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New modular relations for cubic class invariants

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Abstract. In this paper, we establish several new P–Q mixed modular equations involving theta–functions which are similar to those recorded by Ramanujan in his notebooks. As an application, we establish several new explicit evaluations of cubic class invariants and cubic singular moduli.

Keywords: Modular equations, Theta–functions, cubic class invariant, cubic singular moduli.

MSC 2000 classification: primary 33D10, secondary 11A55, 11F27

1 Introduction

Dedekind eta-function is defined by

$$\eta(z) := q^{1/24}(q;q)_\infty, \quad q = e^{2\pi iz}, \quad Im(z) > 0. \tag{1}$$

For $|q| < 1$, Ramanujan's theta function $f(-q)$ [3] is defined by

$$f(-q) := \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n-1)/2} = (q;q)_\infty, \tag{2}$$

this implies

$$f(-q) = q^{-1/24}\eta(z). \tag{3}$$

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where $(a;q)_\infty := \prod_{n=0}^{\infty}(1-aq^n)$.

At scattered places of his second notebook [18], Ramanujan records a total of nine P–Q mixed modular equations of degrees 1, 3, 5 and 15. These equations were proved by B. C. Berndt and L. -C. Zhang in [6], [7]. In [1], C. Adiga, Taekyun Kim and M. S. Mahadeva Naika have also established several cubic modular equations and some P–Q eta-function identities. For more details on P–Q eta-function identities one can refer [14], [15], [16] and [17].

The ordinary hypergeometric series ${}_2F_1(a,b;c;x)$ is defined by

$${}_2F_1(a,b;c;x) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n x^n}{(c)_n n!},$$

where

$$(a)_0 = 1, (a)_n = a(a+1)(a+2)...(a+n-1), \text{ for } n \geq 1, \ |x| < 1.$$

Let

$$z(r) := z(r;x) := {}_2F_1\left(\frac{1}{r}, \frac{r-1}{r}; 1; x\right)$$

and

$$q_r := q_r(x) := \exp\left(-\pi \csc\left(\frac{\pi}{r}\right) \frac{{}_2F_1(\frac{1}{r}, \frac{r-1}{r}; 1; 1-x)}{{}_2F_1(\frac{1}{r}, \frac{r-1}{r}; 1; x)}\right).$$

where r= 2, 3, 4, 6 and $0 < x < 1$.
Let n_1 denote a fixed natural number, and assume that

$$n_1 \frac{{}_2F_1\left(\frac{1}{3}, \frac{2}{3}; 1; 1-\alpha\right)}{{}_2F_1\left(\frac{1}{3}, \frac{2}{3}; 1; \alpha\right)} = \frac{{}_2F_1\left(\frac{1}{3}, \frac{2}{3}; 1; 1-\beta\right)}{{}_2F_1\left(\frac{1}{3}, \frac{2}{3}; 1; \beta\right)}, \tag{4}$$

holds. Then a cubic modular equation of degree n_1 in the theory of elliptic functions of signature 3 is a relation between α and β induced by (4). We say β is of degree n_1 over α and $m = \frac{z(3;\alpha)}{z(3;\beta)}$ is called the multiplier.

Now, cubic class invariant h_n and H_n [13] are defined by

$$h_n = \{\alpha_n(1-\alpha_n)^{-1}\}^{-1/12} \tag{5}$$

and

$$H_n = \{\alpha_n(1-\alpha_n)\}^{-1/12}, \tag{6}$$

where $q = e^{-2\pi\sqrt{n/3}}$, n is a positive rational number and α_n (*cubic singular modulus*) is the unique number between 0 and 1 satisfying

$$\frac{{}_2F_1\left(\frac{1}{3}, \frac{2}{3}; 1; 1-\alpha_n\right)}{{}_2F_1\left(\frac{1}{3}, \frac{2}{3}; 1; \alpha_n\right)} = \sqrt{n}. \tag{7}$$

In [8], Borweins have determined the values of cubic singular moduli α_n for $n = 2, 3, 4, 5$ and 6 using the known values of Ramanujan–Weber class invariants. In [10] and [11], H. H. Chan and W. C. Liaw have also evaluated several cubic singular moduli using cubic Russell type modular equations and Kronecker's limit formula for Euler's convenient numbers. For more details, one can see [9].

In Section 2, we collect some identities which are used in the subsequent sections.

In Section 3, we establish several new P–Q mixed modular equations akin to those recorded by Ramanujan in his notebooks.

In Sections 4 and 5, we establish several explicit values of cubic class invariants h_n and H_n, defined as in the equations (5) and (6), respectively.

In the last section, we evaluate several explicit values of cubic singular modulus α_n.

2 Preliminary results

In this section, we list some results which are useful in establishing our main results.

Lemma 2.1. [5, Ch. 33, p. 104] For $0 < \alpha < 1$, we have

$$f(-e^{-y}) = \sqrt{z}3^{-1/8}\alpha^{1/24}(1-\alpha)^{1/8} \tag{8}$$

and

$$f(-e^{-3y}) = \sqrt{z}3^{-3/8}\alpha^{1/8}(1-\alpha)^{1/24}, \tag{9}$$

where

$$z = {}_2F_1\left(\frac{1}{3}, \frac{2}{3}; 1; \alpha\right) \quad \text{and} \quad y = \pi\frac{{}_2F_1(\frac{1}{3}, \frac{2}{3}; 1; 1-\alpha)}{{}_2F_1(\frac{1}{3}, \frac{2}{3}; 1; \alpha)}.$$

Lemma 2.2. [3, Entry 1(iv), p. 349]

$$\text{If } P := \frac{f^3(-q)}{q^{1/4}f^3(-q^3)} \quad \text{and} \quad Q := \frac{f^3(-q^3)}{q^{3/4}f^3(-q^9)}, \text{ then}$$

$$PQ + \frac{27}{PQ} + 9 = \frac{Q^2}{P^2}. \tag{10}$$

Lemma 2.3. [4, Ch. 25, Entry 56, p. 210]

$$\text{If } P := \frac{f(-q)}{q^{1/3}f(-q^9)} \quad \text{and} \quad Q := \frac{f(-q^2)}{q^{2/3}f(-q^{18})}, \text{ then}$$

$$P^3 + Q^3 = P^2Q^2 + 3PQ. \tag{11}$$

Lemma 2.4. [2] If $P := \dfrac{f(-q)}{q^{1/3}f(-q^9)}$ and $Q := \dfrac{f(-q^3)}{qf(-q^{27})}$, then

$$\begin{aligned}(PQ)^3 + \left(\frac{9}{PQ}\right)^3 + 27\left[\left(\frac{P}{Q}\right)^3 + \left(\frac{Q}{P}\right)^3\right] + 243\left(\frac{1}{P^3} + \frac{1}{Q^3}\right) \\ + 9\left(P^3 + Q^3\right) + 81 = \left(\frac{Q}{P}\right)^6.\end{aligned} \tag{12}$$

Lemma 2.5. [16] If $P := \dfrac{f(-q)f(-q^5)}{q^2f(-q^9)f(-q^{45})}$ and $Q := \dfrac{f(-q)f(-q^{45})}{q^{-4/3}f(-q^9)f(-q^5)}$, then

$$\begin{aligned}Q^3 + \frac{1}{Q^3} = \left(P^2 + \frac{9^2}{P^2}\right) + 5\left(\sqrt{P} + \frac{3}{\sqrt{P}}\right)\left(\sqrt{Q^3} + \frac{1}{\sqrt{Q^3}}\right) \\ + 5\left(P + \frac{9}{P}\right) + 10.\end{aligned} \tag{13}$$

Lemma 2.6. [16] If

$$P := \frac{f(-q)f(-q^7)}{q^{8/3}f(-q^9)f(-q^{63})} \quad \text{and} \quad Q := \frac{f(-q)f(-q^{63})}{q^{-2}f(-q^9)f(-q^7)}$$

then

$$\begin{aligned}Q^4 + \frac{1}{Q^4} - 14\left(Q^3 + \frac{1}{Q^3}\right) + 28\left(Q^2 + \frac{1}{Q^2}\right) + 7\left(Q + \frac{1}{Q}\right) = P^3 + \frac{9^3}{P^3} \\ + 7\left(\sqrt{P^3} + \frac{27}{\sqrt{P^3}}\right)\left[\left(\sqrt{Q^3} + \frac{1}{\sqrt{Q^3}}\right) - \left(\sqrt{Q} + \frac{1}{\sqrt{Q}}\right)\right] + 98.\end{aligned} \tag{14}$$

Lemma 2.7. [16] If

$$P := \frac{f(-q)f(-q^{11})}{q^4f(-q^9)f(-q^{99})} \quad \text{and} \quad Q := \frac{f(-q)f(-q^{99})}{q^{-10/3}f(-q^9)f(-q^{11})}$$

then

$$\begin{aligned}Q^6 + \frac{1}{Q^6} = 165\left(P + \frac{9}{P}\right) + 66\left(P^2 + \frac{9^2}{P^2}\right) + 11\left(P^3 + \frac{9^3}{P^3}\right) + 1848 \\ + \left(P^5 + \frac{9^5}{P^5}\right) + 22\left(Q^3 + \frac{1}{Q^3}\right)\left[2\left(P^2 + \frac{9^2}{P^2}\right) + 3\left(P + \frac{9}{P}\right) + 26\right] \\ + 55\left(\sqrt{P} + \frac{3}{\sqrt{P}}\right)\left(\sqrt{Q^9} + \frac{1}{\sqrt{Q^9}}\right) + 11\left(\sqrt{Q^3} + \frac{1}{\sqrt{Q^3}}\right)\left[\sqrt{P^7}\right. \\ \left. + \frac{3^7}{\sqrt{P^7}} + \left(\sqrt{P^5} + \frac{3^5}{\sqrt{P^5}}\right) + 14\left(\sqrt{P^3} + \frac{3^3}{\sqrt{P^3}}\right) + 15\left(\sqrt{P} + \frac{3}{\sqrt{P}}\right)\right].\end{aligned} \tag{15}$$

Lemma 2.8. [16] If

$$P := \frac{f(-q)f(-q^{13})}{q^{14/3}f(-q^9)f(-q^{117})} \quad \text{and} \quad Q := \frac{f(-q)f(-q^{117})}{q^{-4}f(-q^9)f(-q^{13})},$$

then

$$\begin{aligned}
&Q^7 + \frac{1}{Q^7} - 65\left(Q^6 + \frac{1}{Q^6}\right) + 910\left(Q^5 + \frac{1}{Q^5}\right) - 1417\left(Q^4 + \frac{1}{Q^4}\right)\\
&- 6994\left(Q^3 + \frac{1}{Q^3}\right) + 10049\left(Q^2 + \frac{1}{Q^2}\right) + 6981\left(Q + \frac{1}{Q}\right) = 17472\\
&+ P^6 + \frac{9^6}{P^6} - 13\left(\sqrt{P^9} + \frac{3^9}{\sqrt{P^9}}\right)\left[\left(\sqrt{Q} + \frac{1}{\sqrt{Q}}\right) - \left(\sqrt{Q^3} + \frac{1}{\sqrt{Q^3}}\right)\right]\\
&- 13\left(\sqrt{P^3} + \frac{3^3}{\sqrt{P^3}}\right)\left[139\left(\sqrt{Q} + \frac{1}{\sqrt{Q}}\right) - 179\left(\sqrt{Q^3} + \frac{1}{\sqrt{Q^3}}\right)\right. \qquad (16)\\
&\left. -2\left(\sqrt{Q^5} + \frac{1}{\sqrt{Q^5}}\right) + 52\left(\sqrt{Q^7} + \frac{1}{\sqrt{Q^7}}\right) + 10\left(\sqrt{Q^9} + \frac{1}{\sqrt{Q^9}}\right)\right]\\
&+ 13\left(P^3 + \frac{9^3}{P^3}\right)\left[26 + 5\left(Q^3 + \frac{1}{Q^3}\right) - 12\left(Q^2 + \frac{1}{Q^2}\right)\right.\\
&\left. -6\left(Q + \frac{1}{Q}\right)\right].
\end{aligned}$$

3 Mixed modular equations

In this section, we establish several new mixed modular equations which are useful to establish explicit evaluations of cubic class invariants and cubic singular moduli.
Throughout this section, we set

$$A := \frac{f^2(-q^3)}{q^{1/6}f(-q)f(-q^9)} \quad \text{and} \quad B_r := \frac{f^2(-q^{3r})}{q^{r/6}f(-q^r)f(-q^{9r})}$$

Theorem 3.1. If $P := AB_2$ and $Q := \dfrac{A}{B_2}$, then

$$Q^3 + \frac{1}{Q^3} = P - \frac{3}{P}. \qquad (17)$$

Proof. Equation (10) can be rewritten as

$$X^3 + \frac{27}{X^3} + 9 = A^6, \text{ where } X := \frac{f(-q)}{q^{1/3}f(-q^9)}. \qquad (18)$$

Changing q to q^2 in the equation (18), we get

$$Y^3 + \frac{27}{Y^3} + 9 = B_2^6, \text{ where } Y := \frac{f(-q^2)}{q^{2/3} f(-q^{18})}. \tag{19}$$

From the equation (11), we obtain

$$w = a^2 + 3a, \text{ where } a := XY \text{ and } w := X^3 + Y^3. \tag{20}$$

Solving for a and cubing both sides, we obtain

$$27X^3Y^3 + 6X^6Y^3 + 6X^3Y^6 + X^6Y^6 = X^9 + Y^9. \tag{21}$$

On solving the quadratic equations (18) and (19) for X^3 and Y^3 respectively, and using the resultants in the equation (21), we find that

$$\begin{aligned} &A^6uB_2^6v + 9A^6u + 135A^6B_2^6 + 27A^6v + 27uB_2^6 - 2A^{12}u - 12A^{12}B_2^6 \\ &- 12A^6B_2^{12} - 3uB_2^{12} - 2B_2^{12}v + A^{12}B_2^{12} + A^6uB_2^{12} + A^{12}B_2^6v - 12A^6B_2^6v \\ &- 3uB_2^6v - 12A^6uB_2^6 - 3A^6uv + 27B_2^{12} + 27A^{12} - 2A^{18} - 2B_2^{18} + 81B_2^6 \\ &- 3A^{12}v + 81A^6 + 9B_2^6v + 54v + 54u = 0, \end{aligned} \tag{22}$$

where $u := \pm\sqrt{A^{12} - 18A^6 - 27}$ and $v := \pm\sqrt{B_2^{12} - 18B_2^6 - 27}$.
Collecting the terms containing u on one side of the equation (22) and squaring both sides, we find that

$$C(v,q)F(q)G(q) = 0, \tag{23}$$

where $C(v,q)$ is a non-zero constant, $F(q) = A^6 + 3B_2^2A^2 - B_2^4A^4 + B_2^6$ and

$$G(q) = A^{12} + B_2^4A^{10} - 3B_2^2A^8 + B_2^8A^8 + 5A^6B_2^6 + B_2^{10}A^4 + 9B_2^4A^4 - 3B_2^8A^2 + B_2^{12}.$$

Consider the sequence $\{q_n\} = \left\{\frac{1}{n+1}\right\}$, $n = 1, 2, 3\ldots$, which has a limit in $|q| < 1$. We see that $F(q_n) = 0$, where as $G(q_n) \neq 0$ $\forall n$. Then by zeros of analytic functions, $F \equiv 0$ on $|q| < 1$ [12]. By setting $P := AB_2$ and $Q := \frac{A}{B_2}$, we arrive at (17). This completes the proof. QED

Theorem 3.2. If $P := AB_3$ and $Q := \frac{A}{B_3}$, then

$$\begin{aligned} &\left(Q^3 + \frac{1}{Q^3}\right)\left[-18 + 9\left(P^3 + \frac{1}{P^3}\right)\right] + P^3\left(Q^6 + \frac{1}{Q^6}\right) - \left(P^6 + \frac{81}{P^6}\right) \\ &+ 27\left[\frac{1}{P^9} + \left(P^3 + \frac{4}{P^3}\right)\right] = 82. \end{aligned} \tag{24}$$

Proof. The proof of (24) is similar to the proof of the equation (17), except that in place of the equation (11), (12) is used. QED

Theorem 3.3. If $P := AB_5$ and $Q := \dfrac{A}{B_5}$, then

$$Q^3 + \frac{1}{Q^3} + 5\left(P - \frac{3}{P}\right) = \left(P^2 - \frac{9}{P^2}\right) \tag{25}$$

Proof. The proof of (25) is similar to the proof of the equation (17), except that in place of the equation (11), (13) is used. QED

Theorem 3.4. If $P := AB_7$ and $Q := \dfrac{A}{B_7}$, then

$$\begin{aligned} &Q^4 + \frac{1}{Q^4} + 7\left(Q^3 + \frac{1}{Q^3}\right) + 28\left(Q^2 + \frac{1}{Q^2}\right) + 56\left(Q + \frac{1}{Q}\right) + 77 \\ &= \left(P^3 - \frac{27}{P^3}\right) \end{aligned} \tag{26}$$

Proof. The proof of (26) is similar to the proof of the equation (17), except that in place of the equation (11), (14) is used. QED

Theorem 3.5. If $P := AB_{11}$ and $Q := \dfrac{A}{B_{11}}$, then

$$\begin{aligned} &22\left(P^4 + \frac{3^4}{P^4}\right) + 682\left(P^2 + \frac{3^2}{P^2}\right) + Q^6 + \frac{1}{Q^6} = 176\left(P^3 + \frac{3^3}{P^3}\right) \\ &+ 1551\left(P + \frac{1}{P}\right) + 11\left(Q^3 + \frac{1}{Q^3}\right)\left[4\left(P - \frac{3}{P}\right) - \left(P^2 - \frac{9}{P^2}\right)\right] \\ &+ P^5 + \frac{3^5}{P^5} - 2750. \end{aligned} \tag{27}$$

Proof. The proof of (27) is similar to the proof of the equation (17), except that in place of the equation (11), (15) is used. QED

Theorem 3.6. If $P := AB_{13}$ and $Q := \dfrac{A}{B_{13}}$, then

$$\begin{aligned} &\left(P^3 - \frac{3^3}{P^3}\right)\left[650 + 455\left(Q + \frac{1}{Q}\right) + 143\left(Q^2 + \frac{1}{Q^2}\right) + 13\left(Q^3 + \frac{1}{Q^3}\right)\right] \\ &+ Q^7 + \frac{1}{Q^7} + 13\left(Q^6 + \frac{1}{Q^6}\right) + 130\left(Q^5 + \frac{1}{Q^5}\right) + 741\left(Q^4 + \frac{1}{Q^4}\right) \end{aligned}$$

$$+3042\left(Q^3+\frac{1}{Q^3}\right)+8385\left(Q^2+\frac{1}{Q^2}\right)+14755\left(Q+\frac{1}{Q}\right)+17524$$
$$=P^6+\frac{3^6}{P^6}. \tag{28}$$

Proof. The proof of (28) is similar to the proof of the equation (17), except that in place of the equation (11), (16) is used. QED

4 Explicit evaluations of class invariant h_n

In this section, we obtain some explicit values of cubic class invariant h_n using the above modular equations.

Lemma 4.1. We have

$$\alpha_n = 1-\alpha_{1/n}. \tag{29}$$

Proof. From the definition of α_n (7), we have

$$\sqrt{n}=\frac{{}_2F_1\left(\frac{1}{3},\frac{2}{3};1;1-\alpha_n\right)}{{}_2F_1\left(\frac{1}{3},\frac{2}{3};1;\alpha_n\right)}=\frac{{}_2F_1\left(\frac{1}{3},\frac{2}{3};1;\alpha_{1/n}\right)}{{}_2F_1\left(\frac{1}{3},\frac{2}{3};1;1-\alpha_{1/n}\right)}.$$

By the monotonicity of ${}_2F_1\left(\frac{1}{3},\frac{2}{3};1;\alpha_n\right)$ on $(0,1)$, it follows that, for $0<\alpha_n<1$,

$${}_2F_1\left(\frac{1}{3},\frac{2}{3};1;\alpha_n\right)={}_2F_1\left(\frac{1}{3},\frac{2}{3};1;1-\alpha_{1/n}\right).$$

This completes the proof. QED

Lemma 4.2. We have

$$h_n h_{1/n}=1. \tag{30}$$

Proof. It is easily obtained from the definition of h_n (5) and (29). QED

Lemma 4.3.

$$\frac{f^2(-q^3)}{q^{1/6}f(-q)f(-q^9)} = \frac{h_{9n}}{h_n}, \tag{31}$$

$$\frac{f^2(-q^{3r})}{q^{r/6}f(-q^r)f(-q^{9r})} = \frac{h_{9r^2n}}{h_{r^2n}} \tag{32}$$

Proof. Using the equations (8), (9) and (5), we obtain (31) and (32). QED

Lemma 4.4. If $X := h_n h_{9n}$ and $Y := \dfrac{h_{9n}}{h_n}$, then

$$3\sqrt{3}\left(X^3 + \frac{1}{X^3}\right) + 9 = Y^6. \tag{33}$$

Proof. Using the equation (10) along with the equations (5),(8) and (9), we obtain (33). QED

Theorem 4.5. We have

$$h_6 = (9 + 3\sqrt{6})^{1/6}, \tag{34}$$

$$h_{3/2} = (9 - 3\sqrt{6})^{1/6}, \tag{35}$$

$$h_{1/6} = \left(\frac{3 - \sqrt{6}}{9}\right)^{1/6}, \tag{36}$$

$$h_{2/3} = \left(\frac{3 + \sqrt{6}}{9}\right)^{1/6}. \tag{37}$$

Proof. On employing the equations (31) and (32) in the equation (17) with $n = 1/6$ and using the fact that $h_n h_{1/n} = 1$, we obtain

$$x^2 - 2x - 3 = 0, \tag{38}$$

where $x = h_6^2 h_{3/2}^2$. On solving the equation (38) for x, we get

$$h_6 h_{3/2} = \sqrt{3}. \tag{39}$$

Now, setting $n = 1/6$ in the equation (33) and using (39), we obtain

$$y^2 - 2y\sqrt{3} + 1 = 0, \tag{40}$$

where $y = h_6^3 h_{2/3}^3$. On solving the equation (40) for y, we get

$$h_6^3 h_{2/3}^3 = \sqrt{3} + \sqrt{2}. \tag{41}$$

Using (39), (41) and (30), we obtain (34) to (37). QED

Theorem 4.6. We have

$$h_{1/15} = 3^{-1/4}(\sqrt{5} - 2)^{1/12}\left(\frac{\sqrt{5} - \sqrt{3}}{\sqrt{2}}\right)^{1/6}\left(\frac{3\sqrt{3} - 5}{\sqrt{2}}\right)^{1/6}, \tag{42}$$

$$h_{15} = 3^{1/4}(\sqrt{5} + 2)^{1/12}\left(\frac{\sqrt{5} + \sqrt{3}}{\sqrt{2}}\right)^{1/6}\left(\frac{3\sqrt{3} + 5}{\sqrt{2}}\right)^{1/6}, \tag{43}$$

$$h_{5/3} = 3^{-1/4}(\sqrt{5} - 2)^{1/12}\left(\frac{\sqrt{5} + \sqrt{3}}{\sqrt{2}}\right)^{1/6}\left(\frac{3\sqrt{3} + 5}{\sqrt{2}}\right)^{1/6}, \tag{44}$$

$$h_{3/5} = 3^{1/4}(\sqrt{5}+2)^{1/12}\left(\frac{\sqrt{5}-\sqrt{3}}{\sqrt{2}}\right)^{1/6}\left(\frac{3\sqrt{3}-5}{\sqrt{2}}\right)^{1/6}. \tag{45}$$

Proof. On employing the equations (31) and (32) in the equation (25) with $n = 1/15$ and using the fact that $h_n h_{1/n} = 1$, we deduce that

$$x^4 - 5x^3 - 2x^2 + 15x - 9 = 0, \tag{46}$$

where $x = h_{15}^2 h_{3/5}^2$. On solving the equation (46) for x, we get

$$h_{15}^2 h_{3/5}^2 = \frac{3}{2}(\sqrt{5}+1). \tag{47}$$

Now, setting $n = 1/15$ in the equation (33) and using (47), we obtain

$$y^2 - \sqrt{3}y(5+3\sqrt{5}) + 1 = 0, \tag{48}$$

where $y = h_{15}^3 h_{5/3}^3$. On solving the equation (48) for y, we get

$$h_{15}^3 h_{5/3}^3 = \frac{5\sqrt{3}+3\sqrt{15}+5\sqrt{5}+9}{4}. \tag{49}$$

Using (47), (49) and (30), we obtain (42) to (45). QED

Theorem 4.7.

$$h_{1/21} = 3^{-1/12}\left(\frac{19\sqrt{21}-87}{6}\right)^{1/12}\left(\sqrt{\frac{17+3\sqrt{21}}{8}}-\sqrt{\frac{9+3\sqrt{21}}{8}}\right), \tag{50}$$

$$h_{21} = 3^{1/12}\left(\frac{19\sqrt{21}+87}{2}\right)^{1/12}\left(\sqrt{\frac{17+3\sqrt{21}}{8}}+\sqrt{\frac{9+3\sqrt{21}}{8}}\right), \tag{51}$$

$$h_{3/7} = 3^{1/12}\left(\frac{19\sqrt{21}+87}{2}\right)^{1/12}\left(\sqrt{\frac{17+3\sqrt{21}}{8}}-\sqrt{\frac{9+3\sqrt{21}}{8}}\right), \tag{52}$$

$$h_{7/3} = 3^{-1/12}\left(\frac{19\sqrt{21}-87}{6}\right)^{1/12}\left(\sqrt{\frac{17+3\sqrt{21}}{8}}+\sqrt{\frac{9+3\sqrt{21}}{8}}\right). \tag{53}$$

Proof. On employing the equations (31) and (32) in the equation (26) with $n = 1/21$ and using the fact that $h_n h_{1/n} = 1$, we find that

$$x^2 - 261x - 27 = 0, \tag{54}$$

where $x = h_{21}^6 h_{3/7}^6$. On solving the equation (54), we get

$$h_{21}^6 h_{3/7}^6 = \frac{261 + 57\sqrt{21}}{2}. \tag{55}$$

Now, setting $n = 1/21$ in the equation (33) and using (55), we obtain

$$y^3 - 3y = \left(\frac{27\sqrt{3}}{2} + \frac{19\sqrt{7}}{2}\right), \tag{56}$$

where $y = t + \frac{1}{t}$ and $t = h_{21}h_{7/3}$. On solving the equation (56) for y $(y \in R)$, we get

$$t + \frac{1}{t} = \frac{\sqrt{7} + 3\sqrt{3}}{2}. \tag{57}$$

Again, on solving (57) for t, we get

$$h_{21}h_{7/3} = \frac{\sqrt{7} + 3\sqrt{3} + \sqrt{18 + 6\sqrt{21}}}{4}. \tag{58}$$

Using (55), (58) and (30), we obtain (50) to (53). QED

Theorem 4.8.

$$h_{39} = 3^{1/12}(26\sqrt{3} + 45)^{1/12}(\sqrt{13} + 2\sqrt{3})^{1/12}(2 + \sqrt{3})^{1/4}\left(\frac{a+b}{2}\right)^{1/2}, \tag{59}$$

$$h_{1/39} = 3^{-1/12}\left(\frac{26\sqrt{3} - 45}{3}\right)^{1/12}(\sqrt{13} - 2\sqrt{3})^{1/12}(2 + \sqrt{3})^{1/4}\left(\frac{a-b}{2}\right)^{1/2}, \tag{60}$$

$$h_{13/3} = 3^{-1/12}\left(\frac{26\sqrt{3} - 45}{3}\right)^{1/12}(\sqrt{13} - 2\sqrt{3})^{1/12}(2 + \sqrt{3})^{1/4}\left(\frac{a+b}{2}\right)^{1/2}, \tag{61}$$

$$h_{3/13} = 3^{1/12}(26\sqrt{3} + 45)^{1/12}(\sqrt{13} + 2\sqrt{3})^{1/12}(2 + \sqrt{3})^{1/4}\left(\frac{a-b}{2}\right)^{1/2}. \tag{62}$$

where $a = \sqrt{8 + \sqrt{39}}$ and $b = \sqrt{4\sqrt{3} + \sqrt{39}}$.

Proof. On employing the equations (31) and (32) in the equation (28) with $n = 1/39$ and using the fact that $h_n h_{1/n} = 1$, we obtain

$$u^2 - 1872u - 71604 = 0, \tag{63}$$

where $u = x - \frac{27}{x}$ and $x = h_{39}^6 h_{3/13}^6$. On solving the equation (63) for u, we get

$$x - \frac{27}{x} = 936 + 270\sqrt{13}. \tag{64}$$

Again, on solving (64) for x, we get

$$x = 270\sqrt{3} + 78\sqrt{39} + 135\sqrt{13} + 468. \tag{65}$$

Now, setting $n = 1/39$ in the equation (33) and using (65), we obtain

$$y^3 - 3y = 90 + 26\sqrt{13} + 15\sqrt{39} + 51\sqrt{3}, \tag{66}$$

where $y = t + \frac{1}{t}$ and $t = h_{39}h_{13/3}$. On solving the equation (66) for y, we get

$$t + \frac{1}{t} = \frac{3 + \sqrt{39} + \sqrt{13} + \sqrt{3}}{4} \tag{67}$$

Again, on solving (67) for t, we get

$$h_{39}h_{13/3} = \frac{3 + \sqrt{39} + \sqrt{13} + \sqrt{3} + 2\sqrt{(2\sqrt{3}+3)(\sqrt{13}+4)}}{4}. \tag{68}$$

Using (65), (68) and (30), we obtain (59) to (62). QED

5 Explicit evaluation of class invariant H_n

In this section, we establish several explicit evaluation for H_n.

Lemma 5.1. We have

$$H_n^6 = h_n^6 + \frac{1}{h_n^6}. \tag{69}$$

Proof. From the equations (5) and (6), we obtain (69). QED

Lemma 5.2. We have

$$H_n = H_{1/n}. \tag{70}$$

Proof. From the equation (6), we obtain (70). QED

One can evaluate the explicit evaluation of H_n by using the value of h_n which is established in the previous section and the equation (69).

Theorem 5.3.

$$H_6 = \left(\frac{28}{3} + \frac{26\sqrt{6}}{9}\right)^{1/6}, \tag{71}$$

$$H_{3/2} = \left(\frac{32}{9} + \frac{\sqrt{6}}{3}\right)^{1/6}, \tag{72}$$

$$H_{15} = \left(\frac{\sqrt{3\sqrt{5}-6}}{9}\right)^{1/6} (340\sqrt{3} + 150\sqrt{15} + 254\sqrt{5} + 576)^{1/6}, \tag{73}$$

$$H_{3/5} = \left(\frac{\sqrt{3\sqrt{5}-6}}{9}\right)^{1/6} (340\sqrt{3} + 150\sqrt{15} - 254\sqrt{5} - 576)^{1/6}, \tag{74}$$

$$H_{21} = \left(\frac{19\sqrt{21}-87}{18}\right)^{1/12} \left(\frac{7341\sqrt{3} + 4807\sqrt{7} + 3a(623 + 136\sqrt{21})}{4}\right)^{1/6}, \tag{75}$$

$$H_{3/7} = \left(\frac{19\sqrt{21}-87}{18}\right)^{1/12} \left(\frac{7341\sqrt{3} + 4807\sqrt{7} - 3a(623 + 136\sqrt{21})}{4}\right)^{1/6}, \tag{76}$$

$$H_{39} = \left[\left(\frac{308\sqrt{3}}{9} + \frac{26\sqrt{13}}{3} + 10\sqrt{39} + 30\right) + b\left(\frac{7}{3} + \frac{5\sqrt{39}}{3} + \frac{\sqrt{13}}{3} + 7\sqrt{3}\right)\right]^{1/6} \times \left(270\sqrt{3} + 78\sqrt{39} + 468 + 135\sqrt{13}\right)^{1/12}, \tag{77}$$

$$H_{3/13} = \left[\left(\frac{308\sqrt{3}}{9} + \frac{26\sqrt{13}}{3} + 10\sqrt{39} + 30\right) - b\left(\frac{7}{3} + \frac{5\sqrt{39}}{3} + \frac{\sqrt{13}}{3} + 7\sqrt{3}\right)\right]^{1/6} \times \left(270\sqrt{3} + 78\sqrt{39} + 468 + 135\sqrt{13}\right)^{1/12}. \tag{78}$$

where $a = \sqrt{18 + 6\sqrt{21}}$ and $b = \sqrt{(2\sqrt{3}+3)(\sqrt{13}+4)}$.

Proof. To prove (71), we use (34) and (69).Similarly, to prove (72), (73), (74), (75), (76), (77) and (78), we employ the values of $h_{3/2}$, h_{15}, $h_{3/5}$, h_{21}, $h_{3/7}$, h_{39}, and $h_{3/13}$, respectively in (69). QED

6 Evaluation of cubic singular modulus α_n

In this section, we establish several explicit evaluation of cubic singular modulus α_n for different values of n.

Lemma 6.1. We have

$$\alpha_n = \frac{1}{1 + h_n^{12}}. \tag{79}$$

Proof. From the equation (5), we obtain (79) QED

We now find some values of α_n using (79) and the new values of h_n established in the previous section.

n	α_n
6	$\frac{17}{125}-\frac{27\sqrt{6}}{500}$
3/2	$\frac{108}{125}-\frac{27\sqrt{6}}{500}$
15	$\frac{9863}{1331}-\frac{22815\sqrt{3}}{5324}-\frac{16713\sqrt{5}}{5324}+\frac{4833\sqrt{15}}{2662}$
3/5	$-\frac{8532}{1331}-\frac{22815\sqrt{3}}{5324}+\frac{16713\sqrt{5}}{5324}+\frac{4833\sqrt{15}}{2662}$
21	$\frac{1841863}{2456500}-\frac{160683\sqrt{21}}{2456500}-\frac{\sqrt{18+6\sqrt{21}}}{4913000}(32697+139311\sqrt{3})$
3/7	$\frac{614637}{2456500}+\frac{160683\sqrt{21}}{2456500}-\frac{\sqrt{18+6\sqrt{21}}}{4913000}(32697+139311\sqrt{3})$
39	$\frac{\sqrt{(2\sqrt{3}+3)(\sqrt{13}+4)}}{12167000}(2413773\sqrt{3}-983007+142893\sqrt{13}-731529\sqrt{39})$ $-\frac{171639\sqrt{3}}{6083500}+\frac{938069}{1216700}-\frac{8856\sqrt{13}}{1520875}-\frac{40557\sqrt{39}}{1216700}$
3/13	$\frac{\sqrt{(2\sqrt{3}+3)(\sqrt{13}+4)}}{12167000}(983007-2413773\sqrt{3}-142893\sqrt{13}+731529\sqrt{39})$ $-\frac{171639\sqrt{3}}{6083500}+\frac{938069}{1216700}-\frac{8856\sqrt{13}}{1520875}-\frac{40557\sqrt{39}}{1216700}$

Remark 1. The explicit values of $\alpha_{1/n}$ for $n = 6, 3/2, 15, 3/5, 21, 3/7, 39, 3/13$ can be evaluated by using the equation (29).

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