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Quantitative conditions for the existence of low-order spin-orbit resonances [i]

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Abstract. We consider $(p, q)-$periodic orbits of a dissipative spin-orbit model, i.e. during q revolutions about the central planet the satellite does p rotations on its own spin-axis. L. Biasco and L. Chierchia in [2] give a sufficient condition for the existence of such orbits for $q = 1, 2$ and 4. Here we give explicit upper bounds on the eccentricity and, in the case $q = 4$, also on the dissipation, such that this condition is satisfied.

Keywords: Periodic orbits. Celestial Mechanics. Spinorbit resonances. Dissipative systems.

MSC 2000 classification: 70F15, 70F40, 70E20, 70G70, 32A10, 34C25

1 Introduction

The planar dissipative spin-orbit model considers the movements of an ellipsoidal satellite, orbiting about a central planet on a Keplerian elliptical trajectory. Important assumptions[1] are that the spin-axis of the satellite is perpendicular to the orbital plane and that the formula for dissipation, caused by the non-rigid structure of the satellite, is given by MacDonald's torque [8].

The system has a $(p, q)-$resonance (or equivalently: the two bodies move on a $(p, q)-$periodic orbit) if the satellite makes p rotations on itself exactly when it completes q revolutions about the central planet. In [2] L. Biasco and L. Chierchia study the special cases with $q = 1, 2$ and 4. In particular the cases $q = 1, 2$ are the most interesting cases in Celestial Mechanics since all the satellites of the solar system, observed in a spin-orbit resonance, are actually in a (1,1) resonance (including the Earth-Moon system) with the remarkable exception of Mercury, which is trapped in a (3,2) resonance with the Sun.

In [2] L. Biasco and L. Chierchia give a sufficient condition (see (7)) for the existence of such $(p, q)-$ periodic orbits in the cases $q = 1, 2$ and 4. However, they do not investigate the hypotheses under which such a condition holds.

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[1] The detailed model will be introduced in Subsection 1.1.

By an *asymptotic analysis* we find upper bounds of the eccentricity and, for the case $q = 4$, of the dissipation, such that the condition (7) is satisfied.

1.1 The planar dissipative spin-orbit model[2]

Consider a triaxial ellipsoidal satellite $\mathcal{S}$ moving on a Keplerian elliptic orbit with eccentricity $\mathbf{e} \in [0, 1)$ under the gravitational influence of a central body $\mathcal{P}$, (see Figure 1). We assume that the spin-axis of the satellite coincides with its shortest physical axis and that it is perpendicular to the orbit plane. The equation of motion is then given by

$$\ddot{x} + \eta(\dot{x} - \nu) + \varepsilon f_x(x, t; \mathbf{e}) = 0, \tag{1}$$

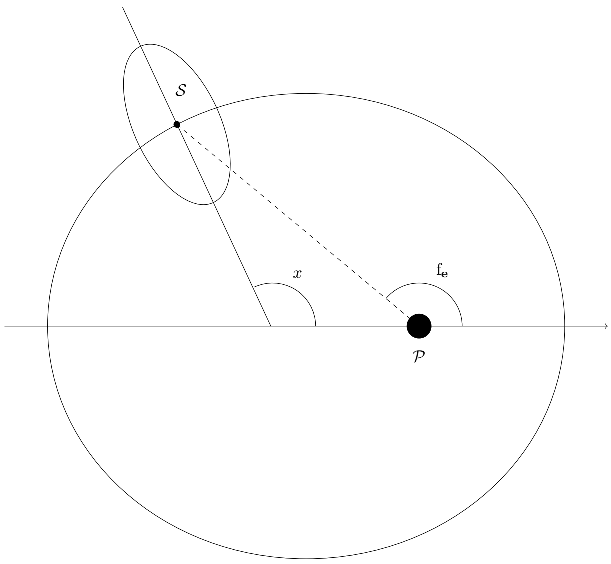

Figure 1. The planar spin-orbit model.

where

- x represents the angle (see Figure 1) formed by the direction of the major physical axis of the satellite with the major axis of the orbit plane;
- the parameters $\eta = \eta(\mathbf{e})$ and $\nu \mp \nu(\mathbf{e}) > 1$ are real-analytic functions of the eccentricity;

[2]According to [6], [7], [11], [12], [4], [3] and most recently to [5].

- the *equatorial ellipticity* ε measures the oblateness of the satellite;
- the Newtonian potential f is given by

$$f(x,t;\mathbf{e}) := -\frac{1}{2\rho_{\mathbf{e}}(t)^3}\cos(2x - 2\mathrm{f}_{\mathbf{e}}(t)). \tag{2}$$

t is the mean anomaly, $\rho_{\mathbf{e}}(t)$ is the (normalized) orbital radius and $\mathrm{f}_{\mathbf{e}}(t)$ is the true anomaly. These are related by the equations

$$\rho_{\mathbf{e}}(t) := 1 - \mathbf{e}\cos(u_{\mathbf{e}}(t)), \tag{3}$$

$$\mathrm{f}_{\mathbf{e}}(t) := 2\arctan\left(\sqrt{\frac{1+\mathbf{e}}{1-\mathbf{e}}}\tan\left(\frac{u_{\mathbf{e}}(t)}{2}\right)\right), \tag{4}$$

