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THE PARTIALLY ORDERED SETS OF MEASURE THEORY AND TUKEY'S ORDERING

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Dedicated to the memory of Professor Gottfried Köthe

In [28], J.W. Tukey introduced an ordering on the class of directed sets, designed to illuminate the theory of Moore-Smith convergence. 1 show how variations of his idea can be used to give information on a wide variety of partially ordered sets arising in measure theory.

INTRODUCTION

In [1], [22], [3] and [9] there is a series of results conceming the additivity and cofinality of a-ideals of measure and category. It tums out that all the main ideas of the proofs can be expressed in the following scheme: to show that add $(\mathscr{I}) \leq$ add $(\mathscr{J})$ and cf $(\mathscr{J}) \leq$ cf $(\mathscr{I})$, where $\mathscr{I}$ and $\mathscr{J}$ are ideals of sets, first show that $\mathscr{J} \leq \mathscr{I}$ in Tukey's sense. (For definitions see 1D, 11 below.) In the present paper 1 seek to develop this idea systematically to cover both known results and some interesting offshoots.

A variety of complications appear, so it is perhaps worth while trying to outline the theory presented below. Tukey defined a transitive, reflexive relation $\leq$, and the associated equivalence relation $\equiv$, on the class of directed sets; the definition can be usefully applied in the class of all partially ordered sets. It is not quite sufficient for our needs and 1 describe similar relations $\leq_{<\omega}$, $\leq_{\omega}$ to cover transitions between directed and undirected sets and between ideals and σ-ideals. In all cases the idea is that if $P \leq_{(..)} Q$ then P is in some way «simpler» than Q . All the relations are large ones and give a rather coarse classifiction of the partially ordered sets discussed; but they nevertheless give some useful information, primarily about additivity and cofinality, but also about such things as cellularity (1 J) .

In § 2 1 deal with partially ordered sets derived from a measure space. Let (X, μ) be a Maharam homogeneous Radon probability space with Maharani type $\kappa \geq \omega$; write $\mathfrak{A}$ for its measure algebra, $\mathfrak{A}^-$ for $\mathfrak{A} \setminus \{1\}$, $\mathcal{N}_\mu$ for its idea1 of negligible sets, and Σ_μ^* for $\{E : E \subseteq X, \mu(E) < 1\}$. Then

$$\Sigma_\mu^* \equiv \mathfrak{A}^- \equiv_{<\omega} \ell^1(\kappa) \equiv L^1(\mu) \equiv L^0(\mu) \equiv_\omega \mathcal{N}_\mu$$

(2C, 2E, 2F, 21). Thus all these partially ordered sets are in some sense at the same leve1 of complexity. Moreover, for many κ (and in simple models of set theory, for all κ) we have

$$\ell^1(\kappa) \equiv_\omega \ell^1(\mathrm{N}) \times [\kappa]^{\leq\omega}$$

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(2J), so that the complexity depends on κ in a manageable way.

The most important case is of course $\kappa = \omega$, $X = [0, 1]$, and μ = Lebesgue measure; in this case write $\mathcal{N}$ for $\mathcal{N}_\mu$. It turns out that the partially ordered sets $\ell^1(\mathbf{N})$, $\mathcal{N}$ have a special place among the partially ordered sets arising in real analysis, even those in which measure theory does not seem to be directly involved. Write $\mathcal{F}$ for the ideal of nowhere dense subsets of $\mathbf{N}^{\mathbf{N}}$ and $\mathcal{M}$ for the σ-ideal of meagre sets. (Any non-empty Polish space without isolated points could take the place of $\mathbf{N}^{\mathbf{N}}$ here.) Then

$$\mathbf{N}^{\mathbf{N}} \leq \mathcal{F} \equiv_\omega \mathcal{M} \leq \mathcal{N}$$

(3B), so that $\mathcal{F}$, $\mathcal{M}$ define a level of complexity somewhere between $\mathbf{N}^{\mathbf{N}}$ and $\mathcal{N}$. In § 3 I seek to locate in this pattern further ideals of interest; for instance, the ideal $\mathcal{Z}$ of subsets of $\mathbf{N}$ with zero asymptotic density also falls somewhere between $\mathbf{N}^{\mathbf{N}}$ and $\ell^1(\mathbf{N})$ (3K), while the ideal of subsets of $\mathbf{R}$ with strong measure zero is dominated by a power of $\mathcal{N}$ (3I).

Note that nearly all these results are theorems of ZFC in a context in which ZFC leaves a great deal of flexibility. In any particular model of set theory - under CH, for instance, or in random real models - the pattern may collapse dramatically, at least in those parts dealing with σ-ideals. I do not attempt to discuss such questions here. Several relevant models of set theory are described in [20].

I should like to thank J. Cichón and S. Todorčević for introducing me to [1] and [28], respectively, and for helpful correspondence since.

1. TUKEY'S ORDERING WITH VARIATIONS

I give the definitions on which this paper is based (1A, 1B, 1D, 1F, 1I) and a variety of elementary consequences, with a brief discussion of «bursting numbers» (1K-1M).

Definition 1A. *Let P be a partially ordered set and A, B two subsets of P. Say that B* dominates *A if for every $a \in A$ there is a $b \in B$ with $a \leq b$. Say that A is* finitely/countably dominated *in P if there is a finite/countable subset of P dominating A.*

Definition 1B. *Let P and Q be partially ordered sets and $f : P \to Q$ a function.*

(a) *f is a* Tukey function *if $\{p : f(p) \leq q\}$ is either empty or bounded above in P for every $q \in Q$.*

(b) *f is a* $<\omega$-Tukey function *if $\{p : f(p) \leq q\}$ is finitely dominated in P for every $q \in Q$.*

(c) *f is an* ω-Tukey function *if $\{p : f(p) \leq q\}$ is countably dominated in P for every $q \in Q$.*

Proposition 1C. (a) *A Tukey function is $<\omega$-Tukey ; a $<\omega$-Tukey function is w-Tukey .*

(b) *The composition of two Tukey functions is a Tukey function; the composition of two $<$ w -Tukey functions is a $<$ w - Tukey function; the wmposition of two w -Tukey functions is an w - Tukey function.*

(c) If P is upwards-directed and f : $P \to Q$ is a $<$ w -Tukey function, tben f is a Tukey function. If non-empty wuntable subsets of P are boundedabove in P, and $f : P \to$ $\to Q$ is an w -Tukey function, tben f is a Tukey function.

Part of proof. Suppose, for instance, that $f : P \to Q$ and $g : Q \to R$ are $<\omega$-Tukey functions. Let $r \in R$. Then $\{q : g(q) \leq r\}$ is dominated by some finite set $B \subseteq Q$. For $b \in B$, $\{p : f(p) \leq b\}$ is dominated by some finite set $A_b \subseteq P$. *Now* it is easy to check that $\bigcup_{b\in B} A_b$ is a finite subset of P dominating $\{p : g(f(p)) \leq r\}$. As r is arbitrary, gf is a $<$ w-Tukey function.

Definition 1D. *Let P and Q bepartially ordered sets.*

(a) *Write $P \leq Q$ if there is a Tukey function from P to Q .*

(b) *Write $P \leq_{<\omega} Q$ if tbere is a $<$ w -Tukey function from P to Q .*

(c) Write $P \leq_{\omega} Q$ if tbere is an w -Tukey function from P to Q .

Proposition 1E. (a) *If $P \leq Q$ then $P \leq_{<\omega} Q$; if $P \leq_{<\omega} Q$ then $P \leq_{\omega} Q$.*

(b) *$\leq, \leq_{<\omega}$ and $\leq_{\omega}$ are transitive andreflexive.*

(c) If $P \leq_{<\omega} Q$ and P is upwards-directed tben $P \leq Q$; if $P \leq_{\omega} Q$ and nonempty countable subsets of P are bounded above in P, then $P \leq Q$.

(d) *If $\langle P_i\rangle_{i\in I}$ and $\langle Q_i\rangle_{i\in I}$ are families of partially ordered sets and $P_i \leq Q_i$ for every $i \in I$, then $\prod_{i\in I} P_i \leq \prod_{i\in I} Q_i$.*

(e) Zf $\langle P_i\rangle_{i\in I}$ and $\langle Q_i\rangle_{i\in I}$ are finite families of partially ordered sets and $P_i \leq_{\omega} Q_i$ for every $i \in I$, then $\prod_{i\in I} P_i \leq_{\omega} \prod_{i\in I} Q_i$.

(f) *If $\langle P_i\rangle_{i\in I}$ is a finite family of partially ordered sets and Q is upwards directed and $P_i \leq Q_i$ foreach $i \in I$, then $\prod_{i\in I} P_i \leq Q$.*

Definition 1F. *Let P and Q be partially ordered sets.*

(a) *Write $P \equiv Q$ if $P \leq Q$ and $Q \leq P$.*

(b) *Wtite $P \equiv_{<\omega} Q$ if $P \leq_{<\omega} Q$ and $Q \leq_{<\omega} P$.*

(c) Write $P \equiv_{\omega} Q$ if $P \leq_{\omega} Q$ and $Q \leq_{\omega} P$.

Proposition 1G. (a) *$\equiv, \equiv_{<\omega}$ and $\equiv_{\omega}$ are equivalence relations on the class of partially ordered sets.*

(b) *If $P \equiv Q$ then $P \equiv_{<\omega} Q$; if $P \equiv_{<\omega} Q$ then $P \equiv_{<\omega} Q$.*

(c) If P is a partially ordered set and Q is a cofinal subset of P, then $P \equiv Q$.

Remark 1H. (a) The phrase «Tukey function» comes from [1]; the concept is due to [28]. The relations $\leq, \equiv$ (for directed sets) are studied at length in [28], [24], [13], [26].

(b) If P and Q are upwards-directed sets then $P \equiv Q$ iff there is a partially ordered set R with cofinal subsets isomorphic to P and Q ([28], chap. 11). This is false for undirected partially ordered sets; for example, $[w]^1 \equiv [\omega]^1 \times \omega$ (where $[w]^1$ is the set of singleton subsets of ω, ordere by $\subseteq$); but a partially ordered set in which a copy of $[\omega]^1$ is cofinal must have maximal elements, while a partially ordered set in which a copy of $[\omega]^1 \times w$ is cofinal cannot have maximal elements.

(c) As is already apparent from the example just given, and will become even clearer in § 2 of this paper, the relations $\leq, \leq_{<\omega}$ and $\leq_\omega$ are large, and relate some unexpected sets; this is the whole point of their use here.

(d) Throughout this paper, partially ordered sets are considered to be active upwards; thus a Tukey function is one such that the inverse image of an *upwards*-bounded set is empty or *upwards*-bounded. When we come to study Boolean algebras this will force us to work with partially ordered sets $\mathfrak{A}^-$ (2Ad) rather than the more familiar sets $\mathfrak{A}^+$ of non-zero elements. It seems easier to make a few such inversions than to devise a languagecapable of dealing with cases in which one partially ordered set is active upwards and another is active downwards.

(e) Evidently the role of w in $\leq_\omega, \equiv_\omega$ can be played by other cardinals (we shall want regular cardinals in place of w in $\leq .., \equiv_{<\omega}$). 1 have no applications for such a generalisation so 1 pass it by.

Cardinal functions 11.1 shall use the following definitions. Let P bea partially ordered sets.

(a) Write

$\mathrm{add}(P) = \min\{\#(A) : 0 \neq A \subseteq P, A$ has no upper bound in $P\}$,

$\mathrm{add}_\omega(P) = \min\{\#(A) : A \subseteq P, A$ is not countably dominated in $P\}$,

$\mathrm{cf}(P) = \min\{\#(B) : B$ is a cofinal subset of $P\}$,

$c(P) = \sup\{\#(A) : A \subseteq P$ is an up-antichain in $P\}$.

(The formulae for $\mathrm{add}(P)$, $\mathrm{add}_\omega(P)$ may call for an interpretation of «min $\emptyset$». 1 shall use «∞» in such cases - e.g. $\mathrm{add}(P) = \infty$ if P is empty or has a greatest member - with the convention that $\kappa < \infty$ for every cardinal κ.) Note that if $\mathrm{add}(P) > w$ then $\mathrm{add}_\omega(P) = = \mathrm{add}(P)$.

(b) *Zf* κ, λ and θ are *cardinals, I say* ***that*** $(n, \lambda, <\theta)$ *is* ***a*** triple precaliber upwards ***of*** P *if* ***for every indexed*** *family* $\langle p_\xi\rangle_{\xi<\kappa}$ ***in*** *P there is* ***an*** $I \in [\kappa]^\lambda$ ***such*** *that* $\{p_\xi : \xi \in J\}$ *is* ***boundedabove in*** *P* ***forevery*** $J \in [I]^{<\theta}$.

Theorem 1J. ***Let*** *P* ***and Q be partially*** *ordered* ***sets.***

(a) ***Zf*** $P \leq Q$ *then*

$\mathrm{add}(Q) \leq \mathrm{add}(P)$;

$\mathrm{add}_\omega(Q) \leq \mathrm{add}_\omega(P)$;

$\mathrm{cf}(P) \leq \mathrm{cf}(Q)$;

if $(\kappa, \lambda, <\infty)$ is a triple precaliber upwards of Q, it is a triple precaliber upwards of P;

$c(P) \leq c(Q)$.

(b) *If $P \leq_\omega Q$ then*

$\mathrm{add}_\omega(Q) \leq \mathrm{add}_\omega(P)$;

$\mathrm{cf}(P) \leq \max(\omega, \mathrm{cf}(Q))$.

Proof. (a) If $P = \emptyset$ the result is trivial ($\mathrm{add}(P) = \mathrm{add}_\omega(P) = \infty$, $\mathrm{cf}(P) = c(P) = 0$, and $(\kappa, \lambda, <\infty)$ is always a triple precaliber of P). So let us suppose that $P \neq \emptyset$. Let $f : P \to Q$ be a Tukey function.

(i) If $\mathrm{add}(P) < \infty$, let $A \in [P]^{\mathrm{add}(P)}$ be a set with no upper bound in P; then $f[A]$ can have no upper bound in Q so $\mathrm{add}(Q) \leq \#(f[A]) \leq \mathrm{add}(P)$.

(ii) Similarly, $\mathrm{add}_\omega(Q) \leq \mathrm{add}_\omega(P)$ because $f[A]$ cannot be countably dominated in Q if $A \subseteq P$ is not countably dominated in P.

(iii) Take $C \in [Q]^{\mathrm{cf}(Q)}$ which is cofinal with Q. For each $c \in C$ choose an upper bound a_c of $\{p : f(p) \leq c\}$. Thcn $A = \{a_c : c \in C\}$ is cofinal with P, so $\mathrm{cf}(P) \leq \#(A) \leq$ $\leq \mathrm{cf}(Q)$.

(iv) If $\langle p_\xi \rangle_{\xi<\kappa}$ is a famili in P, there is an $I \in [\kappa]^\lambda$ such that $\{f(p_\xi) : \xi \in J\}$ is bounded above in Q for every $J \in [I]^{<\theta}$; now $\{p_\xi : \xi \in J\}$ is bounded above in P for every $J \in [I]^{<\theta}$.

(v) $c(P) \leq \kappa$ iff $(\kappa^+, 2, <3)$ is a triple precaliber upwards for p.

(b) Similar to (ii), (iii) of (a).

Remark. «Triple precalibers» look unfriendly. 1 mention them because they provide arguments to show that $P \not\leq Q$ (see 2Ma, 2Nb below).

Bursting numbers 1K. (a) For an upwards-directed partially ordered set P, write

$$bu(P) = \min\{\kappa : \exists \text{ cofinal } C \subseteq P \text{ such that } \#(\{c : c \in C, c \leq b\}) < \kappa \; \forall b \in P\},$$

the principal bursting number of P (see [13], § 4).

(b) Observe that (for directed P)

$$bu(P) = \min\{\kappa : \exists A \in [P]^{\mathrm{cf}(P)} \text{ such that } \#(\{a : a \in A, a \leq b\}) < \kappa \; \forall b \in P\}.$$

Theurem 1L. (a) *$bu([\kappa]^{\leq\omega}) \leq \omega_1$ for every cardinal $\kappa < \omega_\omega$.*

(b) *$bu([\kappa]^{\leq\omega}) \leq \omega_1$ whenever $\mathrm{cf}([\kappa]^{\leq\omega}) = \kappa$.*

(c) Suppose that $\square_\kappa$ *is true and* $\mathrm{cf}([\kappa]^{\le\omega}) \le \kappa^+$ *whenever* $\kappa > \mathrm{cf}(\kappa) = \omega$. *Then* $bu([\kappa]^{\le\omega}) \le \omega_1$ *for every* κ.

(d) *If*

(*) for every $f : [\omega_{\omega+1}]^{<\omega} \to [\omega_\omega]^{\le\omega}$ *there is an uncountable* $A \subseteq \omega_{\omega+1}$ *such that* $\{f(I) : I \in [A]^{<\omega}\}$ *is countable, then* $bu([\omega_\omega]^{\le\omega}) > \omega_1$.

Proof. (a) (i) It will help to begin with the following fact: let D be the class of ordinals Δ such that $bu([\Delta]^{\le\omega}) \le \omega_1$. Suppose that α is an ordinal, that $\mathrm{cf}(\alpha) > \omega$, and that $\alpha = \sup(\alpha \cap D)$; then $\alpha \in D$. For we can choose, for each $\Delta \in \alpha \cap D$, a cofinal set $C_\Delta \subseteq [\Delta]^{\le\omega}$ such that $\{c : c \in C_\Delta, c \subseteq a\}$ is countable for each $a \in [\Delta]^{\le\omega}$. Next, for $a \in [\alpha]^{\le\omega}$, set $\gamma(a) = \min\{\Delta : \Delta \in \alpha \cap D, a \subseteq \Delta\}$; $\gamma(a)$ is well-defined (not ∞!) because $\mathrm{cf}(\alpha) > \#(a)$ and $\alpha = \sup(\alpha \cap D)$. Note that $\gamma(a) \le \gamma(b)$ if $a \subseteq b$. Write

$$C = \left\{a : a \in [\alpha]^{\le\omega},\ a \in C_{\gamma(a)}\right\}.$$

This C witnesses that $bu([\alpha]^{\le\omega}) \le \omega_1$. To see that it is cofinal with $[\alpha]^{\le\omega}$, take any $a \in [\alpha]^{\le\omega}$; then $a \in [\gamma(a)]^{\le\omega}$ so there is a $c \in C_{\gamma(a)}$ such that $a \subseteq c$; because $a \subseteq c \subseteq \gamma(a)$, $\gamma(c) = \gamma(a)$ and $c \in C$. Finally, given $b \in [\alpha]^{\le\omega}$,

$$\{c : c \in C,\ c \subseteq b\} \subseteq \bigcup_{\Delta \in E} \left\{c : c \in C_\Delta,\ c \subseteq b \cap \Delta\right\}$$

where $E = \{\gamma(a) : a \subseteq b\}$. But if $\Delta, \Delta' \in E$ and $\Delta < \Delta'$ then there is an $a \subseteq b$ such that $\gamma(a) = \Delta'$, so that $a \subseteq \Delta'$ but $a \not\subseteq \Delta$; in this case $b \cap \Delta' \setminus \Delta \ne \emptyset$. It follows that E is countable. But also $\{c : c \in C_\Delta,\ c \subseteq b \cap \Delta\}$ must be countable for every $\Delta \in E$, so $\{c : c \in C, c \subseteq b\}$ is countable, as required.

(ii) Since $bu([\kappa]^{\le\omega}) = 2$ for $\kappa \le \omega$ and $bu([\alpha]^{\le\omega}) = bu([\#(\alpha)]^{\le\omega})$ for every ordinal α, we see at once that if there is any cardinal κ such that $bu([\kappa]^{\le\omega}) > \omega_1$ then the lest such cardinal is uncountable and of countable cofinality. In particular, it must be at least ω_ω; which proves (a).

(b) Consider $A = [\kappa]^1 \subseteq [\kappa]^{\le\omega}$; no uncountable subset of A can be bounded above in $[\kappa]^{\le\omega}$; so if $\#(A) = \mathrm{cf}([\kappa]^{\le\omega})$, $bu([\kappa]^{\le\omega}) \le \omega_1$, by 1Kb.

(c) Now suppose that $\mathrm{cf}(\kappa) = \omega < \kappa$, that $bu([\lambda]^{\le\omega}) \le \omega_1$ for every $\lambda < \kappa$, that $\square_\kappa$ is true and that $\mathrm{cf}([\kappa]^{\le\omega}) \le \kappa^+$. By Theorem 8.7 of [25] (see also [27], 2.8), there is a set $M \subseteq \kappa^{\mathbb{N}}$ such that (i) $\#(M) = \kappa^+$ (ii) any subset of M of cardinal ω_1 is expressible as $\bigcup_{n\in\mathbb{N}} F_n$ where each F_n is well-ordered for the lexicographic ordering of $\kappa^{\mathbb{N}}$. Consider

$$A = \{f[\mathbb{N}] : f \in M\} \subseteq [\kappa]^{\le\omega}.$$

If $F \subseteq \kappa^{\mathbf{N}}$ is uncountable and well-ordered for the lexicographic ordering of $\kappa^{\mathbf{N}}$ then $\bigcup\{f[\mathbf{N}] : f \in F\}$ is uncountable; it follows that if $F \subseteq M$ is uncountable so is $\bigcup\{ f[\mathbf{N}] : f \in F\}$; consequently $\#(A) = \#(M) = \kappa^+$ and every uncountable subset of A is unbounded above in $[\kappa]^{\leq\omega}$. By 1Kb, $bu([\kappa]^{\leq\omega}) \leq \omega_1$.

Putting this together with part (i) of the proof of (a) above, we have a proof of (c).

(d) Now suppose that (*) is true, and that C is cofinal with $[\omega_\omega]^{\leq\omega}$. In this case $\#(C) > \omega_\omega$. Let $\langle c_\xi \rangle_{\xi<\omega_{\omega+1}}$ be a family of distinct elements of C and set

$$f(I) = \bigcup_{\xi \in I} c_\xi \in [\omega_\omega]^{\leq\omega}$$

for $I \in [\omega_{\omega+1}]^{\leq\omega}$. By (*) there is an uncountable $A \subseteq \omega_{\omega+1}$ such that $a = \bigcup\{f(I) : I \in [A]^{\leq\omega}\}$ is countable. Now $\{c : c \in C, c \subseteq a\} \supseteq \{c_\xi : \xi \in A\}$ is uncountable. As C is arbitrary, $bu([\omega_\omega]^{\leq\omega}) > \omega_1$.

Remarks 1M. (a) 1L(a-b) come from [13]. 1L(c-d) are due to S. Todorčević.

(b) The condition of 1Lb is satislied by cardinals κ of the form 2^λ, where $\lambda \geq \omega$, and by strong limit cardinals of uncountable cotinality.

(c) The conditions of 1Lc are satisfied e.g. whenever Jensen's Covering Lemma is true (see [5], § V.5).

(d) If it is relatively consistent with ZFC to suppose that there is a 2-huge cardinal, then it is relatively consistent with ZFC to assume (*) of 1Ld; see [18].

2. THE PARTIALLY ORDERED SETS OF MEASURE THEORY

1 apply the ideas of § 1 to function spaces (specifically L^1, L^0 and ℓ^1), measure algebras, and ideals of negligible sets.

2A. Notation. It may be helpful to declare the following, which is supposed to be nearly standard.

(a) If I is any set, then $\ell^1(I)$ is the Banach lattice $\{x : x \in \mathbf{R}^I, \|x\| = \sum_{i\in I} |x(i)| < \infty\}$, ordered by saying that $x \leq y$ iff $x(i) \leq y(i)$ for every $i \in I$. If $x \in \ell^1(I)$ then $x^+ \in \ell^1(I)$ is defined by writing $x^+(i) = \max(x(i), 0)$ for every $i \in I$.

(b) If (X, μ) is a measure space, write $\mathcal{N}_\mu$ for the σ-ideal of negligible subsets of X viz. $\{E : \exists F \supseteq E$ such that $\mu(F) = 0\}$. Write $L^0(\mu)$ for the Dedekind u-complete Riesz space of equivalence classes of μ-measurable real-valued functions on X, and $L^1(\mu)$ for the Banach lattice of equivalence classes of μ-integralbe real-valued functions on X.

(c) Following [7] and [8], I take a *Radon measure* space to be a quadruple $(X, \mathfrak{T}, \Sigma, \mu)$ where (X, Σ, μ) is a complete locally determined (or «saturated») measure space, $\mathfrak{T}$ is a

Hausdorff topology on X, $\mathfrak{T} \subseteq \Sigma$, and μ is locally finite and inner regular for the compact sets. Most of the time we shall be dealing with probability spaces; in this case $\mathcal{N}_\mu = \mathcal{N}_\nu$ where ν is the restriction of μ to the algebra of Bore1 subsets of X .

(d) If $\mathfrak{A}$ is a Boolean algebra, write $\mathfrak{A}^-$ for the partially ordered set $\mathfrak{A} \setminus \{1\}$.

(e) Forcardinals $\kappa \geq \omega$, take $\mathfrak{A}_\kappa$ tobethemeasurealgebraof $[0,1]^\kappa$ when $[0,1]^\kappa$ is given its usual measure (the Radon product measure when each copy of [0, 1] is given Lebesgue measure; see [8], A7E).

(f) Recall that the *Maharam type* of a u-finite measure space (X, μ) is the least cardinal $\tau(\mathfrak{A})$ of any subset of the measure algebra $\mathfrak{A}$ of (X, μ) which completely generates $\mathfrak{A}$ (see [11]). A e-finite measure space (X, μ) is *Maharam* ***homogeneous*** if all its non-negligible measurable subspaces have the same Maharam type, that is, if its measure algebra is a homogeneous Boolean algebra.

(g) Finally, 1 write $\mathcal{N}$ for the ideal of Lebesgue negligible subsets of [0 , 1] ; note that ([0, 1] ,$\mathcal{N}$) is isomorphic to $(X, \mathcal{N}_\mu)$ whenever X is a separable metric space and μ is an atomless, nonzero Radon measure on X .

2B. The first theorem of this section is rather abstract, but its generality enables us to deal sirnultaneously with partially ordered sets like L^0, L^1 and $\mathfrak{A}^-$.

Theorem. ***Let P be a partially ordered set such that $x \vee y = \sup\{x, y\}$ is defined for all $x, y \in P$. Suppose that there is a metric p on P for which P is complete and $\wedge : P \times P \to P$ is uniformly continuous. Let $Q \subseteq P$ be a p-open set, given the induced ordering. Then $Q \leq_{<\omega} \ell^1(\kappa)$ for any $\kappa \geq$ d(Q) , where d(Q) is the topologica1 density of Q .***

Proof. (a) If Q is finite this is elementary, as Q $\leq_{<\omega}$ ***R*** for any non-empty partially ordered set ***R.*** SO let us suppose that Q and κ are infinite. Fix on a family $\langle q_\xi \rangle_{\xi<\kappa}$ in Q such that $\{q_\xi : \xi < \kappa\}$ is dense in Q . For each g $\in$ Q let m(g) $\in$ N be such that

$$\{p : p \in P, \rho(p, q) \leq 2^{-m(q)}\} \subseteq Q.$$

For each $n \in \mathbb{N}$, let $\triangle_n > 0$ be such that

$$\rho(\sup I, \sup J) \leq 2^{-n} \text{ whenever } \emptyset \neq I \subseteq J \subseteq P,\ \#(J) \leq 2^n,$$

$$\sup_{q\in J} \inf_{p\in I} \rho(q,p) \leq 2\triangle_n;$$

such a $\triangle_n$ exists because $\langle p_i \rangle_{i<k} \mapsto \sup_{i<k} p_i : P^k \to P$ is uniformly continuous for every k, and in particular for $k = 2^n$; we may suppose also that $\triangle_{n+1} \leq \triangle_n \leq 2^{-2}$ for every n.

(b) Define $f : Q \to \ell^1(\kappa)$ as follows. Given $p \in Q$, choose $\langle \xi(p, n)\rangle_{n\in\mathbb{N}} \in \kappa^{\mathbb{N}}$ such that $\rho(p, q_{\xi(p,n)}) \le \triangle_{n+1}$ for every $n \in \mathbb{N}$. Take $f(p) \in \ell^1(\kappa)$ such that

$$f(p)(m(p)) \ge 1,\ f(p)(\xi(p,n)) \ge 2^{-n} \forall \in \mathbb{N}$$

(regarding $m(p) \in \mathbb{N}$ as a member of n).

(c) 1 now have to show that f is $<$ w-Tukey . Fix $z \in \ell^1(\kappa)$, and consider $A = \{p : p \in \in Q, f(p) \le z\}$. Set

$$K_i = \left\{q_\xi : \xi < \kappa,\ z(\xi) \ge 2^{-i}\right\}$$

for $i \in \mathbb{N}$. Then there is a $k \in \mathbb{N}$ such that $z(i) < 1$ for $i \in \mathbb{N}$, $i \ge k$, and also $\#(K_i) \le 2^{-i}$ for every $i \ge k$ (for $\| z \| \ge \sum_{i\in\mathbb{N}} 2^{-i-1}\#(K_i)$). For each $r \in K_k$ define $\langle I(r, i)\rangle_{i\ge k}$ by writing

$$I(r,k) = \{r\},$$
$$I(r,i+1) = \{q : q \in K_{i+1}, \exists q' \in I(r,i), \rho(q,q') \le 2\triangle_{i+1}\}.$$

Set $p_{ri} = \sup I(r, i) \in \boldsymbol{P}$ for $i \ge k$; then $\langle p_{ri}\rangle_{i\ge k}$ is an increasing sequence in P. Moreover, if $i \ge k$,

$$\rho\left(p_{r,i+1}, p_{ri}\right) = \rho(\sup I(r,i+1), \sup I(r,i)) \le 2^{-i-1}$$

because $\#(I(r,i+1)) \le \# K_{i+1} \le 2^{i+1}$ and every member of $I(r, i+1)$ is within a distance $2\triangle_{i+1}$ of some member of $I(r, i)$. So $\langle p_{ri}\rangle_{i\ge k}$ is a Cauchy sequence in $\boldsymbol{P}$ and has a limit $p_r \in P$.

Set $B = \{p_r : r \in K_k\} \cap \boldsymbol{Q} \in [Q]^{<\omega}$. 1 claim that B dominates $\boldsymbol{A}$. For suppose that $p \in A$. Then

$$2^{-k} \le f(p)(\xi(p,k)) \le z(\xi(p,k)),$$

SO $r = q_{\xi(p,k)} \in K_k$. Next, $q_{\xi(p,i)} \in I(r, i)$ for every $i \ge k$, because $q_{\xi(p,i+1)} \in K_{i+1}$ and $\rho(q_{\xi(p,i+1)}, q_{\xi(p,i)}) \le \triangle_{i+2} + \triangle_{i+1} \le 2\triangle_{i+1}$ for every $i \ge k$. Finally,

$$1 \le f(p)(m(p)) \le z(m(p)),$$

so $m(p) < k$. It follows that

$$p \wedge p_r = \lim_{i\to\infty} q_{\xi(p,i)} \wedge p_r = \lim_{i\to\infty}\lim_{j\to\infty} q_{\xi(p,i)} \wedge p_{rj} =$$

$$= \lim_{i\to\infty}\lim_{j\to\infty} p_j = p_r,$$

so that $p \leq p_r$; and also that

$$\rho(p_r, p) \leq \rho(r,p) + \sum_{i=k}^{\infty} \rho\left(p_{r,i+1}, p_{ri}\right) \leq$$

$$\leq 2^{-k} + \sum_{i-k}^{\infty} 2^{-i-1} = 2^{-k+1} \leq 2^{-m(p)},$$

so that $p_r \in Q$ and $p_r \in B$.

This shows that B dominates A and that A is finitely dominated in Q . As z is arbitrary, f is $<$ w-Tukey.

Theorem 2C. *Let* (X, μ) ***be an*** *atomless probability* ***space of Maharam type*** $\kappa \geq w$ ***. Then***

$$[\kappa]^{\leq\omega} \times \ell^1(\mathbf{N}) \leq \ell^1(\kappa) \equiv L^1(\mu) \equiv_{<\omega} \mathfrak{A}_\kappa^- .$$

Proof. (a) (i) Choose $f : [\kappa]^{\leq\omega} \rightarrow \ell^1(\kappa)$ such that f(a)(f) $>$ 0 whenever $\xi \in$ a $\in$ $\in [\kappa]^{\leq\omega}$. Then f is a Tukey function so $[\kappa]^{\leq\omega} \leq \ell^1(\kappa)$. (ii) Define g : $\ell^1(\mathbf{N}) \rightarrow \ell^1(\kappa)$ by setting g(x) (n) = z(n) for n $\in$ N , $g(x)(\xi) = 0$ for $\xi \in \kappa \setminus$ N (identifying the set N of natural numbers with the set w of finite ordinals). Then g is a Tukey function so ℓ^1 (N) $\leq$ $\leq \ell^1$ (IC) . (iii) Because ℓ^1(κ) is upwards-directed, it follows that [n] $^{\leq\omega}$ x ℓ^1 (N) $\leq \ell^1(\kappa)$.

(b) Let $\mathfrak{A}$ be the measure algebra of (X, μ) . then there is a disjoint sequence $\langle a_n \rangle_{n\in\mathbf{N}}$ in $\mathfrak{A} \setminus \{0\}$ such that $\sup_{n\in\mathbf{N}} a_n = 1$ in $\mathfrak{A}$ and each relative algebra $\mathfrak{A}_{a_n}$ is Maharam homogeneous ([11], § 3). Set $\kappa_n = \tau(\mathfrak{A}_{a_n})$ for each n; then $\sup_{n\in\mathbf{N}} \kappa_n = \kappa$ (it may be that some or all of the κ_n are equal to n). Let $\langle X_n \rangle_{n\in\mathbf{N}}$ be a disjoint sequence of measurable subsets of X such that the equivalence class $X_n^{\cdot}$ of X_n in $\mathfrak{A}$ is a_n, for each n $\in$ N . Define measures μ_n on X_n by setting

$$\mu_n(E) = \mu(E)/\mu(X_n)$$

for each μ-measurable set $E \subseteq X_n$. Then $L^1(\mu)$ is isomorphic to the ℓ^1 -direct sum

$$\left\{u : u \in \prod_{n\in\mathbf{N}} L^1(\mu_n),\ \| u \| = \sum_{n\in\mathbf{N}} \| u(n) \| < \infty\right\}$$

of $\langle L^1(\mu_n)\rangle_{n\in\mathbf{N}}$.

For each n $\in$ N , $L^1(\mu_n)$ is isomorphic (as Banach lattice) to $L^1(\nu_n)$, where ν_n is the usual measure on $[0,1]^{\kappa_n}$. Let P be the ℓ^1-direct sumof $\langle L^1(\nu_n)\rangle_{n\in\mathbf{N}}$, sothat $P \cong L^1(\mu)$.

For each $\xi<\kappa$ take $m(\xi)\in\mathrm{N}$ such that $\xi<\kappa_{m(\xi)}$, and define $u_\xi\in P$ by writing

$$u_\xi(i)=0 \text{ if } i\in\mathrm{N}\setminus\{m(\xi)\},$$
$$u_\xi(m(\xi))=w_\xi^{\cdot},$$

the equivalence class in $L^1(\nu_{m(\xi)})$ of w_ξ, where $w_\xi:[0,1]^{\kappa_{m(\xi)}}\to\mathrm{R}$ is given by the formula

$$w_\xi(t)=\frac{1}{\sqrt{t(\xi)}} \text{ if } t(\xi)>0,\ 0 \text{ otherwise}$$

Observe that $\|u_\xi\|=\int w_\xi \mathrm{d}\nu_{m(\xi)}=2$ for every $\xi<\kappa$. Observe also that if I is a non-empty finite subset of κ and m(f) = k for every $\xi\in I$, then

$$\|\sup_{\xi\in I}2^{-n}u_\xi\|\geq 2^{-n}\#(I)^{1/2}$$

for every $n\in\mathrm{N}$. For $\|\sup_{\xi\in I}2^{-n}u_\xi\|=\int w\mathrm{d}\nu_k$, where

$$w(t)=2^{-n}\max_{\xi\in I}(t(\xi))^{-1/2}$$

if $t\in[0,1]^{\kappa_k}$ and $\min_{\xi\in I}t(\xi)>0$, w(t) = 0 if $\min_{\xi\in I}t(\xi)=0$. But for each real $\alpha>0$,

$$\nu_k\{t:W(t)\leq\alpha\}=\prod_{\xi\in I}\nu_k\{t:t(\xi)\geq 4^{-n}\alpha^{-2}\}=$$
$$=\left(1-4^{-n}\alpha^{-2}\right)^{\#(I)}\leq 1-\frac{1}{2}\beta^2\alpha^{-2} \quad \text{if} \quad \alpha\geq\beta,$$

where $\beta=2^{-n}\#(I)^{1/2}$. So

$$\nu_k\{t: w(t)\geq\alpha\}\geq\frac{1}{2}\beta^2\alpha^{-2} \quad \text{if} \quad \alpha\geq\beta,$$

and

$$\int w\mathrm{d}\nu_k=\int_0^\infty\nu_k\{t:w(t)\geq\alpha\}\mathrm{d}\alpha\geq\frac{1}{2}\int_\beta^\infty\beta^2\alpha^{-2}\mathrm{d}\alpha+\frac{1}{2}\beta=\beta,$$

as required.

Now let $\langle y_\xi\rangle_{\xi<\kappa}$ run over a norm-dense subset of $\ell^1(\kappa)$. For $x\in\ell^1(\kappa)$ choose inductively a sequence $\langle\xi(x,n)\rangle_{n\in\mathrm{N}}$ in κ such that

$$\|x^+-\sum_{i\leq n}y_{\xi(x,i)}\|\leq 8^{-n}\ \forall n\in\mathrm{N}.$$

Let $f : \ell^1(\kappa) \to P$ be such that

$$f(x) \geq \sum_{n\in\mathbf{N}} 2^{-n} u_{\xi(x,n)} \, , \| f(x) \| \geq \| x \| \; \forall \; x \in \ell^1(\kappa).$$

To see that f is a Tukey function, fix $v \in P$, and set $A = \{x : f(x) \leq v\}$. Set

$$K_{kn} = \left\{\xi : m(\xi) = k, \; 2^{-n} u_\xi \leq v\right\}$$

for k, n $\in$ N . Then, by the calculations in the last paragraph,

$$\| v(k) \| \geq 2^{-n} \; \forall \; k, n \in \mathbf{N}.$$

so

$$\sum_{k\in\mathbf{N}} \# \left(K_{kn}\right) \leq 4^n \sum_{k\in\mathbf{N}} \| v(k) \|^2 \leq 4^n \left(\sum_{k\in\mathbf{N}} \| v(k) \|\right)^2 =$$
$$= 4\text{"} \; \| v \|^2,$$

for every n $\in$ N . Consequently

$$\#(\{\xi(x,n) : \mathrm{x} \in A\}) \leq 4\text{"} \; \| v \|^2 \; \forall \, n \in \mathbf{N},$$

and

$$\sum_{n\in\mathbf{N}} \sum \{\| y_{\xi(x,n)} \| : x \in A\} \leq \sum_{n\in\mathbf{N}} 4^n \| v \|^2 \sup_{x\in A} \| y_{\xi(x,n)} \| \leq$$
$$\leq \| v \|^2 \; \left(1 + \sup_{x\in A} \| x \|\right) + \| v \|^2 \sum_{n=1}^{\infty} 4^n \left(8^{-n+1} + 8^{-n}\right) < \infty$$

(as $\| \mathrm{x} \| \leq \| v \|$ for every x $\in A$). But this means that

$$y = \sum_{n\in\mathbf{N}} \sum \left\{y^+_{\xi(x,n)} : x \in A\right\}$$

is detined in $\ell^1(\kappa)$, and is an upper bound for A in ℓ^1 (n) .

As v is arbitrary, f is a Tukey function, and ℓ^1 (n) $\leq P \cong L^1(\mu)$.

(c) By 2B, with $P = Q = L^1(\mu)$, $\rho(u,v) = \| u - v \|$, we have $L^1(\mu) \leq_{<\omega} \ell^1(\kappa)$. But as $L^1(\mu)$ is upwards-directed, it follows that $L^1(\mu) \leq \ell^1(\kappa)$, so that $L^1(\mu) \equiv \ell^1(\kappa)$.

(d) To see that $\ell^1(\kappa) \leq \mathfrak{A}_\kappa^-$, we can use a simpler version of the argument in (b). Define $f : \ell^1(\kappa) \to \mathfrak{A}_\kappa$ by writing $a_x = \{\xi : x(\xi) > 0\}$,

$$f(x) = \{t : t \in [0,1]^\kappa, \exists \xi \in a_x, t(\xi)\left(1 + x^+(\xi)\right) < x^+(\xi)\}^\cdot .$$

Then $f(x \vee y) = f(x) \cup f(y)$, and

$$\tilde{\nu}(f(x)) = 1 - \prod_{\xi<\kappa}\left(1 - \frac{x^+(\xi)}{1 + x^+(\xi)}\right) < 1,$$

where $\tilde{\nu}$ is the measure of $\mathfrak{A}_\kappa$, because $\sum_{\xi<\kappa} \frac{x^+(\xi)}{1+x^+(\xi)} \leq \| x \| < \infty$. It follows directly that f is a Tukey function. For suppose that $a \in \mathfrak{A}^-$ and that $A = \{x : f(z) \leq a\}$. Then A contains 0 and is upwards-directed. Set $y(\xi) = \sup_{x\in A} x(\xi)$ for each $\xi \in I$ (allowing, notionally, $y(\xi) = 00$). Then

$$\frac{y(\xi)}{1 + y(\xi)} = \sup_{x\in A} \frac{x^+(\xi)}{1 + x^+(\xi)} \quad \text{for every } \xi < \kappa$$

(taking $\infty/(1+\infty)$ as 1), so

$$\prod_{\xi<\kappa}\left(1 - \frac{y(\xi)}{1 + y(\xi)}\right) = \inf_{x\in A}\prod_{\xi<\kappa}\left(1 - \frac{x^+(\xi)}{1 + x^+(\xi)}\right) \geq 1 - \tilde{\nu}(a) > 0;$$

this shows both that $y(f) < \infty$ for every ξ and that $\sum_{\xi<\kappa} \frac{y(\xi)}{1+y(\xi)} < \infty$ so that $y \in \ell^1(n)$. Accordingly y is an upper bound for A in $\ell^1(\kappa)$. As a is arbitrary, f is a Tukey function and $\ell^1(\kappa) \leq \mathfrak{A}_\kappa^-$.

(e) Finally, $\mathfrak{A}_\kappa^- \leq_{<\omega} \ell^1(\kappa)$, by 2B, using $P = \mathfrak{A}_\kappa$ and $Q = \mathfrak{A}_\kappa^-$, with $\rho(a,b) = \tilde{\nu}(a \triangle b)$. So $\ell^1(\kappa) \equiv_{<\omega} \mathfrak{A}_\kappa^-$.

Theorem 2D. ***Let*** (X, μ) ***be an atomless Radon probability space of Maharam type*** $\kappa \geq w$ ***. Then***

$$[\kappa]^{\leq\omega} \times \mathcal{N} \leq \mathcal{N}_\mu \leq \ell^1(\kappa).$$

Proof. (a) (i) By [9], Lemma 14 (repeated in [Il], 6.10) there is a family $\langle E_\xi\rangle_{\xi<\kappa}$ in $\mathcal{N}_\mu$ such that $\{\xi : E_\xi \subseteq E\}$ is countable for every $E \in \mathcal{N}_\mu$. Now $a \mapsto \bigcup_{\xi\in a} E_\xi : [\kappa]^{\leq\omega} \to \mathcal{N}_\mu$ is a Tukey function, so $[\kappa]^{\leq\omega} \leq \mathcal{N}_\mu$. (ii) There is an inverse-measure-preserving function

$h : X \to [0,1]$ ([8], A6I); now $E \mapsto h^{-1}[E] : \mathcal{N} \to \mathcal{N}_\mu$ is a Tukey function, by the argument of [9], §§ 7-8, or [Il], 6.12. Thus $\mathcal{N} \leq \mathcal{N}_\mu$. (i) Because $\mathcal{N}_\mu$ is upwards-directed, it follows that $[\kappa]^{\leq\omega} \times \mathcal{N} \leq \mathcal{N}_\mu$.

(b) Let $\mathfrak{A}$ be the measure algebra of (X, μ), and $\tilde{\mu}$ the measure of $\mathfrak{A}$. Set

$$P = \left\{p : p \in \mathfrak{A}^{\mathbf{N}}, \lim_{n\to\infty} \tilde{\mu}(p(n)) = 0\right\}.$$

For $p, q \in P$ set

$$\rho(p,q) = \sup_{n\in\mathbf{N}} \tilde{\mu}(p(n)\triangle q(n)).$$

Then P satisfies the conditions of 2B, and its topological density is $\max(\omega, \tau(\mathfrak{A})) = \kappa$; so $P \leq_{<\omega} \ell^1(\kappa)$; because P is upwards-directed, $P \leq \ell^1(\kappa)$.

Now define $f : \mathcal{N}_\mu \to P$ by writing $f(E) = \langle G^\cdot_{En}\rangle_{n\in\mathbf{N}}$, where for $E \in \mathcal{N}_\mu, n \in \mathbf{N}$, G_{En} is an open set of measure $\leq 2^{-n}$ including E. I claim that f is a Tukey function. For take $p \in P$. For each $n \in \mathbf{N}$ set

$$\mathscr{H}_n = \{H : H \subseteq X \text{ is open}, H^\cdot \subseteq p(n) \text{ in } \mathfrak{A}\},$$
$$H_n = \bigcup \mathscr{H}_n.$$

Then $\mu(H_n) = \sup\{\mu(H) : H \in \mathscr{H}_n\}$ (because every compact subset of H_n is included in some member of $\mathscr{H}_n$), so $H^\cdot_n = \sup\{H^\cdot : H \in \mathscr{H}_n\} \subseteq p(n)$. Set $F = \bigcap_{n\in\mathbf{N}} H_n$; then $F \in \mathcal{N}_\mu$ because $\inf_{n\in\mathbf{N}} \tilde{\mu}(p(n)) = 0$. But if $E \in \mathcal{N}_\mu$ and $f(E) \leq p$, then $G_{En} \subseteq H_n$ for every $n \in \mathbf{N}$, so $E \subseteq F$. Thus $\{E : f(E) \leq p\}$ is bounded above in $\mathcal{N}_\mu$, and f is a Tukey function.

Accordingly $\mathcal{N}_\mu \leq P$ and $\mathcal{N}_\mu \leq \ell^1(\kappa)$.

Remark. The argument of [11], 6.5, can also be used to show that under the conditions of this theorem, $\mathcal{N}_\mu \leq \mathfrak{A}^-_\kappa$.

Theorem 2E. *Let (X, μ) be a Maharam homogeneous probability space of Maharam type $\kappa \geq \omega$. Then $L^0(\mu) \equiv \ell^1(\kappa)$.*

Proof. (a) As in 2C, we have $L^0(\mu) \leq \ell^1(\kappa)$, because $L^0(\mu)$ carries a complete metric ρ for which its density is κ and $\wedge$ is uniformly continuous (set $\rho(u,v) = \int \frac{|u-v|}{1+|u-v|}$).

(b) To check that $\ell^1(\kappa) \leq L^0(\mu)$, it is enough to consider the case in which $X = [0,1]^\kappa$ with its usual measure, since the Riesz space $L^0(\mu)$ depends only on the measure algebra of X ([7], 62L). In this case, for $\xi < \kappa$, $t \in [0,1]^\kappa$, $t \in [0,1]^\kappa$ set $w_\xi(t) = t(\xi)^{-1/2}$ if $t(\xi) > 0$, 0 otherwise; set $u_\xi = w_\xi^\bullet \in L^0(\mu)$. For $x \in \ell^1(\kappa)$ observethat

$$\mu\left(\left\{t : \sup_{\xi<\kappa} \left(x^+(\xi)\right)^{1/2} w_\xi(t) \le \alpha\right\}\right) =$$
$$\mu\left(\left\{t : x^+(\xi) \le \alpha^2 t(\xi)\ \forall\, \xi < \kappa\right\}\right) =$$
$$= \prod_{\xi<\kappa} \left(1 - \alpha^{-2} x^+(\xi)\right) \ge 1 - \alpha^{-2} \parallel x \parallel \to 1$$

as $\alpha \to \infty$. Consequently

$$\sup_{\xi<\kappa} \sqrt{x^+(\xi)}\, w_\xi(t) < \infty \quad \mu - \text{a.e.}\ (t),$$

and $f(x) = \sup_{x(\xi)>0} \sqrt{x(\xi)}\, u_\xi$ is defined in $L^0(\mu)$.

TO see that $f : \ell^1(\kappa) \to L^0(\mu)$ is a Tukey function, take $v \in L^0(\mu)^+$. Write $A = \{x : f(x) \le v\}$ and set $y(\xi) = \sup_{x\in A} x(\xi)$ for $\xi < \kappa$; because $\sqrt{x^+(\xi)}\, u_\xi \le v$ for every $x \in A$, $y(\xi)$ is finite. Take $\alpha > 0$ such that $\mu(\{t : w(t) \le \alpha\}) = \beta > 0$, where w is a measurable function with w' = v. Then the calculation just above shows that

$$\prod_{\xi<\kappa} \left(1 - \alpha^{-2} x^+(\xi)\right) \ge \beta$$

for every $x \in A$; as A is upwards-directed, it follows that

$$\prod_{\xi<\kappa} \left(1 - \alpha^{-2} y(\xi)\right) \ge \beta,$$

so that $\sum_{\xi<\kappa} \alpha^{-2} y(\xi) < \infty$ and $y \in \ell^1(\kappa)$. Thus y is an upper bound for A in $\ell^1(\kappa)$. As v is arbitrary, f is a Tukey function, and ℓ^1 (n) $\le L^0(\mu)$.

Theorem **2F.** *Let (X, μ) be a Maharam homogeneous Radon probability space of Maharam type $\kappa \ge$ w. Then $\mathcal{N}_\mu \equiv_\omega \ell^1(\kappa)$.*

Proof. (a) We know already from 2D that $\mathcal{N}_\mu \le \ell^1$ (n) .

(b) (i) Let $\langle H_\xi \rangle_{\xi<\kappa}$ be a family in $\mathcal{N}_\mu$ such that $\{\xi : H_\xi \subseteq E\}$ is countable for every $E \in \mathcal{N}_\mu$ (see (a) (i) of the proof of 2D). Let $\langle G_{\xi n} \rangle_{\xi<\kappa, n\in\mathbb{N}}$ be a stochastically independent family of measurable sets in X such that $\mu(G_{\xi n}) = 2^{-n}$ forall $\xi < \kappa, n \in \mathbf{N}$; such a family exists because the measure algebra of X is isomorphic, as measure algebra, to that of $[0,1]^{\kappa\times\mathbf{N}}$. For $x \in \ell^1(\kappa)$ set $J(x,m) = \{\xi : 2^{-m+1} > x(\xi) \geq 2^{-m}\}$,

$$f(x) = \bigcup_{x(\xi)>0} H_\xi \cup \bigcap_{n\in\mathbf{N}} \bigcup_{m\geq n} \bigcup_{\xi\in J(x,m)} G_{\xi m}.$$

Because $\{\xi : x(\xi) > 0\}$ is countable, f(z) is a measurable set; because

$$\sum_{m\in\mathbf{N}} \sum_{\xi\in J(x,m)} \mu\left(G_{\xi m}\right) \leq 2\, \| x \| < \infty,$$

$f(x) \in \mathcal{N}_\mu$.

(ii) Let $E \in \mathcal{N}_\mu$ and let $A = \{x : f(x) \subseteq E\}$. *Note* that $I = \{\xi : H_\xi \subseteq E\}$ is countable, and that $x(\xi) \leq 0$ whenever $x \in A$ and $\xi \in \kappa \setminus I$. Because I is countable, there is a compact set $F_0 \subseteq X \setminus E$ such that $\mu(F_0) > 0$ and $F_0 \cap G_{\xi n}$, $F_0 \setminus G_{\xi n}$ are compact for all $\xi \in I$, $n \in \mathbf{N}$; now there is a compact set $F \subseteq F_0$ such that $\mu(F) = \mu(F_0)$ and $\mu(F \cap G) > 0$ whenever G is an open set meeting F ([8], A7Bg). Define $\pi : F \to \{0,1\}^{I\times\mathbf{N}}$ by saying that $\pi(t)(\xi,n) = 1$ if $t \in$ G,,, 0 otherwise; then π is continuous. Write $\mathscr{U}$ for the family of open-and-closed subsets of $\{0,1\}^{I\times\mathbf{N}}$ meeting $\pi[F]$; then $\mathscr{U}$ is countable. If $U \in \mathscr{U}$, then $\pi^{-1}[U]$ is a non-empty relatively open subset of F, so $\mu(\pi^{-1}[U]) > 0$.

For $U \in \mathscr{U}$, $n \in \mathbf{N}$ set

$$K(U,n) = \left\{\xi : \xi \in I, \pi^{-1}[U] \cap G_{\xi n} = \emptyset\right\}$$

Then

$$0 < \mu\left(\pi^{-1}[U]\right) \leq \mu\left(\bigcap_{n\in\mathbf{N}} \bigcup_{\xi\in K(U,n)} X \setminus G_{\xi n}\right) =$$

$$= \prod_{n\in\mathbf{N}} \prod_{\xi\in K(U,n)} \mu\left(X \setminus G_{\xi n}\right) = \prod_{n\in\mathbf{N}} \left(1 - 2^{-n}\right)^{\#(K(U,n))}.$$

so $\sum_{n\in\mathbf{N}} 2^{-n}\#(K(U,n)) < \infty$. Take $z_U \in \ell^1(\kappa)^+$ suchthat $z_U(\xi) \geq 2^{-n}$ if $n \in \mathbf{N}$ and $\xi \in K(U,n)$, and $z_U(\xi) > 0$ forevery $\xi \in I$. Set

$$B = \{k z_U : k \in \mathbf{N},\ U \in \mathscr{U}\};$$

then B is a countable subset of ℓ^1 (n) .

(iii) I claim that B dominates A. For suppose that $x \in A$. Then f(z) $\subseteq E$ so $F \cap$ $\cap f(x) = \emptyset$. But

$$F \cap f(x) \supseteq \bigcap_{n\in \mathrm{N}} \bigcup_{m\geq n} \bigcup_{\xi\in J(x,m)} G_{\xi m} \cap F =$$

$$= \pi^{-1}\left[\bigcap_{n\in \mathrm{N}} \bigcup_{m\geq n} W_m\right],$$

where

$$W_m = \{t : t \in \{0, 1\}^{I\times \mathrm{N}},\ \exists \xi \in J(x,m) \text{ such that } t\{(\xi, m) = 1\}$$

is an open set in $\{0,1\}^{I\times \mathrm{N}}$ for each $m \in \mathrm{N}$. Now $\pi[F]$ is a compact non-empty set not meeting the G_Δ set $\bigcap_{n\in \mathrm{N}} \bigcup_{m\geq n} W_m$. So there are an $n \in \mathrm{N}$ and a $U \in \mathscr{U}$ suchthat $\pi[F] \cap U \cap \bigcup_{m\geq n} W_m = 0$. In this case, $\pi^{-1}[U] \cap \bigcup_{m\geq n} \pi^{-1}[W_m] = \emptyset$ i.e.

$$\pi^{-1}[U] \cap \bigcup_{m\geq n} \bigcup_{\xi\in J(x,m)} G_{\xi m} = \emptyset,$$

and $J(x, \mathrm{m}) \subseteq K(U, \mathrm{m})$ for every $m \geq n$. But this means that $x(\xi) \leq 2 z_U(\xi)$ whenever $x(\xi) < 2^{-n+1}$. So $\{\xi : x(\xi) > 2 z_U(\xi)\}$ is finite. Also $Z_U(\xi) > 0$ whenever $x(\xi) > 0$, because such ξ belong to I. So there is an $k \in \mathrm{N}$ such that $x \leq k z_U \in B$. As x is arbitrary, B dominates A.

(iv) As E is arbitrary, f is w-Tukey and $\ell^1(\kappa) \equiv_\omega \mathscr{N}_\mu$.

Remark. Versions of this argument may be found in [[1], [22], [10], [9] and [11]. The form here is based on that of [22].

Corollary 2G. $\mathscr{N} \equiv_\omega \ell^1$ **(N) .**

Remark. Theorem 5 of [1] states, in effect, that $\mathrm{add}(\mathscr{N}) = \mathfrak{C}$ iff $\mathrm{add}_\omega(\ell^1(\mathrm{N})) = \mathfrak{C}$; the arguments there include everything necessary to prove that $\mathscr{N} \equiv_\omega \ell^1$ (N) .

Corollary 2H. ***If (X, μ) and (Y, ν) are Radon measure spaces with Boolean isomorphic measure algebras then*** $\mathscr{N}_\mu \equiv \mathscr{N}_\nu$.

Proof. (a) Suppose first that the measure algebras are homogeneous (as Boolean algebras). (i) If both are $\{0\}$ then $\mu X = \mu Y = 0$, $\mathscr{N}_\mu = \mathscr{P}X$ and $\mathscr{N}_\nu = \mathscr{P}Y$, and $\mathscr{N}_\mu \equiv \{\emptyset\} \equiv$ $\equiv \mathscr{N}_\nu$. (ii) If both are $\{0,1\}$ then there are $t \in X$, $u \in Y$ such that $\mathscr{N}_\mu = \mathscr{P}(X \setminus \{t\})$, $\mathscr{N}_\mu = \mathscr{P}(\mathrm{Y} \setminus \{u\})$ so again $\mathscr{N}_\mu \equiv \mathscr{N}_\nu$. g If both are atomless, not $\{0\}$, then both must be *ccc* so there are probability measures ν', μ' on X, Y respectively with the same measurable sets and the same negligible sets as μ, ν. Now the measure algebras of (X, μ') and (Y, Y') are still Boolean isomorphic; suppose that their Maharam type is κ; then

$$\mathcal{N}_\mu = \mathcal{N}_{\mu'} \equiv_\omega \quad \ell^1(\kappa) \equiv_\omega \mathcal{N}_{\nu},$$

because μ' and ν' are still Radon measures, so $\mathcal{N}_\mu \equiv_\omega \mathcal{N}_\nu$ and (because both are σ-ideals) $\mathcal{N}_\mu \equiv \mathcal{N}_\nu$.

(b) In general, the measure algebras $\mathfrak{A}$, $\mathfrak{B}$ of (X, μ), (Y, ν) are semi-finite, so there is a partition $\langle a_\xi \rangle_{\xi<\kappa}$ of 1 in $\mathfrak{A}$ such that $\mathfrak{A}\,|a_\xi$ is homogeneous for each $\xi < \kappa$. Because (X, μ) is decomposable ([7], 72B, or [11], 1.1O), there is a partition $\langle X_\xi \rangle_{\xi<\kappa}$ of X into measurable sets such that $X_\xi^\cdot = a_\xi$ for every $\xi < \kappa$ and $\mathcal{N}_\mu = \{E : E \subseteq X, E \cap X_\xi \in \in \mathcal{N}_\mu \forall \xi < \kappa\}$. Let μ_ξ be the restriction of μ to subsets of X_ξ for each $\xi < \kappa$; then $\mathcal{N}_\mu \cong \cong \prod_{\xi<\kappa} \mathcal{N}_{\mu_\xi}$. If we repeat the argument in Y with the family $\langle \phi(a_\xi) \rangle_{\xi<\kappa}$, where $\phi : \mathfrak{A} \to \to \mathfrak{B}$ is an Boolean isomorphism, we obtain a partition $\langle Y_\xi \rangle_{\xi<\kappa}$ of Y with a corresponding family $\langle \nu_\xi \rangle_{\xi<\kappa}$ of measures. By (a) above, $\mathcal{N}_{\mu_\xi} \equiv \mathcal{N}_{\nu_\xi}$ for every $\xi < \kappa$; consequently $\mathcal{N}_\mu \equiv \mathcal{N}_\nu$, by 1E d.

Proposition ***21. Let (X, μ) be a measure space with measure algebra $\mathfrak{A}$, Write Σ_μ^* for $\{E : E \in \mathrm{dom}(\mu), X \setminus E \notin \mathcal{N}_\mu\}$. Then***

(a) $\mathfrak{A}^- \leq \Sigma_\mu^*$;

(b) ***if (X, μ) is a Radon measure space, $\mathfrak{A}^- \equiv \Sigma_\mu^*$.***

Proof. (a) Choose any function $f : \mathfrak{A}^- \to \Sigma_\mu^*$ such that a = f(a)' forevery $a \in \mathfrak{A}^-$. Then f is a Tukey function, so $\mathfrak{A}^- \leq \Sigma_\mu^*$.

(c) For $E \in \Sigma_\mu^*$ choose an open set $G_E \supseteq E$ such that $G_E \in \Sigma_\mu^*$. Write f(E) = $G_E^\cdot \in \in \mathfrak{A}^-$. If f(E) $\subseteq a \in \mathfrak{A}^-$ then $E \subseteq H_a = \bigcup\{H : H \subseteq X$ is open, $H' \subseteq a\}$. But $H_a^\cdot \subseteq a$ so $H_a \in \Sigma_\mu^*$. Thus f is a Tukey function and $\Sigma_\mu^* \leq \mathfrak{A}^-$.

Theorem **2J.** ***Suppose that $\kappa \geq w$ and that*** $\mathrm{bu}([\kappa]^{\leq\omega}) \leq$ add $(\mathcal{N})$ ***(see IK-IM). Then***

(a) $\ell^1(\kappa) \equiv_\omega [\kappa]^{\leq\omega} \times \ell^1(\mathrm{N})$;

(b) *for any atomless space (X, μ) of Maharam type κ, $L^0(\mu) \equiv_\omega \ell^1(\kappa)$;*

(c) for any atomless Radon probability space (X, μ) of Maharam type κ, $\mathcal{N}_\mu \equiv [\kappa]^{\leq\omega} \times \mathcal{N}$.

Proof. (a) We know already that $[n]^{\leq\omega} \times \ell^1(\mathrm{N}) \leq \ell^1(\kappa)$ (2C) and that $\ell^1(\mathrm{N}) \equiv_\omega \mathcal{N}$ (2G) so that $\mathrm{add}_\omega(\ell^1(\mathrm{N})) = \mathrm{add}_\omega(\mathcal{N}) = \mathrm{add}(\mathcal{N})$ (1Jb).

If $a \subseteq \kappa$ and $u \in \ell^1(a)$, write u^* for the member of $\ell^1(\kappa)$ defined by setting $u^*(\xi) = u(\xi)$ for $\xi \in a$, 0 for $\xi \in \kappa \setminus a$. Now let $C \subseteq [\kappa]^{\leq\omega}$ be a cofinal set such that $\#(\{c : c \in C, c \subseteq a\}) < \mathrm{add}(\mathcal{N})$ for every $a \in [\kappa]^{\leq\omega}$; of course we may suppose that every member of C is infinite. For $c \in C$ let $\pi_c : \mathbf{N} \to c$ be a bijection. For $x \in \ell^1(\kappa)$ choose $c(x) \in C$ such that $\{\xi : x(\xi) \neq 0\} \subseteq c(x)$. Define $f : \ell^1(\kappa) \to [\kappa]^{\leq\omega} \times \ell^1(\mathbf{N})$ by writing

$$f(x) = \left(c(x), x \cdot \pi_{c(x)}\right).$$

I claim that f **is an ω-Tukey** function. For let $(a, z) \in [\kappa]^{\leq\omega} \times \ell^1(\mathbf{N})$ and set $A = \{x : f(x) \leq (a, z)\}$. Set $D = \{d : d \in C, d \subseteq a\}$ and for $d \in D$ set $A_d = \{x : x \in A, c(x) = d\}$, so that $\#(D) < \mathrm{add}(\mathcal{N})$ and $A = \bigcup_{d\in D} A_d$. If $d \in D$ and $x \in A_d$, then $x \cdot \pi_d \leq z$ so $x \leq y_d = (z \cdot \pi_d \leq z$ so $x \leq y_d = (z \cdot \pi_d^{-1})^*$. Now consider $B = \{y_d | a : d \in D\} \subseteq \ell^1(a)$. As $\#(B) \leq \#(D) < \mathrm{add}_\omega(\ell^1(\mathbf{N}))$, B is countably dominated in $\ell^1(a)$; let B_1 be a countable subset of $\ell^1(a)$ dominating B; then $\{u^* : u \in B_1\}$ is a countable subset of $\ell^1(\kappa)$ dominating A. As (a, z) is arbitrary, f is an ω-Tukey function.

Accordingly $\ell^1(\kappa) \leq_\omega [\kappa]^{\leq\omega} \times \ell^1(\mathbf{N})$ and $\ell^1(\kappa) \equiv_\omega [\kappa]^{\leq\omega} \times \ell^1(\mathbf{N})$.

(b) As in the proof of 2Cb, we can find a sequence $\langle(X_n, \mu_n)\rangle_{n\in\mathbf{N}}$ of Maharam homogeneous probability spaces such that $L^0(\mu) \cong \prod_{n\in\mathbf{N}} L^0(\mu_n) \equiv \prod_{n\in\mathbf{N}} \ell^1(\kappa_n)$ (by 2E), where κ_n is the Maharam type of (X_n, μ_n), so that $\kappa = \sup_{n\in\mathbf{N}} \kappa_n$. Now $[\kappa_n]^{\leq\omega} \leq \ell^1(\kappa_n)$ for each $n \in \mathbf{N}$, so

$$[\kappa]^{\leq\omega} \cong \prod_{n\in\mathbf{N}} [\kappa_n]^{\leq\omega} \leq \prod_{n\in\mathbf{N}} \ell^1(\kappa_n) \equiv L^0(\mu);$$

also $\ell^1(\mathbf{N}) \leq \ell^1(\kappa_0) \equiv L^0(\mu_0) \leq L^0(\mu)$. So $[\kappa]^{\leq\omega} \times \ell^1(\mathbf{N}) \leq L^0(\mu)$. By (a), it follows that $\ell^1(\kappa) \leq_\omega L^0(\mu)$. But also $L^0(\mu) \leq_{<\omega} \ell^1(\kappa)$, by 2B. So $L^0(\mu) \equiv_\omega \ell^1(\kappa)$.

(c) Putting 2D and (a) above together, we have

$$[\kappa]^{\leq\omega} \times \mathcal{N} \leq \mathcal{N}_\mu \leq \ell^1(\kappa) \equiv_\omega [\kappa]^{\leq\omega} \times \ell^1(\mathbf{N}) \equiv_\omega [\kappa]^{\leq\omega} \times \mathcal{N}.$$

So $[\kappa]^{\leq\omega} \times \mathcal{N} \equiv \mathcal{N}_\mu$.

Additivity and cofinality 2K. The original impetus for this work was an investigation of the additivity and cofinality of the partially ordered sets involved. I list some consequences of the results above in this direction.

(a) If $\kappa \geq \omega$ is a cardinal, then

$$\begin{aligned} \mathrm{add}_\omega(\mathfrak{A}_\kappa^-) = \mathrm{add}_\omega(\ell^1(\kappa)) &= \omega_1 \text{ if } \kappa > \omega, \\ &= \mathrm{add}(\mathcal{N}) \text{ if } \kappa = \omega, \end{aligned}$$

$$\mathrm{cf}(\ell^1(\kappa)) = \mathrm{cf}(\mathfrak{A}_\kappa^-) = \max\left(\mathrm{cf}(\mathcal{N}), \mathrm{cf}([\kappa]^{\leq\omega})\right).$$

For 2C tells us that

$$\mathrm{add}_\omega\left(\mathfrak{A}_\kappa^-\right) = \mathrm{add}_\omega\left(\ell^1(\kappa)\right) \le \min\left(\mathrm{add}\left([\kappa]^{\le\omega}\right), \mathrm{add}_\omega\left(\ell^1(\mathcal{N})\right)\right),$$

$$\mathrm{cf}\left(\ell^1(\kappa)\right) = \mathrm{cf}\left(\ell^1(\kappa)\right) \ge \max\left(\mathrm{cf}\left([\kappa]^{\le\omega}\right), \mathrm{cf}\left(\ell^1(\mathcal{N})\right)\right),$$

and 2G that

$$\mathrm{add}_\omega\left(\ell^1(\mathbb{N})\right) = \mathrm{add}(\mathcal{N}), \ \mathrm{cf}\left(\ell^1(\mathbb{N})\right) = \mathrm{cf}(\mathcal{N}).$$

Of course $\mathrm{add}([\kappa]^{\le\omega}) = \omega_1$ if $\kappa > \omega$, if $\kappa = \omega$. The only point remaining is to check that $\mathrm{cf}(\ell^1(\kappa)) \le \max(\mathrm{cf}(\mathcal{N}), \mathrm{cf}([\kappa]^{\le\omega}))$. But this is straightforward, because if $Q \subseteq [\kappa]^{\le\omega}$ and $C \subseteq \ell^1(\mathbb{N})$ are cofinal, then $\{x_{qz} : q \in Q, z \in C\}$ is cofinal with $\ell^1(\kappa)$, where for each $q \in Q$ we choose an injection $\pi_q : q \to \mathbb{N}$ and set $x_{qz}(\xi) = z(\pi_q(\xi))$ if $\xi \in q$, 0 if $\xi \in \kappa \setminus q$; so that $\mathrm{cf}(\ell^1(\kappa)) \le \max(\#(Q), \#(C))$.

(b) If (X, μ) is a probability space of Maharam type $\kappa \ge \omega$, then

$$\begin{aligned}\mathrm{add}_\omega\left(L^1(\mu)\right) = \mathrm{add}_\omega\left(L^0(\mu)\right) &= \omega_1 \text{ if } \kappa > \omega,\\ &= \mathrm{add}(\mathcal{N}) \text{ if } \kappa = \omega,\end{aligned}$$

$$\mathrm{cf}\left(L^1(\mu)\right) = \mathrm{cf}\left(L^0(\mu)\right) = \max\left(\mathrm{cf}(\mathcal{N}), \mathrm{cf}\left([\kappa]^{\le\omega}\right)\right).$$

For 2C tells us that $\mathrm{add}_\omega(L^1(\mu)) = \mathrm{add}_\omega(\ell^1(\kappa))$ and that $\mathrm{cf}(L^1(\mu)) = \mathrm{cf}(\ell^1(\kappa))$. As for L^0, we saw in part (b) of the proof of 21 that $L^0(\mu) \equiv \prod_{n\in\mathbb{N}} \ell^1(\kappa_n)$ for some sequence $\langle\kappa_n\rangle_{n\in\mathbb{N}}$ of infinite cardinals with supremum κ. So

$$\omega_1 \le \mathrm{add}_\omega\left(L^0(\mu)\right) \le \min_{n\in\mathbb{N}} \mathrm{add}_\omega\left(\ell^1(\kappa_n)\right) = \omega_1$$

if $\kappa > \omega$; while if $\kappa = \omega$ then $\mathrm{add}_\omega(L^0(\mu)) = \mathrm{add}_\omega(\ell^1(\kappa))$ by 2E. Finally, $\mathrm{cf}(L^0(\mu)) \le$ $\le \mathrm{cf}(\ell^1(\kappa))$ by 2B, while also

$$\mathrm{cf}(\mathcal{N}) \le \mathrm{cf}\left(\ell^1(\kappa_0)\right) \le \mathrm{cf}\left(L^0(\mu)\right),$$

$$[\kappa]^{\le\omega} \equiv \prod_{n\in\mathbb{N}} [\kappa_n]^{\le\omega} \le \prod_{n\in\mathbb{N}} \ell^1(\kappa_n) \equiv L^0(\mu)$$

so $\mathrm{cf}([\kappa]^{\le\omega}) \le \mathrm{cf}(L^0(\mu))$. Putting these facts together with those of (a), we have fixed $\mathrm{add}_\omega(L^0(\mu))$ and $\mathrm{cf}(L^0(\mu))$.

(c) If (X, μ) is an atomless Radon probability space of Maharam tupe κ, then

$$\begin{aligned}\mathrm{add}\left(\mathcal{N}_\mu\right) &= \omega_1 \text{ if } \kappa > \omega,\\ &= \mathrm{add}(\mathcal{N}) \text{ if } \kappa = \omega,\end{aligned}$$

$$\mathrm{cf}\left(\mathcal{N}_\mu\right) = \max\left(\mathrm{cf}\left([\kappa]^{\le\omega}\right), \mathrm{cf}(\mathcal{N})\right);$$

this is immediate from 2D and (a) above (this is the main result of [9]).

The Tukey classification of $\mathcal{N}$, $[\kappa]^{\leq\omega}$ 2L. In the work above 1 have attempted to describe various partially ordered sets in terms of the basic sets $[\kappa]^{\leq\omega}$, $\mathcal{N}$. Neither of these is quite straightforward. For instance, when is $[\kappa]^{\leq\omega} \equiv [X]^{\leq\omega}$? For $\omega_1 \leq \kappa \leq \omega_\omega$, we have $\mathrm{cf}([\kappa]^{\leq\omega}) = \kappa$, so all the $[\kappa_n]^{\leq\omega}$ are distinct (see [13], 4.5). But if, for instance, $bu([\omega_\omega]^{\leq\omega}) = \omega_1$ (see 1Lc), then $[\kappa]^{\leq\omega} \equiv [\omega_\omega]^{\leq\omega}$ wheneverw, $\leq \kappa \leq \mathrm{cf}([\omega_\omega]^{\leq\omega})$, while $\mathrm{cf}([\omega_\omega]^{\leq\omega}) > \omega_\omega$ (for if $\langle c_\xi\rangle_{\xi<\kappa}$ is a family in $[\omega_\omega]^{\leq\omega}$ such that no uncountable subfamily is bounded above in $[\omega_\omega]^{\leq\omega}$, then $a \mapsto \bigcup_{\xi\in a} c_\xi : [n]^{\leq\omega} \rightarrow [\omega_\omega]^{\leq\omega}$ is a Tukey function. See [13], 4.9; also [26], pp. 713-4).

As for $\mathcal{N}$, we see that if $\mathrm{add}(\mathcal{N}) = \mathrm{cf}(\mathcal{N}) = \kappa$ (e.g. because add $(\mathcal{N}) = \mathfrak{C}$, as under Martin's axiom, or $\mathrm{cf}(\mathcal{N}) = \omega_1$), then $\mathcal{N} \equiv \kappa$. On the other hand, in a random-real mode1 of set theory, we can expect a family $\langle t_\xi\rangle_{\xi<\mathfrak{C}}$ in $[0, 1]$ such that no negligible set contains more than countably many t_ξ ([13], 3.18), while $\mathfrak{C}$ can be large. In this case, $\mathcal{N} \equiv [\mathfrak{C}]^{\leq\omega}$.

On the structure of $\ell^1(\kappa), \mathfrak{A}_\kappa^-, L^1(\mu), L^0(\mu)$ 2M. (a) From 2C we see that if $\kappa \geq$ w there is an upwards-directed partially ordered set (viz. $\ell^1(\kappa)$) such that $\mathfrak{A}_\kappa^- \equiv_{<\omega} \ell^1$ (n); which implies, for instance, that $(\mathfrak{A}_\kappa^-)^2 \equiv_{<\omega} \mathfrak{A}_\kappa^-$. As it happens, it is easy to show that $(\mathfrak{A}_\kappa^-)^n \equiv \mathfrak{A}_\kappa^-$ for every n ≥ 1. This is not to be taken for granted. Suppose, for instance, that T is a Souslin tree. Then $(\omega_1$, w, $<\omega_1)$ is a triple precaliber upwards for T (the point is that any uncountable subset of T, with its induced ordering, is again a Souslin tree, so has elements of infinite rank). Consequently, any partially ordered set P such that $P \leq_{<\omega} T$ must be upwards-ccc (for if $A \subseteq P$ is an up-antichain, no infinite subset of A can be finitely dominated in P, and the image of A under a $<$ w-Tukey function must be countable). In particular, $T^2 \not\leq_{<\omega} T$, and there is no upwards-directed P such that $T \equiv_{<\omega} P$.

(b) Similarly, from 2F we see that for every $\kappa \geq$ w there is a μ such that ℓ^1 (n) $\not\leq\not\leq$ w $\mathcal{N}_\mu$. Here it is easy to see that ℓ^1 (n) $\equiv_\omega \mathcal{I}$ for some σ-ideal of sets 9. For try $\mathcal{I}$ the ideal of countably-dominated subsets of $\ell^1(\kappa)$. If $\langle x_n\rangle_{n\in\mathbf{N}}$ is any sequence in ℓ^1 (n), there is a sequence $\langle \varepsilon_n\rangle_{n\in\mathbf{N}}$ of strictly positive real numbers such that $\sum_{n\in\mathbf{N}} \varepsilon_n x_n^+$ is defined in ℓ^1 (n). It follows that if $A \in \mathcal{I}$ then there is a $z_A \in \ell^1(\kappa)^+$ such that A is dominated by $\{kz_A : k \in \mathrm{N}\}$. Now x $\mapsto$ {x} : $\ell^1(\kappa) \rightarrow \mathcal{I}$ and A *t-i* $z_A : \mathcal{I} \rightarrow \ell^1(\kappa)$ are w -Tukey functions, so that $\mathcal{I} \equiv_\omega \ell^1(\kappa)$. If $bu([\kappa]^{\leq\omega}) \leq \mathrm{add}(\mathcal{N})$, then $\mathcal{I} \equiv [\kappa]^{\leq\omega} \times \mathcal{N}$.

(c) In the case of ℓ^1 (N) there is another relatively familiar space involved. Write $s_0 =$ $= \{x : x \in \mathbf{R}^{\mathbf{N}}, \{i : x(i) \neq 0\}$ is finite $\}$. Then it is easy to sec that the Riesz space quotient $\ell^1(\mathrm{N})/s_0 \equiv \mathcal{I}$, the ideal of countably-dominated subsets of ℓ^1 (N), so that ℓ^1 (N) $/s_0 \equiv$ $\equiv \mathcal{N}$.

(d) Similar arguments apply to L^1 and L^0, because both have the property that a countably-dominated set is dominated by the set of multiples of a fixed element. For atomless probability spaces (X, μ) we find that

$$L^1(\mu) \equiv_\omega L^1(\mu)/L^\infty(\mu) \equiv \mathcal{N}_\nu$$

if ν is a Maharam homogeneous Radon probability measure with the same Maharani type as μ, whileforany measurespace (X,μ) wehave $L^0(\mu) \equiv_\omega L^0(\mu)/L^\infty(\mu)$.

Further remarks 2N. (a) The outstanding problem left open above is: is it consistent to suppose that for some $\kappa \geq$ w and for some Radon probability space (X, μ) of Maharam type κ, $\mathcal{N}_\mu \not\equiv [\kappa]^{\leq\omega}$ x $\mathcal{N}$? The first case left open by 2J is κ = w,, and the arguments there make it clear tbat it is enough to consider X = $[0, 1]^\kappa$. It seems that we can have $bu([\kappa]^{\leq\omega}) > \omega_1$ in a context which allows ω_1 = $\mathfrak{C}$ = add($\mathcal{N}$) (1Ld); but it is not clear what happens to ℓ^1 (n) and $\mathcal{N}_\mu$ under these circumstances. Note that their additivities and cofinalities are what they ought to be (2K).

(b) Some subsidiary questions present themselves. In 2Jb, can the result be sharpened to $L^0(\mu) \equiv \ell^1(\kappa)$? This is a problem only when cf(κ) = w and (X, μ) has no Maharam homogeneous subspace of Maharam type κ (a similar question arises in 2Ja; but here it is quite easy to show that $\ell^1(\omega_1) \not\equiv [\omega_1]^{\leq\omega}$ x.@(N), because $(\omega_1, \omega, < \omega_1)$ isatripleprecaliber upwards for $\omega_1 \times \ell^1(\mathrm{N})$ but not for $\ell^1(\omega_1)$).

(c) S. Todorčević (private communication, December 1989) has given an example in ZFC of a partially ordered set P such that $P^2 \not\equiv_\omega P$.

3. FURTHER RESULTS

1 show that a wide variety of partially ordered sets arising in analysis are amenable to the methods of this paper.

3A. Notation. Apart from $\ell^1(\mathrm{N})$ and $\mathcal{N}$, which dominated § 2,1 shall be referring often to the ideal $\mathcal{F}$ of nowhere dense subsets of N^N and the ideal σ-ideal $\mathcal{M}$ of meagre subsets of N^N.

Theorem 3B. (a) $\mathcal{F} \cong \mathcal{F}^\mathrm{N}$.

(b) $\mathcal{F} \equiv_\omega \mathcal{M}$.

(c) $\mathcal{F} \leq \ell^1(\mathrm{N})$.

(d) $\mathcal{M} \leq \mathcal{N}$.

(e) $\mathrm{N}^\mathrm{N} \leq \mathcal{F}$.

Proof. Throughout this proof and the next, write Seq = $\bigcup_{n\in\mathrm{N}} N^n$, taking each member of N as the set of its predecessors, so that dom(cr) = n if cr $\in$ N ^{n}. For $\sigma \in$ Seq write $d(\sigma) = \sum_{i\in \mathrm{dom}(\sigma)}(\sigma(i) + 1)$ and $I_\sigma = \{\alpha : \sigma \subseteq \alpha \in \mathrm{N}^\mathrm{N}\}$. For $\sigma \in$ Seq, $\alpha \in \mathrm{N}^\mathrm{N}$ write $g_\sigma(\alpha) = \sigma^\frown\alpha$ (sothat $g_\sigma(\alpha)(i)$ = o(i) ifi $<$ dom (cr), $\alpha(i - \mathrm{dom}(\sigma))$ if $i \geq \mathrm{dom}(\sigma)$).

(a) The map $F \mapsto \langle g_{\langle i\rangle}^{-1}[F]\rangle_{i\in\mathbb{N}}$ (where $\langle i\rangle \in \mathbb{N}^1$ is the one-term sequence with value i) is an isomorphism from $\mathcal{F}$ to $\mathcal{F}^{\mathbb{N}}$.

(b) (i) For each $M \in \mathcal{M}$ choose a sequence $\langle F_{Mi}\rangle_{i\in\mathbb{N}}$ in $\mathcal{F}$ covering M. Then $M \mapsto$ $\mapsto \langle F_{Mi}\rangle_{i\in\mathbb{N}} : \mathcal{M} \to \mathcal{F}^{\mathbb{N}}$ is a Tukey function, so $\mathcal{M} \leq \mathcal{F}^{\mathbb{N}}$ and $\mathcal{M} \leq \mathcal{F}$, by (a).

(ii) Enumerate Seq as $\langle \sigma_n\rangle_{n\in\mathbb{N}}$. For $F \in \mathcal{F}$ choose inductively $\langle \tau(F,n)\rangle_{n\in\mathbb{N}}$, $\langle \nu(F,n)\rangle_{n\in\mathbb{N}}$ in Seq as follows. The inductive hypothesis is that

$$I_{\nu(F,j)} \cap \left(F \cup g_{\tau(F,i)}[F]\right) = \emptyset \quad \forall\ i,j < n.$$

Supposing that this is satisfied at level n, set

$$E = I_{\sigma_n} \cap \left(F \cup \bigcup_{i<n} g_{\tau(F,i)}[F]\right).$$

If $E = \emptyset$ set $\nu(F,n) = \sigma_n$, $\tau(F,n) = \emptyset$ (so that $g_{\tau(F,n)}[F] = F$). If $E \neq \emptyset$ then E is still nowhere dense, so there is an $\nu(F,n) \supseteq \sigma_n$ such that $E \cap I_{\nu(F,n)} = \emptyset$. Next, $\bigcup_{j\leq n} I_{\nu(F,j)}$ is a closed set not including I_{σ_n}, so there is a $\tau(F,n) \supseteq \sigma_n$ such that $I_{\tau(F,n)} \cap$ $\cap \bigcup_{j\leq n} I_{\nu(F,j)} = \emptyset$. Evidently this construction of $\nu(F,n)$, $\tau(F,n)$ will satisfy the inductive hypothesis at the next level.

On completing the induction, set

$$f(F) = \overline{F \cup \bigcup_{i\in\mathbb{N}} g_{\tau(F,i)}[F]}.$$

Then $f(F) \cap I_{\nu(F,n)} = \emptyset$ for every $n \in \mathbb{N}$, so $f(F) \in \mathcal{F} \subseteq \mathcal{M}$. Also, if $\sigma \in$ Seq is such that $f(F) \cap I_\sigma \neq \emptyset$, then there is a $\tau \supseteq \sigma$ such that $g_\tau[F] \subseteq f(F)$; for take $n \in \mathbb{N}$ such that $\sigma = \sigma_n$; then $\nu(F,n)$ cannot be σ_n, so $\tau(F,n) \supseteq \sigma$ and $g_{\tau(F,n)}[F] \subseteq f(F)$.

This defines $f : \mathcal{F} \to \mathcal{M}$. Now take $M \in \mathcal{M}$ and set

$$\mathcal{A} = \{F : F \in \mathcal{F},\ f(F) \subseteq M\},$$

$$\mathcal{B} = \{g_\tau^{-1}[E_k] : \tau \in Seq,\ k \in \mathbb{N}\} \in [\mathcal{F}]^{\leq\omega}$$

where $\langle E_k\rangle_{k\in\mathbb{N}}$ is a sequence of closed nowhere dense sets covering M. I claim that $\mathcal{B}$ dominates $\mathcal{A}$. For if $F \in \mathcal{A} \setminus \{\emptyset\}$ then by Baire's theorem there are $\sigma \in$ Seq, $k \in \mathbb{N}$ such that $\emptyset \neq f(F) \cap I_\sigma \subseteq E_k$, and now there is a $\tau \supseteq \sigma$ such that $g_\tau[F] \subseteq f(F)$ and $F \subseteq g_\tau^{-1}[E_k] \in \mathcal{B}$.

So f is an w-Tukey function, and $\mathscr{F} \leq_\omega \mathscr{M}$ and $\mathscr{F} \equiv_\omega \mathscr{M}$.

(c) (i) Note first that if X is any non-empty second-countable topological space, then for each $n \in \mathbb{N}$ there is a countable family $\mathscr{H}_n$ of open subsets of X such that

(α) $\bigcap \mathscr{H} \neq \emptyset$ whenever $\mathscr{H} \subseteq \mathscr{H}_n$, $\#(\mathscr{H}) \leq n$,

(β) if $F \subseteq X$ is nowhere dense then there is an $H \in \mathscr{H}_n$ such that $F \cap H = \emptyset$.

To see this, induce on n. For $n = 0$ take $\mathscr{H}_0 = \{\emptyset\}$. For the inductive step, let $\langle H_i \rangle_{i \in \mathbb{N}}$ be a sequence running over $\mathscr{H}_n$, and let $\mathscr{U}$ be a countable base for the topology of X which is closed under finite unions. Set

$$\mathscr{H}_{n+1} = \Big\{ U \cup H_i : i \in \mathbb{N}, U \in \mathscr{U}, U \cap \bigcap_{j \in I} H_j \neq \emptyset$$

$$\text{whenever } I \subseteq \{0, \dots, i\} \text{ and } \bigcap_{j \in I} H_j \neq \emptyset \Big\}.$$

Then $\mathscr{H}_{n+1}$ is a countable family of open sets satisfying (β). If $\mathscr{H} \subseteq \mathscr{H}_{n+1}$ and $0 < \#(\mathscr{H}) \leq n+1$, express $\mathscr{H}$ as $\{U_j \cup H_{i(j)} : j \leq n\}$ where each $U_j \cup H_{i(j)}$ is as described in the formula for $\mathscr{H}_{n+1}$. Suppose these are arranged so that $i(j) \leq i(n)$ for every $j \leq n$. By the inductive hypothesis, $H = \bigcap_{j<n} H_{i(j)} \neq \emptyset$; now $U_n \cap H \neq \emptyset$, so $\bigcap \mathscr{H} \supseteq U_n \cap H \neq \emptyset$.

(ii) Now let $\langle U_n \rangle_{n \in \mathbb{N}}$ enumerate a countable base for the topology of $\mathbb{N}^{\mathbb{N}}$ consisting of non-empty sets, and for each $n \in \mathbb{N}$ choose a countable family $\mathscr{H}_n$ of open subsests of U_n such that

(α) $\bigcap \mathscr{H} \neq \emptyset$ whenever $\mathscr{H} \subseteq \mathscr{H}_n$, $\#(\mathscr{H}) \leq 2^n$,

(β) if $F \subseteq \mathbb{N}^{\mathbb{N}}$ is nowhere dense then there is an $H \in \mathscr{H}_n$ such that $F \cap H = \emptyset$.

Write $K = \bigcup_{n \in \mathbb{N}} \{n\} \times \mathscr{H}_n$ and choose any function $f : \mathscr{F} \to \ell^1(K)$ such that for every $F \in \mathscr{F}$, $n \in \mathbb{N}$ there is an $H \in \mathscr{H}_n$ such that $F \cap H = \emptyset$ and $f(F)(n, H) \geq 2^{-n}$.

I claim that f is a Tukey function. For take any $x \in \ell^1(K)$. Let $m \in \mathbb{N}$ such that

$$\sum_{H \in \mathscr{H}_n} |x(n, H)| \leq 1 \quad \forall\, n \geq m.$$

Then $\mathscr{H}'_n = \{H : H \in \mathscr{H}_n, x(n, H) \geq 2^{-n}\}$ has cardinal at most 2^n and $G_n = \bigcap \mathscr{H}'_n$ is a non-empty open subset of U_n for each $n \geq m$, so that $G = \bigcup_{n>m} G_n$ is a dense open subset of $\mathbb{N}^{\mathbb{N}}$. If $F \in \mathscr{F}$ and $f(F) \leq x$, then for every $n \geq m$ we have an $H \in \mathscr{H}_n$ such that $f(F)(n, H) \geq 2^{-n}$ and $F \cap H = \emptyset$; in this case, $H \in \mathscr{H}'_n$ so $G_n \subseteq H$ and

$F \cap G_n = \emptyset$; as n is arbitrary, $F \cap G = \emptyset$. Thus $\mathbf{N}^{\mathbf{N}} \setminus G$ is an upper bound in $\mathscr{F}$ for $\{F : f(F) \le x\}$. As x is arbitrai-y, f is a Tukey function.

(iii) Accordingly $\mathscr{F} \le \ell^1(K) \le \ell^1(\mathbf{N})$.

(d) Now $\mathscr{M} \equiv_\omega \mathscr{F} \le \ell^1(\mathbf{N}) \equiv_\omega \mathscr{N}$ so $\mathscr{M} \le_\omega \mathscr{N}$ and (because $\mathscr{M}$ is a σ-ideal) $\mathscr{M} \le \mathscr{N}$.

(e) $\mathbf{N} \le \mathscr{F}$ because add$(\mathscr{F})$ = w (if $\langle \alpha_n \rangle_{n \in \mathbf{N}}$ enumerates any dense subset of $\mathbf{N}^{\mathbf{N}}$, then $n \mapsto \{\alpha_i : i \le n\}$ is a Tukey function from N to $\mathscr{F}$). So $\mathbf{N}^{\mathbf{N}} \le \mathscr{F}^{\mathbf{N}} \cong \mathscr{F}$.

Remarks. The main result of [1] is that add$(\mathscr{N}) \le$ add$(\mathscr{M})$. [22] shows in addition that cf$(\mathscr{M}) \le$ cf$(\mathscr{N})$. The result that $\mathscr{M} \le \mathscr{N}$ is explicit in [IO]; it depended on an idea of J. Pawlikowski. [20] showed that add$_\omega(\mathbf{N}^{\mathbf{N}}) \ge$ add$(\mathscr{M})$; the result that cf$(\mathbf{N}^{\mathbf{N}}) \le$ cf$(\mathscr{M})$ seems to be folklore (I leamt it from J. Cichón). Note that add, $(\mathbf{N}^{\mathbf{N}})$ and cf $(\mathbf{N}^{\mathbf{N}})$ are called $\mathfrak{b}$ and $\mathfrak{d}$ in [10] and [6]. An w-Tukey function from $\mathbf{N}^{\mathbf{N}}$ to $\mathscr{M}$ is constructed in [10]; see also [2], 2.2. The fact that $\mathscr{F} \equiv \mathscr{F}^{\mathbf{N}}$ is mentioned in [14]. The argument in part (c) of the proof above is taken from [2a], 1.2.3-4.

Proposition ***3C. Let X be a set and*** $\mathscr{V}$ ***a counrable family of subsets of X. Write***

$$\mathscr{D} = \{D : D \subseteq X, \ \forall V \in \mathscr{V} \exists W \in \mathscr{V} \text{ such that } W \subseteq V \setminus D\}.$$

Then $\mathscr{D} \le \mathscr{F}$.

Proof. (For notation see the proof of 3B). If $\mathscr{V} = \emptyset$ then $\mathscr{D} = \mathscr{P}X$ and the result is trivial. So suppose that $\mathscr{V} \ne 0$. Construct a function h : Seq $\to \mathscr{V} \cup \{X\}$ such that $h(\emptyset) = X$ and

$$\{h(\tau) : \sigma \subseteq \tau \in \mathbf{N}^{n+1}\} = \{V : h(\sigma) \supseteq V \in \mathscr{V}\}$$

whenever $n \in \mathbf{N}$, $\sigma \in \mathbf{N}^n$.

For $D \in \mathscr{D}$ choose a family $\langle \tau(D, \sigma) \rangle_{\sigma \in \text{Seq}}$ in Seq such that $\tau(D,\sigma) \supseteq \sigma$, dom$(\tau(D, \sigma)) \ge$ d(a) and $D \cap h(\tau(D,\sigma)) = \emptyset$ for every $\sigma \in$ Seq; this is possible because

$$\{h(\tau) : \sigma \subseteq \tau \in \mathbf{N}^k\} = \{V : V \in \mathscr{V}, V \in h(\tau)\}$$

whenever $\sigma \in$ Seq and $k >$ dom (a). Set

$$f(D) = \mathbf{N}^{\mathbf{N}} \setminus \bigcup_{\sigma \in \text{Seq}} I_{\tau(D,\sigma)};$$

then $f(D) \in \mathscr{F}$ because $\tau(D,\sigma) \supseteq \sigma$ for every $\sigma \in$ Seq.

If $F \in \mathscr{F}$ set

$$D_0 = \bigcup\{D : D \in \mathscr{D}, f(D) \subseteq F\}.$$

If $V \in \mathscr{V}$ take $\sigma \in \mathbb{N}^1$ such that $h(\sigma) = V$. Take $v \supseteq \sigma$ such that $I_v \cap F = \emptyset$. If $D \in \mathscr{D}$ and $f(D) \subseteq F$ then $I_v \subseteq \bigcup_{\sigma \in \mathrm{Seq}} I_{\tau(D,\tau)}$; it follows that there is a $\sigma \in$ Seq such that $\tau(D, \sigma) \subseteq$ u. (Consider any $\alpha \in I_v$ such that $\alpha(i) \neq \tau(D,$ a)(i) whenever $i \geq$ dom(u) and $\sigma \in$ Seq and $\mathrm{dom}(\tau(D,\sigma)) = i + 1$; such exists because $\mathrm{dom}(\tau(D,\sigma)) \geq$ d(a) for every $\sigma \in$ Seq. Now if $\alpha \in I_{\tau(D,\sigma)}$ this implies that $\mathrm{dom}(\tau(D,\sigma)) \leq \mathrm{dom}(v)$ and $\tau(D, \sigma) \subseteq v$.) In this case h(u) $\subseteq h(\tau(D,$ a)) $\subseteq X \setminus D$. As D isarbitrary, $h(v) \cap D_0 = 0$ and $h(v) \subseteq V \setminus D_0$. As V is arbitrary, $D_0 \in \mathscr{D}$. As F is arbitrary, $f : \mathscr{D} \to \mathscr{F}$ is a Tukey function and $\mathscr{D} \leq \mathscr{F}$, as required.

Corollary 3D. ***Let X be a topologica1 space with a countable n-base. Write $\mathscr{F}_X$ for the idea1 of nowhere dense subsets of X and $\mathscr{M}_X$ for the σ-ideal of meagre subsets of X. Then***

(a) $\mathscr{F}_X \leq \mathscr{F}$;

(b) $\mathscr{M}_X \leq \mathscr{M}$;

(c) $\mathrm{add}_\omega(\mathscr{F}_X) \geq \mathrm{add}(\mathscr{M})$.

Proof. (a) Let $\mathscr{V}$ be a countable n-base for the topology of X . Then $\mathscr{F}_X$ is precisely

$$\{F : F \subseteq X, \forall V \in \mathscr{V} \exists W \in \mathscr{V} \text{ suchthat } W \subseteq V \setminus F\}.$$

So $\mathscr{F}_X \leq \mathscr{F}$ by 3C.

(b) Now $\mathscr{M}_X \leq \mathscr{F}_X{}^{\mathbb{N}}$ (as in part (b-i) of the proof of 3B) so $\mathscr{M}_X \leq \mathscr{F}^{\mathbb{N}} \cong \mathscr{F} \equiv_\omega \mathscr{M}$ and $\mathscr{M}_X \leq \mathscr{M}$.

(c) Because $\mathscr{F} \equiv_\omega \mathscr{M}$ we have

$$\text{add, } (\mathscr{F}_X) \geq \mathrm{add}_\omega(\mathscr{F}) = \mathrm{add}_\omega(\mathscr{M}) = \mathrm{add}(\mathscr{M}).$$

Remark. In the language of [8], $\mathscr{F}_X$ has the «($<$ add $(\mathscr{M}), \omega)$ -covering property». Thus Theorem 22B of [8] can be deduced in ZFC from the theorem of [19] that $\mathrm{add}(\mathscr{M}) \geq \mathfrak{p}$.

Corollary 3E. ***Let X be a second-countable topologica1 space and μ a a-finite Bore1 measure on X. Let $\mathscr{E}_\mu$ be the ideal $\{E : E \subseteq X, \mu(\overline{E}) = \emptyset\}$. Then $\mathscr{E}_\mu \leq \mathscr{F}$.***

Proof. (a) If $\mu($ X) = 0 this is trivial. Otherwise (because μ is o-finite) there is a probability measure ν with the same measurable sets and the same null sets as μ, so that $\mathscr{E}_\nu = \mathscr{E}_\mu$. Let $\mathscr{U}$ be a countable base for the topology of X, containing X and 0, and closed under finite unions. For $k \in \mathbb{N}$, let $\mathscr{V}_k$ be the countable set

$$\{V : V \in \mathscr{V}, \mu(V) > 1 - 2^{-k}\},$$

and write

$$\mathscr{D}_k = \{D : D \subseteq X, \forall V \in \mathscr{V}_k \exists W \in \mathscr{V}_k \text{ such that } W \subseteq V \setminus D\},$$

so that $\mathscr{D}_k \leq \mathscr{F}$, by 3C.

(b) The point of this is that $\mathscr{E}_\nu = \bigcap_{k \in \mathbb{N}} \mathscr{D}_k$. To see this, argue as follows. (i) If $E \in \mathscr{E}_\nu$, $k \in \mathbb{N}$ and $V \in \mathscr{V}_k$ then

$$\nu(V \setminus \overline{E}) = \nu(V) > 1 - 2^{-k}.$$

Because $\mathscr{U}$ is a base for the topology of X and is closed under finite unions, there is an increasing sequence $\langle U_i \rangle_{i \in \mathbb{N}}$ in $\mathscr{U}$ with union $V \setminus \overline{E}$. Now there is an $i \in \mathbb{N}$ such that $\nu(U_i) > 1 - 2^{-k}$, in which case $U_i \in \mathscr{V}_k$ and $U_i \subseteq V \setminus E$. As V is arbitrary, $E \in \mathscr{D}_k$; as k and E are arbitrary, $\mathscr{E}_\nu \subseteq \bigcap_{k \in \mathbb{N}} \mathscr{D}_k$. (ii) If $E \in \bigcap_{k \in \mathbb{N}} \mathscr{D}_k$, take any $k \in \mathbb{N}$. Then $X \in \mathscr{V}_k$ and $E \in \mathscr{D}_k$, so there is a $W \in \mathscr{V}_k$ such that $W \subseteq X \setminus E$. As W is open, $W \cap \overline{E} = \emptyset$ and $\nu(\overline{E}) \leq \nu(X \setminus W) \leq 2^{-k}$. As k is arbitrary, $\nu(\overline{E}) = 0$ and $E \in \mathscr{E}_\nu$.

(c) Consequently the map $E \mapsto \langle E \rangle_{k \in \mathbb{N}}$ is a Tukey function from $\mathscr{E}_\nu$ to $\prod_{k \in \mathbb{N}} \mathscr{D}_k$ and

$$\mathscr{E}_\mu = \mathscr{E}_\nu \leq \prod_{k \in \mathbb{N}} \mathscr{D}_k \leq \mathscr{F}^{\mathbb{N}} \cong \mathscr{F}.$$

Remark. Compare [8], 22G, where it is shown that $\operatorname{add}_\omega(\mathscr{E}_\mu) \geq \mathfrak{p}$; also Theorem 2.1 of [20], and 3K below.

3F. I now give three results on «cross-ideals», mixing measure and category, in $[0,1]^2$.

Theorem. *Let $\mathscr{I}$ be the ideal of subsets of $[0,1]^2$ generated by the Borel sets $E \subseteq [0,1]^2$ such that*

$$\{t : t \in [0,1], E[\{t\}] \notin \mathscr{M}_I\} \in \mathscr{N},$$

where $\mathscr{M}_I$ is the ideal of meagre subsets of $[0,1]$ and $E[\{t\}] = \{u : (t,u) \in E\}$. Then $\mathscr{I} \equiv \mathscr{N}$.

Proof. (a) The map $H \mapsto H \times [0,1] : \mathscr{N} \to \mathscr{I}$ is a Tukey function, so $\mathscr{N} \leq \mathscr{I}$.

(b) For the reverse inequality, we need to know some facts about the structure of $\mathscr{I}$ which may be of independent interest. Let $\mathscr{D}$ be the ideal of subsets D of $[0,1]^2$ such that $\overline{D}$ has nowhere dense vertical sections, $\mathscr{I}_0$ the σ-ideal of subsets of $[0,1]^2$ generated by $\mathscr{D}$, and $\mathscr{I}_1$ the ideal of subsets of $[0,1]^2$ generated by $\mathscr{I}_0 \cup \{H \times [0,1] : H \in \mathscr{N}\}$. Let Σ be the family of those sets $E \subset [0,1]^2$ for which there is some Borel set $G \subseteq [0,1]^2$ with open vertical sections such that $G \triangle E \in \mathscr{I}_1$.

(c) Let F be a Borel subset of $[0,1]^2$ with closed vertical sections. Then $F \in \sum$. To see this, let $\langle U_n \rangle_{n \in \mathrm{N}}$ enumerate a base for the topology of $[0,1]$. For each $n \in \mathrm{N}$, set

$$B_n = \{t : t \in [0,1], U_n \cap F[\{t\}] = \emptyset\},$$
$$C_n = \{t : t \in [0,1], U_n \subseteq F[\{t\}]\}.$$

Then B_n and C_n are coanalytic, therefore Lebesgue measurable ([15], 2.9.2). For each $k \in \mathrm{N}$, there is a closed set H_k c $[0,1]$ such that $H_k \setminus B$,, $H_k \setminus C_n$ *are* closed for every $n \in \mathrm{N}$ andthemeasureof H_k isatleast $1 - 2^{-k}$. Set

$$G = \bigcup_{k,n \in \mathrm{N}} (C_n \cap H_k) \times U_n,$$

so that G is a Borel set with open vertical sections and $G \subseteq F$. 1 wish to show that $F \setminus G \in \in \mathscr{I}_1$. For $k \in N$ set

$$F_k = \overline{(H_k \times [0,1]) \cap F \setminus G}.$$

Then $F \setminus G \subseteq (H \times [0,1]) \cup \bigcup_{k,n \in \mathrm{N}} F_k$, where $H = [0,1] \setminus \bigcup_{k,n \in \mathrm{N}} H_k \in \mathscr{N}$, so it will be enough if 1 can prove that every $F_k \in \mathscr{D}$. First note that $F_k \subseteq F$. For if $(t,u) \in [0,1]^2 \setminus F$ there is an $m \in \mathrm{N}$ such that $u \in U_m$ and $U_m \cap F[\{t\}] = \emptyset$ i.e. $t \in B_m$; now

$$([0,1] \setminus (H_k \setminus B_m)) \times U_m$$

is a neighbourhood of (t,u) not meeting $(H_k \times [0,1]) \cap F$, so $(t,u) \notin F_k$. Now examine $F_k[\{v\}]$ for $v \in [0,1]$. 1 need to show that this is nowhere dense. If it were not, there would be an $n \in \mathrm{N}$ such that $\emptyset \neq U_n \subseteq F_k[\{v\}]$. In this case $U_n \subseteq F[\{v\}]$ so $v \in C_n$. But now

$$([0,1] \setminus (H_k \setminus C_n)) \times U_n$$

is an open set not meeting $(H_k \times [0,1]) \setminus G$, so cannot meet F_k, and $(\{v\} \times U_n) \cap F_k = \emptyset$, which is impossible.

Thus every F_k belongs to $\mathscr{D}$ and $F \triangle G = F \setminus G \in \mathscr{I}_1$ and $F \in \sum$, as required.

(d) It follows al once (because $\mathscr{I}_1$ is an ideal) that the complement in $[0,1]^2$ of any member of $\sum$ belong to $\sum$; because $\mathscr{I}_1$ is a cr-ideal, $\sum$ is closed under countable unions, so is a a-algebra of sets. Open subsets of $[0,1]^2$ bclongs to $\sum$, so every Bore1 subset of $[0,1]^2$ mustbelongto $\sum$.

Consequently $\mathscr{I}_1$ is actually equal to 9. For evidently $\mathscr{I}_1 \subseteq \mathscr{I}$. On the other hand, if $A \in \mathscr{I}$, there is a Borel set $E \subseteq [0,1]^2$ such that $A \subseteq E$ and $\{t : E[\{t\}] \notin \mathscr{M}_I\} \in \mathscr{N}$.

Now $E \in \Sigma$; let $G \subseteq [0,1]^2$ be a Borel set with open **vertical** sections such that $G \triangle E \in \mathscr{I}_1$. **In this case** $G \in \mathscr{I}$. **But this must be because** $\{t : G[\{t\}] \neq \emptyset\} \in \mathscr{N}$. So $G \in \mathscr{I}_1$ and $E \in Ir_1$ and $A \in \mathscr{I}_1$.

(e) Now we can find a function

$$A \mapsto (H_A, \langle D_{An}\rangle_{n\in\mathbb{N}}) : \mathscr{I} \to \mathscr{N} \times \mathscr{D}^{\mathbb{N}}$$

such that $A \subseteq (H_A \times [0,1]) \cup \bigcup_{n\in\mathbb{N}} D_{An}$ for every $A \in \mathscr{I} = \mathscr{I}_1$. Clearly this is a Tukey function. So

$$\mathscr{I} \leq \mathscr{N} \times \mathscr{D}^{\mathbb{N}}.$$

(f) But also $\mathscr{D} \leq \mathscr{F}$. To see this, let $\mathscr{U}$ be a countable base for the topology of $[0,1]^2$, closed under finite unions, and let $\mathscr{V}$ be

$$\{V : V \in \mathscr{U},\ \pi_1[V] = [0,1]\},$$

where $\pi_1 : [0,1]^2 \to [0,1]$ is the first-coordinate map. If $D \in \mathscr{D}$ and $V \in \mathscr{V}$ then $\pi_1[V \setminus D] = [0,1]$; because $[0,1]$ is compact, there is a $W \in \mathscr{V}$ such that $W \subseteq V \setminus D$. It follows easily that

$$\mathscr{D} = \{D : D \subseteq [0,1]^2,\ \forall\, V \in \mathscr{V},\ \exists W \in \mathscr{V} \text{ such that } W \subseteq V \setminus D\},$$

so that $\mathscr{D} \leq \mathscr{F}$, by 3C.

(g) Now $\mathscr{D}^{\mathbb{N}} \leq \mathscr{F}^{\mathbb{N}} \cong \mathscr{F}$ and

$$\mathscr{I} \leq \mathscr{N} \times \mathscr{F} \leq_\omega \mathscr{N} \times \mathscr{N} \cong \mathscr{N}.$$

As $\mathscr{I}$ is a σ-ideal, $\mathscr{I} \leq \mathscr{N}$ and $\mathscr{I} \equiv \mathscr{N}$. This completes the proof.

Remark. Some of the arguments above were worked out in the course of correspondence with J. Cichoń.

Proposition **3G.** ***Let $\mathscr{B}$ be the σ-algebra of Borel subsets of $[0,1]^2$ and $\mathscr{I}$ the ideal described in 3F. Write $\mathfrak{A}$ for the quotient algebra $\mathscr{B}/\mathscr{B} \cap \mathscr{I}$. Then $\mathfrak{A}^- \equiv \mathfrak{A}_\omega^-$.***

Proof. (a) Let $\mathfrak{B}$ be the measure algebra of Lebesgue measure on $[0,1]$, so that $\mathfrak{B} \cong \mathfrak{A}_\omega$. We have an order-continuous embedding of $\mathfrak{B}$ in $\mathfrak{A}$ given by sending $H^\cdot$ to $(H \times [0,1])^\cdot$ for any Borel set $H \subseteq [0,1]$. It is easy to check that this induces a Tukey function from $\mathfrak{B}^-$ to $\mathfrak{A}^-$, so that $\mathfrak{B}^- \leq \mathfrak{A}^-$.

(b) Let $\langle U_n\rangle_{n\in\mathbb{N}}$ enumerate a base for the topology of $[0,1]$ with every U_n non-empty. Let $\langle b_n\rangle_{n\in\mathbb{N}}$ be a disjoint sequence of non-zero elements of $\mathfrak{B}$ with $sup_{n\in\mathbb{N}} b_n = 1$ in $\mathfrak{B}$. For each $n \in \mathbb{N}$ let $\mathfrak{B}_n$ be the relative algebra $\mathfrak{B}|b_n$ and let $\phi_n : \mathfrak{B} \to \mathfrak{B}_n$ be a Boolean isomorphism. For each $a \in \mathfrak{A}^-$ choose $E_a \in \mathscr{B}$ such that $E_a^{\cdot} = 1 \setminus a$ in $\mathfrak{A}$. Then E_a belongs to the algebra $\sum$ described in the proof of 3F; let G_a be a Borel set in $[0,1]^2$ with open vertical sections such that $G_a \triangle E_a \in \mathscr{I}$, so that $1 \setminus a = G_a^{\cdot}$. As $G_a \notin \mathscr{I}$ there is an m(a) $\in \mathbb{N}$ such that

$$H_a = \{t : U_{m(a)} \subseteq G_a[\{t\}] \notin \mathscr{N}\}.$$

But H_a is a coanalytic set, therefore Lebesgue measurable, and $H_a^{\cdot}$ is defined in $\mathfrak{B}$. Set

$$f(a) = 1 \setminus \phi_{m(a)}(H_a^{\cdot}) \in \mathfrak{B}^-.$$

(c) I claim that f is a Tukey function. For let $b \in \mathfrak{B}^-$ and consider $A = \{a : f(a) \subseteq b\}$. There is an $n \in \mathbb{N}$ such that $b_n \setminus b \neq 0$; set $c = \phi_n^{-1}(b_n \setminus b)$, and take a Borel set $H \subseteq [0,1]$ such that $H^{\cdot} = c$ in $\mathfrak{B}$. Set $a_0 = 1 \setminus (H \times U_n)^{\cdot} \in \mathfrak{A}^-$.

If $a \in A$ then $\phi_{m(a)}(H_a^{\cdot}) \cup b = 1$ so $b \cup b_{m(a)} = 1$ and m(a) = n; also $H_a^{\cdot} = \phi_n^{-1}(b_n \setminus f(a)) \supseteq c$, so $H \setminus H_a \in \mathscr{N}$. Now $H_a \times U_n = H_a \times U_{m(a)} \subseteq G_a$ so

$$(H \times U_n)^{\cdot} \subseteq (H_a \times U_n)^{\cdot} \subseteq G_a = 1 \setminus a$$

in $\mathfrak{A}$, and

$$a \subseteq 1 \setminus a(H \times U_n)^{\cdot} = a_0$$

in $\mathfrak{A}$. This shows that a_0 is an upper bound for A. As b is arbitrary, f is a Tukey function and $\mathfrak{A}^- \leq \mathfrak{B}^-$.

(d) Thus $\mathfrak{A}^- \equiv \mathfrak{B}^- \cong \mathfrak{A}_\omega^-$.

Remark. