\left(b_1 - a_1'\right)\overline{S}_{kl} + b_2 \overline{K}_{kl} = 0, \tag{23}$$

$$a_2 F_{labk} + a_1 W_{labk} + \frac{1}{2} b_2 \left(\widehat{K}_{kl} g_{ab} + \widehat{K}_{bl} g_{ak} + \widehat{K}_{al} g_{bk} + \overline{K}_{ak} g_{bl}\right) + b_1 g_{bl} \overline{S}_{ak} + a_1'(g_{kl}\overline{S}_{ab} + g_{al}\overline{S}_{bk}) = 0. \tag{24}$$

Proof. Replacing in (III_3) the indices (p,q) with (a,b), alternating in (a,l), then again in (k,l) and adding to the first equation we get

$$a_2 F_{labk} + a_1 W_{labk} + \frac{1}{2} b_2 \left(\widehat{K}_{kl} g_{ab} + 2K_{bl} g_{ak} + \widehat{K}_{al} g_{bk} + \overline{K}_{ak} g_{bl}\right) + b_1 (g_{bl}\overline{S}_{ak} + g_{ak}\overline{S}_{bl}) + a_1'(-g_{ak}\overline{S}_{bl} + g_{kl}\overline{S}_{ab} + g_{al}\overline{S}_{bk}) = 0.$$

Alternating in (a,b) we find

$$g_{bl} T_{ak} - g_{bk} T_{al} - g_{al} T_{bk} + g_{ak} T_{bl} = 0,$$

whence $(n-2)T_{ak} = 0$ results. Then (24) is obvious. QED

Lemma 9. *Under hypothesis (9) suppose* $\dim M > 1$. *Then*

$$(n-1)\beta Y_l = 0$$

on M *holds, where*

$$\beta = 2A(b_1^2 - a_1'^2 - a_1 b_1') + (a_1 b_2 - 2a_2 b_1)(3b_2 + 2a_2') + 2a_2\left[2a_1'(b_2 + a_2') + a_2 b_1'\right].$$

Proof. First, replace in (III_4) the indices (p,q,r) with (a,b,c). Alternating an equation obtained in such a way in (a,l), then in (k,l), and adding the result to the first one, we get

$$\begin{aligned}&a_2 G_{labck} + a_1 Z_{labck} + \\ &\frac{1}{2} b_2 \left[(E_{kcl} - E_{lck}) g_{ab} + (E_{kbl} - E_{lbk}) g_{ac} + 2(E_{bcl} + E_{cbl}) g_{ak} + (E_{acl} - E_{lac}) g_{bk} + \right. \\ &\quad (E_{ack} + 2E_{cak} + E_{kac})\, g_{bl} + (E_{abl} - E_{lba}) g_{ck} + (E_{abk} + 2E_{bak} + E_{kab})\, g_{cl}] + \\ &\quad b_1 (M_{bcl} g_{ak} + M_{ack} g_{bl} + M_{abk} g_{cl}) + a_1' (-M_{bcl} g_{ak} + M_{bck} g_{al} + M_{abc} g_{kl}) + \\ &\quad b_1' \left[(g_{bl} g_{ck} + g_{bk} g_{cl}) Y_a + 2 g_{ak} \left(g_{cl} Y_b + g_{bl} Y_c \right) + (g_{ac} g_{bl} + g_{ab} g_{cl}) Y_k - \right. \\ &\qquad\qquad (g_{ac} g_{bk} + g_{ab} g_{ck}) Y_l] = 0.\end{aligned}$$

Alternating in (k,b) and contracting with $g^{ab} g^{kc}$ we obtain

$$b_2 \left[(n-2)\, E_{rls} + n E_{lrs} \right] g^{rs} + (n-1)(b_1 - a_1') M_{rls} g^{rs} + (n+2)(n-1) b_1' Y_l = 0,$$

which, using (15) and (17), yields the result. QED

Remark 1. In ([3]) it is stated that the Killing vector field on TM with the Cheeger-Gromoll metric g^{CG} depends on three generators X, Y and P. By the Lemma 9, the vector field Y vanishes everywhere on M.

Lemma 10. *Under hypothesis (9)*

$$\begin{aligned}&3AF_{lkmn} + 3a_2 W_{lkmn} + B \left(g_{kl} \overline{K}_{mn} + g_{lm} \overline{K}_{kn} + g_{ln} \overline{K}_{km} \right) + \\ &\quad (b_1 - a_1')\left(Y_{n,l} g_{km} + Y_{m,l} g_{kn} + Y_{k,l} g_{mn} \right) + \\ &\quad 2(b_2 + a_2') \left(g_{kl} \overline{S}_{mn} + g_{lm} \overline{S}_{kn} + g_{ln} \overline{S}_{km} \right) + \\ &\quad 2b_2 \left[g_{km} (X_{n,l} + S_{ln}) + g_{kn} (X_{m,l} + S_{lm}) + g_{mn} (X_{k,l} + S_{lk}) \right] = 0 \qquad (25)\end{aligned}$$

is satisfied on M.

Proof. Differentiating covariantly (13) and subtracting from (II_3) we get

$$\begin{aligned}&AF_{lkmn} + a_2 W_{lkmn} + B \left(g_{lm} K_{nk} + g_{ln} K_{mk} \right) + \\ &\quad (b_1 - a_1') \left(Y_{n,l} g_{km} + Y_{m,l} g_{kn} - Y_{k,l} g_{mn} \right) - \\ &\quad a_1 \left(K^r_n R_{rlkm} + K^r_m R_{rlkn} \right) + 2a_2' g_{kl} \overline{S}_{mn} + \\ &\quad b_2 \left[g_{km} (X_{n,l} + S_{ln}) + g_{kn} (X_{m,l} + S_{lm}) + g_{ln} \overline{S}_{km} + g_{lm} \overline{S}_{kn} \right] = 0. \qquad (26)\end{aligned}$$

Antisymmetrizing in (k,m) and symmetrizing in (k,n) we have

$$\begin{aligned}&B \left[g_{kl} (K_{mn} - 2K_{nm}) + g_{lm} (K_{kn} + K_{nk}) + g_{ln} (K_{mk} - 2K_{km}) \right] + \\ &\quad 2(b_1 - a_1') \left(2Y_{m,l} g_{kn} - Y_{n,l} g_{km} - Y_{k,l} g_{mn} \right) + 3a_1 \left(K^r_n R_{rlmk} + K^r_k R_{rlmn} \right) + \\ &\quad b_2 \left[2g_{kn} (X_{m,l} + S_{lm}) - g_{km} (X_{n,l} + S_{ln}) - g_{mn} (X_{k,l} + S_{lk}) \right] + \\ &\quad (b_2 - 2a_2') \left(2g_{lm} \overline{S}_{kn} - g_{ln} \overline{S}_{km} - g_{kl} \overline{S}_{mn} \right). \qquad (27)\end{aligned}$$

Exchanging in (26) the indices k and m, then multiplying by 3 and adding to the last equation we obtain (25). This completes the proof. $\boxed{QED}$

Lemma 11. *Under hypothesis (9) relation*

$$
\begin{aligned}
&3a_2\left[E^p_{bc}\left(R_{pkal}+R^p_{lak}\right)+E^p_{ac}\left(R_{pkbl}+R^p_{lbk}\right)+E^p_{ab}\left(R_{pkcl}+R^p_{lck}\right)\right]+\\
&\quad 6A'g_{kl}(T_{abc}+T_{bca}+T_{cab})+g_{bc}K_{kal}+g_{ca}K_{kbl}+g_{ab}K_{kcl}+\\
&\quad\quad g_{cl}L_{abk}+g_{al}L_{bck}+g_{bl}L_{cak}+g_{ck}L_{abl}+g_{ak}L_{bcl}+g_{bk}L_{cal}=0 \qquad (28)
\end{aligned}
$$

holds on M, where

$$
\begin{aligned}
K_{kal}=K_{lak}=&\\
&-2b_2\left(S_{ka,l}+S_{la,k}+X_{a,kl}+X_{a,lk}\right)-(b_1-a'_1)(Y_{a,kl}+Y_{a,lk}), \qquad (29)
\end{aligned}
$$

$$
L_{abk}=L_{bak}=2B\overline{K}_{ab,k}+3BT_{kab}+(b_2-2a'_2)\overline{S}_{ab,k}+3B'(g_{ka}Y_b+g_{kb}Y_a). \qquad (30)
$$

Proof. To prove the lemma it is enough to differentiate covariantly (25) and eliminate covariant derivatives of F and W from (I_4). $\boxed{QED}$

Lemma 12. *Under hypothesis (9) suppose* $\dim M>2$. *Then the relation*

$$
\begin{aligned}
&a_1\left[2E^p_{ab}R_{plck}-E^p_{bk}R_{plac}+E^p_{bc}R_{plak}-E^p_{ak}R_{plbc}+E^p_{ac}R_{plbk}\right]+\\
&\quad B\left[\left(E_{ckb}-E_{kcb}\right)g_{al}+\left(E_{cak}-E_{kac}\right)g_{bl}+\right.\\
&\quad\left.\left(E_{abk}+E_{bak}\right)g_{cl}-\left(E_{abc}+E_{bac}\right)g_{kl}\right]+\\
&\quad (b_1-a'_1)\left[\nabla_l\overline{S}_{bc}g_{ak}-\nabla_l\overline{S}_{bk}g_{ac}\right]+\\
&b_2\left[\nabla_l\widehat{K}_{kc}g_{ab}+g_{ak}\left(\frac{3}{2}\nabla_lK_{bc}+\frac{1}{2}\nabla_lK_{cb}\right)-g_{ac}\left(\frac{3}{2}\nabla_lK_{bk}+\frac{1}{2}\nabla_lK_{kb}\right)\right]+\\
&\quad b_2\left(\nabla_lK_{ac}g_{bk}-\nabla_lK_{ak}g_{bc}\right)+\\
&\left(b_2-2a'_2\right)\left(M_{abk}g_{cl}-M_{abc}g_{kl}\right)+b_2\left[g_{bk}T_{lac}-g_{bc}T_{lak}+g_{ak}T_{lbc}-g_{ac}T_{lbk}\right]+\\
&\quad 2b'_2\left[\left(g_{bk}g_{cl}-g_{bc}g_{kl}\right)Y_a+\left(g_{ak}g_{cl}-g_{ac}g_{kl}\right)Y_b+\right.\\
&\quad\quad\left.\left(g_{al}g_{bk}+g_{ak}g_{bl}\right)Y_c-\left(g_{al}g_{bc}+g_{ac}g_{bl}\right)Y_k\right]=0
\end{aligned}
$$

holds on M.

Proof. Firstly, we change in (24) the indices (l,a,b,k) into (k,a,b,c) and differentiate covariantly with respect to ∂_l. Setting in (II_4) (a,b,c) instead of (p,q,r), subtracting the just obtained equation and, finally, alternating in (k,c) we get the result. $\boxed{QED}$

Lemma 13. *Under hypothesis (9) relations*

$$\mathbf{A}_{km} = (3a_1B - a_2b_2)\nabla_k X_m + (-2a_2b_1 + \frac{3}{2}a_1b_2 + 2a_2a_1' - 3a_1a_2')\nabla_k Y_m + a_2B(K_{km} - 2K_{mk}) + (3a_1B - 2a_2b_2 + 2a_2a_2')S_{km} + (-a_2b_2 + 2a_2a_2')S_{mk} = 0, \quad (31)$$

$$\mathbf{F}_{kl} + \mathbf{B}_{kl} = 2a_2b_2(L_Xg)_{kl} + (4a_2b_1 - 3a_1b_2 - 4a_2a_1')(L_Yg)_{kl} + 2\left(3a_2b_2 + 3a_1A' - 4a_2a_2'\right)\overline{S}_{kl} + 2a_2B\overline{K}_{kl} = 0$$

hold on M.

Proof. First, we change in (14) the indices (l,k,p) into (l,m,n), then differentiate covariantly with respect to ∂_k and symmetrize in (k,l). Next, change in (I_3) the indices (p,q) into (m,n) and subtract the former equality to obtain

$$\frac{1}{2}\left(b_2 - 2a_2'\right)(Y_{n,l}g_{km} + Y_{m,l}g_{kn} + Y_{n,k}g_{lm} + Y_{m,k}g_{ln}) - b_2(Y_{k,l} + Y_{l,k})g_{mn} - a_2\left[K_n^r(R_{rklm} + R_{rlkm}) + K_m^r(R_{rkln} + R_{rlkn})\right] + 2A'g_{kl}\overline{S}_{mn} + B\left[g_{ln}(X_{m,k} + S_{km}) + g_{lm}(X_{n,k} + S_{kn}) + g_{kn}(X_{m,l} + S_{lm}) + g_{km}(X_{n,l} + S_{ln})\right] = 0.$$

Eliminating between (27) and the last equation the terms containing curvature tensor we obtain

$$g_{mn}\mathbf{B}_{kl} + g_{kl}\mathbf{F}_{mn} + g_{ln}\mathbf{A}_{km} + g_{kn}\mathbf{A}_{lm} + g_{lm}\mathbf{A}_{kn} + g_{km}\mathbf{A}_{ln} = 0,$$

where

$$\mathbf{F}_{mn} = 2a_2B\overline{K}_{mn} + 2(2a_2b_2 + 3a_1A' - 4a_2a_2')\overline{S}_{mn},$$

$$\mathbf{B}_{kl} = 2a_2b_2(L_Xg)_{kl} + (4a_2b_1 - 3a_1b_2 - 4a_2a_1')(L_Yg)_{kl} + 2a_2b_2\overline{S}_{kl}.$$

Now, the result is a simple consequence of Lemma 15. QED

4 On the classification

To simplify further considerations put for a moment $\overline{X} = \nabla_k X_l + \nabla_l X_k$, $\overline{Y} = \nabla_k Y_l + \nabla_l Y_k$, $\overline{S} = P_{kl} + P_{lk} + \nabla_k X_l + \nabla_l X_k$, $\overline{K} = K_{kl} + K_{lk}$. Symmetrizing indices in (II_1) and taking into consideration equations (I_1), (III_1) and (23) we obtain a homogeneous system of linear equations in $\overline{X}$, $\overline{Y}$, $\overline{S}$, $\overline{K}$:

$$\begin{bmatrix} A & a_2 & 0 & 0 \\ a_2 & a_1 & a_2 & A \\ 0 & 0 & a_1 & a_2 \\ 0 & 0 & 2b & b_2 \end{bmatrix} \begin{bmatrix} \overline{X} \\ \overline{Y} \\ \overline{S} \\ \overline{K} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \tag{32}$$

where $b = b_1 - a_1'$. The system has a unique solution if and only if

$$a(2ba_2 - a_1 b_2) \neq 0,$$

where $a = a_1 A - a_2^2$.

Suppose $a \neq 0$ and $2ba_2 - a_1 b_2 = 0$.

If $a_2 b_2 \neq 0$, then multiplying the third equation by b_2 and the fourth one by a_2 we transform the whole system to

$$\begin{bmatrix} A & a_2 & 0 \\ a_2^2 & a_1 a_2 & -a \\ 0 & 0 & a_1 \end{bmatrix} \begin{bmatrix} \overline{X} \\ \overline{Y} \\ \overline{S} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -a_2\overline{K} \end{bmatrix}$$

with determinant equal to $a_1 a_2 a$.

Therefore, if $a_1 \neq 0$ and $a_2 b_2 \neq 0$, we get

$$\overline{X} + \overline{S} = 0, \quad \overline{Y} = \frac{A}{a_2}\overline{S}, \quad \overline{K} = -\frac{a_1}{a_2}\overline{S}. \tag{33}$$

On the other hand, if $a_1 = 0$ and $a_2 b_2 \neq 0$, then $b = 0$ and (32) yields

$$\begin{bmatrix} A & a_2 & 0 & 0 \\ a_2 & 0 & a_2 & A \\ 0 & 0 & 0 & a_2 \\ 0 & 0 & 0 & b_2 \end{bmatrix} \begin{bmatrix} \overline{X} \\ \overline{Y} \\ \overline{S} \\ \overline{K} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

whence

$$\overline{X} + \overline{S} = 0, \quad A\overline{X} + a_2\overline{Y} = 0, \quad \overline{K} = 0.$$

Now suppose $a_2 = 0$. Then by $2ba_2 - a_1 b_2 = 0$ we have either $a_1 = 0$ or $b_2 = 0$. But $a_1 = a_2 = 0$ would give $a = 0$. On the other hand $a_2 = b_2 = 0$ reduce the system (32) to

$$\begin{bmatrix} A & 0 & 0 & 0 \\ 0 & a_1 & 0 & A \\ 0 & 0 & a_1 & 0 \\ 0 & 0 & 2b & 0 \end{bmatrix} \begin{bmatrix} \overline{X} \\ \overline{Y} \\ \overline{S} \\ \overline{K} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Since $a_2 = 0$ and $a \neq 0$ hold if $a_1 A \neq 0$, we obtain

$$\overline{X} = 0, \quad \overline{S} = 0, \quad A\overline{K} + a_1\overline{Y} = 0.$$

Finally, if $b_2 = 0$ but $a_2 \neq 0$, we have $b = b_1 - a_1' = 0$ and from (32) we easily get (33). Thus we have proved

Lemma 14. *Under assumption $a = a_1 A - a_2^2 \neq 0$ the system (32) has the following solutions:*

(1) If $2ba_2 - a_1 b_2 \neq 0$, then $\overline{X} = \overline{Y} = \overline{S} = \overline{K} = 0$.

(2) If $[2ba_2 - a_1 b_2 = 0]$ and [either $(a_1 a_2 b_2 \neq 0)$ or $(b_2 = 0$ and $a_2 \neq 0)$], then $\overline{X} + \overline{S} = 0$, $\overline{Y} = \frac{A}{a_2}\overline{S}$, $\overline{K} = -\frac{a_1}{a_2}\overline{S}$.

(3) If $a_1 = b = 0$, then $\overline{X} + \overline{S} = 0$, $A\overline{X} + a_2\overline{Y} = 0$, $\overline{K} = 0$.

(4) If $a_2 = b_2 = 0$, then $\overline{X} = 0$, $\overline{S} = 0$, $A\overline{K} + a_1\overline{Y} = 0$.

Conversely, if $a \neq 0$, then the above four cases give the only possible solutions to (32).

Combining the above lemma with (I_1), (II_1), (III_1) and (23) we obtain the following

Theorem 2. *Let (TM, G) be a tangent bundle of a Riemannian manifold (M, g), $dim M > 2$, with non-degenerate $g-$ natural metric G. Let Z be a Killing vector field on TM with its Taylor series expansion around a point $(x, 0) \in TM$ given by (10). Then for each such a point there exists a neighbourhood $U \subset M$ of x such that one of the following cases occurs:*

(1) $2ba_2 - a_1 b_2 \neq 0$. Then

$$\nabla_k X_l + \nabla_l X_k = 0, \quad \nabla_k Y_l + \nabla_l Y_k = 0, \tag{34}$$

$$P_{kl} + P_{lk} = 0, \quad K_{kl} + K_{lk} = 0. \tag{35}$$

(2) $[2ba_2 - a_1 b_2 = 0]$ and [either $(a_1 a_2 b_2 \neq 0)$ or $(a_2 \neq 0$ and $b_2 = 0)$]. Then

$$P_{kl} + P_{lk} + 2(\nabla_k X_l + \nabla_l X_k) = 0, \tag{36}$$

$$a_2(\nabla_k Y_l + \nabla_l Y_k) + A(\nabla_k X_l + \nabla_l X_k) = 0, \tag{37}$$

$$a_2(K_{kl} + K_{lk}) - a_1(\nabla_k X_l + \nabla_l X_k) = 0. \tag{38}$$

(3) $a_2 b_2 \neq 0$ and $a_1 = b = 0$. Then

$$P_{kl} + P_{lk} + 2(\nabla_k X_l + \nabla_l X_k) = 0, \tag{39}$$

$$a_2(\nabla_k Y_l + \nabla_l Y_k) + A(\nabla_k X_l + \nabla_l X_k) = 0, \tag{40}$$

$$K_{kl} + K_{lk} = 0. \tag{41}$$

(4) $a_2 = b_2 = 0$. *Then*

$$\nabla_k X_l + \nabla_l X_k = 0, \quad P_{kl} + P_{lk} = 0, \quad AK_{lk} + a_1 \nabla_l Y_k = 0. \tag{42}$$

In the above theorem we have put $a_j = a_j(r^2)_{|(x,0)\in TM}$, $b_j = b_j(r^2)_{|(x,0)\in TM}$, $a'_j = a'_j(r^2)_{|(x,0)\in TM}$, $A = a_1 + a_3$.

Proof of the Theorem 1:

It is clear that the above results together with Proposition 3 yield Theorem 1.

5 Appendix

5.1 An algebraic lemma

Lemma 15. *Let on a manifold* (M, g), $dimM > 2$, $(0,2)-$ *tensors* A, B, F *satisfying the condition*

$$g(X,Y)F(U,V) + g(U,V)B(X,Y) +$$
$$g(Y,V)A(X,U) + g(X,V)A(Y,U) + g(Y,U)A(X,V) + g(X,U)A(Y,V) = 0$$

for arbitrary vectors X, Y, U, V *be given.*

Then F *and* B *are symmetric. Moreover,* $A = 0$, $B + F = 0$ *and* $nF - (TrF)\, g = nB - (TrB)\, g = 0$.

Proof. In local coordinates $(U, (x^a))$ the condition writes

$$g_{kl}F_{mn} + g_{mn}B_{kl} + g_{ln}A_{km} + g_{kn}A_{lm} + g_{lm}A_{kn} + g_{km}A_{ln} = 0.$$

By contractions with g^{kl}, g^{mn}, g^{km} we obtain in turn

$$2(A_{mn} + A_{nm}) + nF_{mn} + B^p_p g_{mn} = 0,$$

$$2(A_{kl} + A_{lk}) + nB_{kl} + F^p_p g_{kl} = 0,$$

$$(n+2)A_{ln} + B_{nl} + F_{ln} + A^p_p g_{ln} = 0. \tag{43}$$

Now, the symmetry of F and B results from the first two equations. Contracting the first equation with g^{mn} and the third one with g^{ln} we get

$$\begin{aligned} 4A^p_p + n(B^p_p + F^p_p) &= 0, \\ 2(n+1)A^p_p + B^p_p + F^p_p &= 0, \end{aligned}$$

whence $TrA = TrF + TrB = 0$ results. Applying these to the first system we easily get

$$\begin{aligned} 4(A_{mn} + A_{nm}) + n(F_{mn} + B_{mn}) &= 0, \\ (n+2)(A_{mn} + A_{nm}) + 2(F_{mn} + B_{mn}) &= 0, \end{aligned}$$

whence $F + B = 0$ and $A_{mn} + A_{nm} = 0$. Now (43) yields $A = 0$. The further statements are obvious. QED

5.2 Complete lift X^C

If $X = X^r \partial_r$ is a vector field on M, then $X^C = X^r \partial_r + u^s \partial_s X^r \delta_r = X^r \partial_r^h + u^s \nabla_s X^r \partial_r^v$ is said to be the complete lift of X to TM.

Lemma 16. *Let X be a vector field on (M, g) satisfying*

$$L_X g = fg, \tag{44}$$

f being a function on M, and X^C be its complete lift to (TM, G) with non-degenerate g-natural metric G. Then

$$\begin{aligned} (L_{X^C} G)\left(\partial_k^h, \partial_l^h\right) &= \left[a_2 \partial f + f(A + A' r^2)\right] g_{kl} + f(2B + B' r^2) u_k u_l + \\ &\quad \frac{1}{2} b_2 r^2 \left(\nabla_k f u_l + \nabla_l f u_k\right), \end{aligned}$$

$$\begin{aligned} (L_{X^C} G)\left(\partial_k^v, \partial_l^h\right) &= \frac{1}{2} a_1 \left(\nabla_l f u_k - \nabla_k f u_l + \partial f g_{kl}\right) + \\ &\quad f(a_2 + a_2' r^2) g_{kl} + f(2b_2 + b_2' r^2) u_k u_l + \frac{1}{2} b_1 r^2 \nabla_l f u_k, \end{aligned}$$

$$(L_{X^C} G)\left(\partial_k^v, \partial_l^v\right) = f(a_1 + a_1' r^2) g_{kl} + f(2b_1 + b_1' r^2) u_k u_l,$$

where $\partial f = u^r \nabla_r f$.

Proof. Straightforward calculations with the use of (6) - (8) . Relations (1) and (4) are useful. QED

Theorem 3. *Let X be a vector field on (M, g) such that (44) is satisfied. Then X^C is a Killing vector field on (TM, G) with non-degenerate g-natural metric G if and only if $f = 0$ on M.*

Proof. If $f = 0$, then the theorem is obvious by the previous lemma.

Suppose that $L_{X^C} G = 0$ on TM holds for some $f \neq 0$. At first, contracting the third equation with g^{kl}, next transvecting with $u^k u^l$, we easily find

$$f(a_1 + a_1' r^2) = 0,$$
$$f(2b_1 + b_1' r^2) = 0.$$

Consider now $x \in M$ such that $f(x) \neq 0$. The restriction to $T_x M$ of the first equation gives $f(x)\left[a_1(r^2) + a_1'\left(r^2\right) r^2\right] = 0$ for all $(x, u) \in T_x M$, where $r^2 = g_x(u, u)$. Using the fact that $f(x) \neq 0$, we obtain $a_1(r^2) + a_1'\left(r^2\right) r^2 = 0$ for all $(x, u) \in T_x M$. We deduce then that $a_1(t) + a_1'(t)t = 0$ for all $t \in < 0, \infty)$, whence, by continuity, we get $a_1(0) = 0$. Consequently, $a_1(t) = 0$ for all $t \in < 0, \infty)$. By the same argumentation we obtain $b_1(t) = 0$ for all $t \in < 0, \infty)$. The second equation of Lemma 16 yields, by contraction with g^{kl} and then by transvection with $u^k u^l$,

$$f(a_2 + a_2' r^2) = 0,$$
$$f(2b_2 + b_2' r^2) = 0.$$

By the same argumentation as before, we get $a_2(t) = 0$, for all $t \in < 0, \infty)$. Consequently, $a(t) = a_1(t)\left[a_1(t) + a_3(t)\right] - a_2^2(t) = 0$, for all $t \in < 0, \infty)$, which is a contradiction. This completes the proof. QED

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