where the eccentric anomaly $u = u_{\mathbf{e}}(t)$ is given by the Kepler equation

$$t = u - \mathbf{e}\sin(u). \tag{5}$$

Remark 1. By [10] we know that for $\mathbf{e} \in \mathbb{C}$ and $t \in \mathbb{R}$ the function $u_{\mathbf{e}}(t)$ is real-analytic for $|\mathbf{e}| < r_\star$, where

$$r_\star := \max_{y\in\mathbb{R}} \frac{y}{\cosh(y)} = \frac{y_\star}{\cosh(y_\star)} = 0.6627434\ldots \text{ and } y_\star = 1.1996786\ldots.$$

1.2 Results

We define $\alpha_j = \alpha_j(\mathbf{e})$ to be the Fourier coefficients of the Newtonian potential f given in equation (2), i.e.

$$f(x,t;\mathbf{e}) = \sum_{j\neq 0, j\in\mathbb{Z}} \alpha_j(\mathbf{e})\cos(2x - jt). \tag{6}$$

From Theorem 1.2 and Proposition 2.10 of [2] L. Biasco and L. Chierchia we can formulate the following theorem:

Theorem 1. *Let p and q be positive coprime integers*[3]*, with $q = 1, 2$ or 4 and fix $0 < \kappa < 1$. Then, there exists $\varepsilon_0 > 0$ and $\eta_0 > 0$ such that for any $0 < \varepsilon \leq \varepsilon_0$ and $0 < \eta \leq \eta_0$ the spin-orbit problem modelled by equations* (1)-(6) *has periodic solutions x_{pq} of type*[4] (p,q)*, provided*

$$\left|\nu - \frac{p}{q}\right| < \begin{cases} 2\kappa\frac{\varepsilon}{\eta}|\alpha_{2p}|, & \text{if } q = 1,\\ 2\kappa\frac{\varepsilon}{\eta}|\alpha_p|, & \text{if } q = 2,\\ 64\kappa\frac{\varepsilon^2}{\eta}\left|\sum_{j\in\mathbb{Z}, j\neq 0,p}\frac{\alpha_{p-j}\alpha_j}{4(p-2j)^2+\hat{\eta}^2}\right|, & \text{if } q = 4,\end{cases}$$

[3] Equivalently $(p,q) = 1$.

[4] Let $p, q \in \mathbb{Z}$. (p,q)-periodic orbits are solutions $x_{pq}(t)$ of (1), which satisfy the condition $x_{pq}(t + 2\pi q) = x_{pq}(t) + 2\pi p$, i.e. q revolutions of the satellite about $\mathcal{P}$ take the same time as p revolutions on his own axis.

where $\hat{\eta} := q\eta$.

Remark 2. Theorem 1 makes sense if we can prove that the following condition holds:

$$0 \neq \begin{cases} \alpha_{2p}, & \text{for } q = 1, \\ \alpha_p, & \text{for } q = 2, \\ \sum_{j\in\mathbb{Z}, j\neq 0,p} \frac{\alpha_{p-j}\alpha_j}{4(p-2j)^2+\hat{\eta}^2}, & \text{for } q = 4. \end{cases} \tag{7}$$

For $0 < b < 1$ let

$$r(b) := \frac{b}{\cosh b} \quad \text{and} \quad M(b) := \frac{2}{(1-b)^5}\Big((1+r(b))(1+\cosh b) + 1 - b\Big)^2. \tag{8}$$

Moreover define

$$c_{p,1} := \frac{|2p|^{|2p-2|}}{2^{|2p-1|+1}|2p-2|!}, \tag{9}$$

$$c_{p,2} := \frac{|p|^{|p-2|}}{2^{|p-1|+1}|p-2|!}, \tag{10}$$

$$c_{p,4} := \begin{cases} \frac{(2-p)^{4-p}}{2^{11-p}(4-p)^2(4-p)!}, & \text{for } p \leq 1, \\ -\frac{1}{16}, & \text{for } p = 3, \\ \frac{(p-2)^{p-2}}{2^{p-2}(p-4)^2(p-2)!}, & \text{for } p \geq 5. \end{cases} \tag{11}$$

Note that

$$c_{p,1} \approx \frac{\left(\frac{e}{2}\right)^{|2p|}}{\sqrt{p}} \to \infty, c_{p,2} \approx \frac{\left(\frac{e}{2}\right)^{|p|}}{\sqrt{p}} \to \infty \quad \text{and} \quad c_{p,4} \approx \frac{\left(\frac{e}{2}\right)^{|p|}}{p^2\sqrt{p}} \to \infty \tag{12}$$

as $p \to \pm\infty$.

The main result of this paper is stated in the following theorem.

Theorem 2. *Besides the assumptions of Theorem 1 on* p, q, ε, η, *for* $0 < b < 1$ *let* $r(b)$, $M(b)$, $c_{p,q}$ *be as in equations* (8) - (11).

- *If* $q = 1, 2$, *assume*

$$0 < \mathbf{e} \leq c_{p,q}\frac{1}{M(b)}\left(\frac{r(b)}{2}\right)^{|\frac{2p}{q}-2|+1}. \tag{13}$$

- *If* $q = 4$, *assume*

$$0 < \mathbf{e} \leq \frac{r(b)}{\frac{\pi^2 M^2(b)}{6|c_{p,q}|r(b)^{|p-4|}} + |p-4|}, \tag{14}$$

$$\hat{\eta} < \frac{6\sqrt{5|c_{p,q}|}}{\pi M(b)}|\mathbf{e}|^{|p-4|/2}. \tag{15}$$

Then there exists a $(p, q)-$*periodic orbit of the spin-orbit problem modelled by equations* (1)-(6).

Remark 3. (i) Condition (7) depends only on $\mathbf{e}$ when $q = 1, 2$ while it depends also on $\hat{\eta}$ when $q = 4$. This is why, for $q = 4$, we have to assume also the condition (15).

(ii) Conditions on the existence of $(p, q)-$periodic orbits are obtained combining Theorem 1 and Theorem 2. For the case $p/q = 1/1$ equation (13) is equivalent to $0 < \mathbf{e} \leq 0.000412055$, choosing $b = 0.148642$. This condition on the eccentricity is satisfied by Tethys (a satellite of Saturn), which has an eccentricity approximately equal to 0.0001 and is known (from astronomical observation) to be locked in a $(1, 1)-$periodic spin-orbit.

For the case $p/q = 3/2$ equation (13) is equivalent to $0 < \mathbf{e} \leq 0.0000609886$, choosing $b = 0.253122$. Mercury (seen as satellite of the Sun), which represents the only observed example of the $(3, 2)$-periodic spin-orbit in our Solar system, doesn't fulfil this condition, since its eccentricity is approximately equal to 0.2056.

In [1] we focus on the cases $p/q = 1/1, 3/2$ improving the estimate for the existence of $(p, q)-$periodic orbits in order to cover the values of the eccentricity of all satellites in our Solar system, which are observed to be in a $(1, 1)-$ or a $(3, 2)-$periodic orbit.

(iii) We also note that the proof of the theorem above is quite simple in the cases $q = 1, 2$: one has to prove that some single analytic function of $\mathbf{e}$ (namely α_{2p} and α_p, respectively) is not identically zero. Since the behavior of such functions is well known (see Lemma 6), this goal is easily obtained. On the other hand the case $q = 4$ is more difficult (already in the case $\hat{\eta} = 0$), since one has to prove that a series of functions with changing signs is not identically zero and compensations could occur.

To prove Theorem (2) we have to find a lower bound for the absolute value of the right hand side of (7). Since this quantity is a function $\mathbf{e}$, we evaluate the leading term of the $\mathbf{e}$ expansion and we prove that, if $\mathbf{e}$ satisfies the conditions (13)-(15), the difference between the leading term and the rest doesn't change sign.

2 Proof of Theorem 2

In order to prepare and to understand the proof of Theorem 2 we need some intermediate results.
Lemma 1, 2 and 3 are well known results (see, for example, Appendix A of [2] and Appendix B in [1]).

Lemma 1. *$\alpha_0 = \alpha_0(\mathbf{e}) = 0$ for $\mathbf{e} \in [0, 1)$.*

Lemma 2. *Let $G(t) = G_{\mathbf{e}}(t) := -\frac{e^{2if_{\mathbf{e}}(t)}}{2\rho_{\mathbf{e}}(t)^3}$. Then we have*

$$G(t) = \sum_{j \neq 0} \alpha_j(\mathbf{e}) e^{ijt},$$

where $\alpha_j(\mathbf{e})$ are defined in (6).

Lemma 3. *The coefficients $\alpha_j = \alpha_j(\mathbf{e})$ defined in (6) satisfy*

$$\alpha_j = -\frac{1}{4\pi} \int_0^{2\pi} \frac{1}{(1-\mathbf{e}\cos u)^2 (w^2+1)^2} \Big[(w^4 - 6w^2 + 1) c_j(u) - 4w(w^2-1) s_j(u) \Big] du \tag{16}$$

for all $j \in \mathbb{Z}$, where $w := w(u; \mathbf{e}) := \sqrt{\frac{1+\mathbf{e}}{1-\mathbf{e}}} \tan\left(\frac{u}{2}\right)$ and

$$c_j(u) := \cos(ju - j\mathbf{e}\sin u)\,, \qquad s_j(u) := \sin(ju - j\mathbf{e}\sin u)\,.$$

The formula (16) will be suitable for further estimations. The following lemma gives an upper bound for the Fourier coefficients α_j. For the proof we refer to [1] (see Lemma 2).

Lemma 4. *Let $0 < b < 1$, $r(b)$ and $M(b)$ as in Theorem 2. The solution $u_{\mathbf{e}}(t)$ of the Kepler equation (5) is a holomorphic function in the ball*

$$|\mathbf{e}| < r(b) = \frac{b}{\cosh b}$$

satisfying

$$\sup_{t \in \mathbb{R}} |u_{\mathbf{e}}(t) - t| \leq b\,.$$

The functions $\rho_{\mathbf{e}}(t)$ in (3) and $G_{\mathbf{e}}(t)$ in Lemma 3 satisfy

$$|\rho_{\mathbf{e}}(t)| \geq 1 - b\,, \qquad \forall\, t \in \mathbb{R}\,,\ |\mathbf{e}| < r(b),$$

and

$$|G_{\mathbf{e}}(t)| \leq \frac{2}{(1-b)^5} \Big(|1-\mathbf{e}|(1+\cosh b) + 1 - b \Big)^2, \qquad \forall\, t \in \mathbb{R}\,,\ |\mathbf{e}| < r(b)\,,$$

respectively.

Remark 4. From Lemma 2 we know that $\alpha_j = \frac{1}{2\pi}\int_0^{2\pi} G(t)e^{-ijt}dt$ holds. Using Lemma 4 for $0 < b < 1$ it follows

$$|\alpha_j(\mathbf{e})| \leq \frac{2}{(1-b)^5}\Big(|1-\mathbf{e}|(1+\cosh b) + 1 - b\Big)^2 \leq M(b)\,, \quad \forall j \in \mathbb{Z} \quad \forall\, |\mathbf{e}| < r(b), \tag{17}$$

where $r(b)$ and $M(b)$ are defined in (8).

We premise the following result due to Hansen (see Chapter XV of [9]), which will be used to prove Lemma 6 below.

Lemma 5 (Hansen's lemma)**.** *Let $u, t, \mathrm{f}_{\mathbf{e}}(t)$ and $\rho_{\mathbf{e}}(t)$ be as in the Section 1.1, $m \in \mathbb{N}$, $n \in \mathbb{Z}$ and*

$$\beta \ := \ \frac{\mathbf{e}}{1+\sqrt{1-\mathbf{e}^2}} = \frac{1-\sqrt{1-\mathbf{e}^2}}{\mathbf{e}}. \tag{18}$$

Furthermore define $P_0^{n,m} = Q_0^{n,m} := 1$ and for $k \geq 1$ define

$$P_k^{n,m}(\nu) \ := \ \frac{\nu^k}{k!} + \sum_{l=0}^{k-1} \frac{\nu^l}{l!} \frac{\tilde{n}(\tilde{n}-1)\ldots(\tilde{n}-(k-l)+1)}{(k-l)!}, \tag{19}$$

$$Q_k^{n,m}(\nu) \ := \ (-1)^k\frac{\nu^k}{k!} + \sum_{l=0}^{k-1} \frac{\nu^l}{l!}(-1)^l \frac{\hat{n}(\hat{n}-1)\ldots(\hat{n}-(k-l)+1)}{(k-l)!}, \tag{20}$$

where

$$\nu := \frac{j\mathbf{e}}{2\beta}, \qquad \tilde{n} = n+m+1 \qquad \text{and} \qquad \hat{n} = n-m+1. \tag{21}$$

Then the coefficients $X_l^{n,m}$ in the expansion

$$(\rho_{\mathbf{e}}(t))^n e^{im\mathrm{f}_{\mathbf{e}}(t)} =: \sum_{j\in\mathbb{Z}} X_j^{n,m} e^{ijt}, \tag{22}$$

are given, for $j \neq 0$, by the following formula:

$$X_j^{n,m} \ = \ \begin{cases} (1+\beta^2)^{-n-1}(-\beta)^{j-m}\sum_{k=0}^{\infty} Q_{j-m+k}^{n,m}(\nu)P_k^{n,m}(\nu)\beta^{2k}, & \text{if } j > m, \\ (1+\beta^2)^{-n-1}(-\beta)^{m-j}\sum_{k=0}^{\infty} P_{m-j+k}^{n,m}(\nu)Q_k^{n,m}(\nu)\beta^{2k}, & \text{if } j \leq m. \end{cases} \tag{23}$$

The next lemma gives a result for the asymptotic behaviour of $\alpha_j(\mathbf{e})$. Its proof follows by a straightforward algebraic computation from Lemma 5.

Lemma 6. *Let $\alpha_j(\mathbf{e})$ be as in* (16)*. Then for all $0 \neq j \in \mathbb{Z}$ we have*

$$\alpha_j(\mathbf{e}) = \bar{\alpha}_j \mathbf{e}^{|j-2|} + O(\mathbf{e}^{|j-2|+1}), \tag{24}$$

where

$$\bar{\alpha}_j = \begin{cases} P_{2-j}(j), & \text{for } j \leq 2, \\ Q_{j-2}(j), & \text{for } j > 2. \end{cases} \tag{25}$$

$(P_k)_{k\geq 0}$ and $(Q_k)_{k\geq 0}$ are two families of polynomial functions given by the relations

$$P_k(x) := \left(\frac{-1}{2}\right)^{k+1} \frac{x^k}{k!}, \tag{26}$$

$$Q_k(x) := -\frac{1}{2^{k+1}} \sum_{l=0}^{k} \frac{1}{l!} \binom{k-l+3}{3} x^l. \tag{27}$$

In the following lemma we determine the sign of the leading term in the $\mathbf{e}$ expansion of $\alpha_j(\mathbf{e})$.

Lemma 7. *$\bar{\alpha}_j < 0$ for all $j \in \mathbb{Z} \setminus \{0,1\}$, $\bar{\alpha}_0 = 0$ and $\bar{\alpha}_1 = \frac{1}{4} > 0$.*

Proof. $\bar{\alpha}_1 = \frac{1}{4} > 0$ and $\bar{\alpha}_0 = 0$ follow directly from (26). If $j \geq 2$ we see from (27) that all coefficients $(q_i)_{0\leq i\leq k}$ of $Q_k(x) = q_0 + q_1 x + \ldots + q_k x^k$ are strictly negative for every $k \geq 0$. By (25) we have $\bar{\alpha}_j = Q_{j-2}(j) < 0$. Otherwise if $j \leq -1$ we have from (25), (26) that

$$\bar{\alpha}_j = \left(\frac{-1}{2}\right)^{3-j} \frac{j^{2-j}}{(2-j)!} = \frac{(-1)^{3-j}(-1)^{2-j}}{2^{3-j}} \frac{|j|^{2-j}}{(2-j)!} = \frac{-1}{2^{3-j}} \frac{|j|^{2-j}}{(2-j)!} < 0.$$

$\boxed{QED}$

In the next lemma we estimate the absolute value of the difference between $\alpha_j(\mathbf{e})$ and its leading term in the $\mathbf{e}$ expansion.

Lemma 8. *Let $r(b)$, $M(b)$ and $\alpha_j(\mathbf{e})$ be defined as in* (8) *and* (16)*. Moreover let*

$$\bar{r}_j(b) := \frac{|j|^{|j-2|}}{2^{|j-1|+1}|j-2|!} \frac{1}{M(b)} \left(\frac{r(b)}{2}\right)^{|j-2|+1}. \tag{28}$$

Then

$$|\alpha_j(\mathbf{e}) - \bar{\alpha}_j \mathbf{e}^{|j-2|}| \leq \frac{1}{2}|\bar{\alpha}_j||\mathbf{e}|^{|j-2|} \tag{29}$$

holds for all $|\mathbf{e}| \leq \bar{r}_j(b)$. Moreover it follows

$$sgn(\alpha_j(\mathbf{e})) = sgn(\bar{\alpha}_j)$$

for $0 \leq \mathbf{e} \leq \bar{r}_j(b)$.

Remark 5. Note that $\bar{r}_j(b) \approx \left(\frac{er(b)}{4}\right)^j \to 0$ as $j \to \infty$.

Proof. For $j = 0$ this lemma is trivial, since by Lemma 1 and equation (25) we know that $\alpha_0(\mathbf{e}) = \bar{\alpha}_0 = 0$ holds. Fix b and $r(b)$ as in Lemma 4 and set $\rho_j(b) := r(b) - \bar{r}_j(b)$. Then by the Cauchy estimates and (17) we obtain

$$\left|\frac{d^n}{d\mathbf{e}^n}\alpha_j(\mathbf{e})\right| \quad \leq \quad \frac{n!M(b)}{\rho_j(b)^n}\,, \qquad \forall\,|\mathbf{e}| \leq \bar{r}_j(b), j \in \mathbb{Z}, n \in \mathbb{N}\,. \tag{30}$$

By (24) and the integral form for the remainder in the Taylor series we have

$$\alpha_j(\mathbf{e}) \quad = \quad \bar{\alpha}_j \mathbf{e}^{|j-2|} + R_{|j-2|}(\mathbf{e}), \tag{31}$$

$$R_{|j-2|}(\mathbf{e}) \quad := \quad \frac{\mathbf{e}^{|j-2|+1}}{|j-2|!}\int_0^1 \frac{d^{|j-2|+1}\alpha_j(\mathbf{e}s)}{d\mathbf{e}^{|j-2|+1}}(1-s)^{|j-2|}\,ds. \tag{32}$$

Using (30) and (32) with $n = |j-2|+1$ we have

$$|R_{|j-2|}(\mathbf{e})| \leq \frac{|\mathbf{e}|^{|j-2|+1}}{|j-2|!}\frac{M(b)(|j-2|+1)!}{\rho_j(b)^{|j-2|+1}}\int_0^1 (1-s)^{|j-2|}\,ds = \frac{M(b)|\mathbf{e}|^{|j-2|+1}}{\rho_j(b)^{|j-2|+1}} \tag{33}$$

for all $j \in \mathbb{Z}$ and $|\mathbf{e}| \leq \bar{r}_j(b)$. Notice that by (26) and (27) we have

$$|Q_{j-2}(j)| \geq \frac{|j|^{|j-2|}}{2^{|j-1|}|j-2|!} = |P_{2-j}(j)|.$$

By the last inequality, (25) and (26) we have

$$|\bar{\alpha}_j| \geq \min\left(|P_{2-j}(j)|, |Q_{j-2}(j)|\right) = \frac{|j|^{|j-2|}}{2^{|j-1|}|j-2|!}. \tag{34}$$

By (31), (33) and (34) we have that (29) follows if we prove that

$$\frac{M(b)\bar{r}_j(b)}{\rho_j(b)^{|j-2|+1}} \leq \frac{|j|^{|j-2|}}{2^{|j-1|+1}|j-2|!}\,. \tag{35}$$

Since $\rho_j(b) \geq r(b)/2$ (with $r(b)$ defined in (8)), (35) follows by the definition of $\bar{r}_j(b)$ in (28). $\boxed{QED}$

Remark 6. The estimate found in (29) is not optimal[5] for $\bar{r}_j(b)$.

The following lemma determines the order of the leading term of the product $\alpha_j(\mathbf{e})\alpha_{p-j}(\mathbf{e})$.

[5] Optimal estimates in the cases of astronomical interest, as for example Earth-Moon or Mercury-Sun system, can be found in [1].

Lemma 9. *Let $p \in \mathbb{Z}$. It follows*

$$|p-4| \leq |p-j-2| + |j-2|, \qquad \forall j \in \mathbb{Z}$$

and equality holds if one of the following two cases occurs:

- *Case 1:* $p \geq 4$ *and* $2 \leq j \leq p-2$*;*
- *Case 2:* $p \leq 4$ *and* $p-2 \leq j \leq 2$*.*

Proof. Let us define

$$v(p,j) := |p-j-2| + |j-2| = |j-(p-2)| + |j-2|.$$

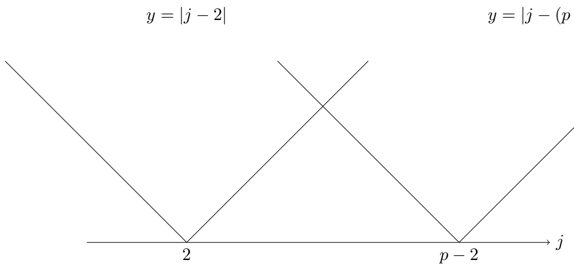

Figure 2. Sketch of the function $v(p,j)$ in the case $p \geq 4$ (the case $p \leq 4$ is analogous).

In the first case, when $p \geq 4$, the function $v(p,j)$ takes its minimum for $2 \leq j \leq p-2$ and for these values of j we have $|p-4| = |j-(p-2)| + |j-2|$ (see Figure 2).
Analogously, when $p \leq 4$, the function $v(p,j)$ takes its minimum for $p-2 \leq j \leq 2$ and, as in the first case, for these values of j we have $|p-4| = |j-(p-2)| + |j-2|$. $\boxed{QED}$

Lemma 10. *Let $p \in \mathbb{Z}$ be odd. Then we have*

$$\begin{cases} \sum_{j=p-2}^{2} \frac{\bar{\alpha}_{p-j}\bar{\alpha}_j}{4(p-2j)^2} > \frac{(2-p)^{4-p}}{2^{11-p}(4-p)^2(4-p)!} > 0, & \text{for } p \leq 1, \\ \sum_{j=p-2}^{2} \frac{\bar{\alpha}_{p-j}\bar{\alpha}_j}{4(p-2j)^2} = \frac{-1}{16} < 0, & \text{for } p = 3, \\ \sum_{j=2}^{p-2} \frac{\bar{\alpha}_{p-j}\bar{\alpha}_j}{4(p-2j)^2} > \frac{(p-2)^{p\mp 2}}{2^{p-2}(p-4)^2(p-2)!} > 0, & \text{for } p \geq 5. \end{cases} \tag{36}$$

Proof. If $p \leq 1$ using Lemma 7 we have

$$\begin{aligned}\sum_{j=p-2}^{2} \frac{\bar{\alpha}_{p-j}\bar{\alpha}_j}{4(p-2j)^2} &= 2\sum_{j=\frac{p+1}{2}}^{2} \frac{\bar{\alpha}_{p-j}\bar{\alpha}_j}{4(p-2j)^2} > \frac{1}{2}\left(\frac{\bar{\alpha}_{p-1}\bar{\alpha}_1}{(p-2)^2} + \frac{\bar{\alpha}_{p-2}\bar{\alpha}_2}{(p-4)^2}\right) \\ &> \frac{1}{32}\frac{\bar{\alpha}_{p-2}\bar{\alpha}_2}{(p-4)^2}.\end{aligned}$$

The last estimate is obtained as follows. Since $\bar{\alpha}_2 = -\frac{1}{2}, \bar{\alpha}_1 = \frac{1}{4}$ it is equivalent to

$$15\,\bar{\alpha}_{p-2}(2-p)^2 \quad < \quad 8\,\bar{\alpha}_{p-1}(4-p)^2. \tag{37}$$

Using (25) and (26) we know that

$$\bar{\alpha}_p = \left(\frac{-1}{2}\right)^{3-p} \frac{p^{2-p}}{(2-p)!}, \qquad \text{for } p \leq 1 \tag{38}$$

holds. Since $p \leq 1$ and p is odd from (37) and (38) we get

$$\frac{15(p-2)^{4-p}}{2^{5-p}(4-p)!}(2-p)^2 \quad < \quad -\frac{8(p-1)^{3-p}}{2^{4-p}(3-p)!}(4-p)^2.$$

This is true for all $p \leq 1$, p odd. So we proved that

$$\sum_{j=p-2}^{2} \frac{\bar{\alpha}_{p-j}\bar{\alpha}_j}{4(p-2j)^2} > \frac{1}{32}\frac{\bar{\alpha}_{p-2}\bar{\alpha}_2}{(p-4)^2} \overset{(38)}{=} \frac{(2-p)^{4-p}}{2^{11-p}(4-p)^2(4-p)!}, \qquad \text{for } p \leq 1, p \text{ odd.}$$

By Lemma 6 we have $\bar{\alpha}_2 = -\frac{1}{2}$ and $\bar{\alpha}_1 = \frac{1}{4}$. Therefore the case $p = 3$ reduces to a direct computation, i.e.

$$\sum_{j=p-2}^{2} \frac{\bar{\alpha}_{p-j}\bar{\alpha}_j}{4(p-2j)^2} = \frac{\bar{\alpha}_1\bar{\alpha}_2}{2} = \frac{-1}{16}.$$

By (25) and (27) we know that

$$\bar{\alpha}_p = \frac{-1}{2^{p-1}}\sum_{l=0}^{p} \frac{1}{l!}\begin{pmatrix} p-l+1 \\ 3 \end{pmatrix} p^l, \qquad \text{for } p \geq 3. \tag{39}$$

Furthermore $\bar{\alpha}_2 = -\frac{1}{2}$ holds. So in case $p \geq 5$ and p odd, by (39) we get

$$\sum_{j=2}^{p-2} \frac{\bar{\alpha}_{p-j}\bar{\alpha}_j}{4(p-2j)^2} \geq \frac{\bar{\alpha}_{p-2}\bar{\alpha}_2}{2(p-4)^2} > \frac{(p-2)^{p-2}}{2^{p-2}(p-4)^2(p-2)!}.$$

This ends the proof of Lemma 10. $\boxed{QED}$

At this point the proof of Theorem 2 will be given.

Proof. **(Theorem 2)** The cases $q = 1$ and $q = 2$ of the equation (13) follows from Lemma 7 and 8 taking

$$\begin{aligned} |\mathbf{e}| &\leq \bar{r}_{2p} = c_{p,1}\frac{1}{M(b)}\left(\frac{r(b)}{2}\right)^{|2p-2|+1}, \text{ for } q=1, \\ |\mathbf{e}| &\leq \bar{r}_{p} = c_{p,2}\frac{1}{M(b)}\left(\frac{r(b)}{2}\right)^{|p-2|+1}, \text{ for } q=2, \end{aligned}$$

where $c_{p,1}, c_{p,2}$ and $\bar{r}_j$ are defined in (9), (10) and (28), respectively.
The case $q = 4$ remains to be proved. Since $(p, q) = 1$ it follows that p is odd, therefore also $p - 2$ is odd. Let us define

$$s(\mathbf{e}, \zeta) := \sum_{j\in\mathbb{Z}, j\neq 0,p} \frac{\alpha_{p-j}(\mathbf{e})\alpha_j(\mathbf{e})}{4(p-2j)^2+\zeta}\,. \tag{40}$$

Recalling condition (7), we have to prove that $s(\mathbf{e}, \hat{\eta}^2) \neq 0$ for $\mathbf{e}$ and $\hat{\eta}$ satisfying (14) and (15), respectively.
Let us start with the case $\hat{\eta} = 0$. For $0 < b < 1$ and $|\mathbf{e}| < r(b) = b/\cosh b$ by (17) we get

$$|s(\mathbf{e},0)| = \left|\sum_{j\in\mathbb{Z}, j\neq 0,p} \frac{\alpha_{p-j}(\mathbf{e})\alpha_j(\mathbf{e})}{4(p-2j)^2}\right| \leq \frac{M^2(b)}{4}\sum_{k\neq 0, k\in\mathbb{Z}}\frac{1}{k^2} = \frac{M^2(b)\pi^2}{12}\,. \tag{41}$$

Furthermore by (24)

$$\begin{aligned} \alpha_{p-j}(\mathbf{e})\alpha_j(\mathbf{e}) &= \left[\bar{\alpha}_{p-j}\mathbf{e}^{|p-j-2|} + O(\mathbf{e}^{|p-j-2|+1})\right]\left[\bar{\alpha}_j\mathbf{e}^{|j-2|} + O(\mathbf{e}^{|j-2|+1})\right] \\ &= \bar{\alpha}_{p-j}\bar{\alpha}_j\mathbf{e}^{|p-j-2|+|j-2|} + O(\mathbf{e}^{|p-j-2|+|j-2|+1}). \end{aligned} \tag{42}$$

Using Lemma 9 and (42) we have

$$s(\mathbf{e},0) = \begin{cases} \sum_{j=p-2}^{2} \frac{\bar{\alpha}_{p-j}\bar{\alpha}_j}{4(p-2j)^2}\mathbf{e}^{|p-4|} + R_p(\mathbf{e}), & \text{for } p \leq 3, \\ \sum_{j=2}^{p-2} \frac{\bar{\alpha}_{p-j}\bar{\alpha}_j}{4(p-2j)^2}\mathbf{e}^{|p-4|} + R_p(\mathbf{e}), & \text{for } p > 4\,, \end{cases} \tag{43}$$

where $R_p(\mathbf{e}) = O(\mathbf{e}^{|p-4|+1})$ is the Taylor remainder. Take

$$0 < r' < r(b)\,.$$

By the Cauchy estimates and (41) we get

$$|R_p(\mathbf{e})| \leq \frac{\pi^2 M^2(b)}{12(r-r')^{|p-4|+1}} |\mathbf{e}|^{|p-4|+1}, \qquad \forall\ |\mathbf{e}| \leq r'(b)\,. \tag{44}$$

Let $c_{p,4}$ be as in (11). Choosing $r' = r'_p(b)$ as the unique solution of

$$\tilde{r} = \frac{6|c_{p,4}|}{\pi^2 M^2(b)} (r(b) - \tilde{r})^{|p-4|+1} =: h(\tilde{r})\,, \tag{45}$$

by (44) we get

$$|R_p(\mathbf{e})| \leq \frac{|c_{p,4}|}{2} |\mathbf{e}|^{|p-4|} \qquad \forall\ |\mathbf{e}| \leq r'_p(b)\,. \tag{46}$$

Then by (36), (43) and (46) we have

$$|s(\mathbf{e},0)| \geq \frac{|c_{p,4}|}{2} |\mathbf{e}|^{|p-4|} \qquad \forall\ |\mathbf{e}| \leq r'_p(b)\,. \tag{47}$$

Let us now consider the case $\hat{\eta} > 0$. For $|\mathbf{e}| < r(b)$ and $\zeta > 0$ we have by (17) and (40)

$$|\partial_\zeta s(\mathbf{e},\zeta)| \leq \sum_{j \in \mathbb{Z}, j \neq 0, p} \frac{|\alpha_{p-j}(\mathbf{e}) \alpha_j(\mathbf{e})|}{(4(p-2j)^2 + \zeta)^2} \leq \frac{M^2(b)}{16} \sum_{0 \neq k \in \mathbb{Z}} \frac{1}{k^4} \leq \frac{\pi^2 M^2(b)}{720}\,. \tag{48}$$

Then by (47) and (48), for $0 < |\mathbf{e}| \leq r'_p(b)$, we obtain

$$\begin{aligned} |s(\mathbf{e},\hat{\eta}^2)| &\geq |s(\mathbf{e},0)| - \frac{\pi^2 M^2(b)}{720} \hat{\eta}^2 \geq \frac{|c_{p,4}|}{2} |\mathbf{e}|^{|p-4|} - \frac{\pi^2 M^2(b)}{720} \hat{\eta}^2 \\ &\geq \frac{|c_{p,4}|}{4} |\mathbf{e}|^{|p-4|} > 0 \end{aligned}$$

if we assume that

$$\hat{\eta} \leq \frac{6\sqrt{5|c_{p,4}|}}{\pi M(b)} |\mathbf{e}|^{|p-4|/2}\,.$$

Equation (14) follows if we prove that

$$r'' := \frac{r(b)}{\frac{\pi^2 M^2(b)}{6|c_{p,4}| r(b)^{|p-4|}} + |p-4|} \leq r'_p(b). \tag{49}$$

From (45)

$$\begin{aligned} h(\tilde r) &\geq h(0) + h'(0)\tilde r \\ &= \frac{6|c_{p,4}|}{\pi^2 M^2(b)}(r(b))^{|p-4|+1} - \frac{6|p-4||c_{p,4}|}{\pi^2 M^2(b)}(r(b))^{|p-4|}\tilde r =: g(\tilde r) \end{aligned}$$

holds for $0 \leq \tilde r \leq r(b)$. Since r'' (defined in the left hand side of (49)) is the unique solution of the equation $\tilde r = g(\tilde r)$ and r' is the unique solution of the equation $\tilde r = h(\tilde r)$, the inequality (49) is true (see Figure 3). This completes the proof of Theorem 2.

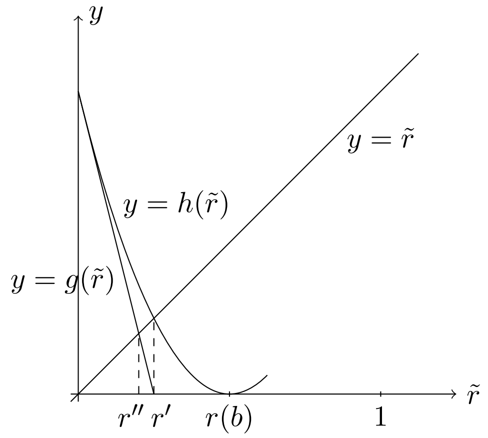

Figure 3. Geometric explanation of equation (49).

$\boxed{QED}$

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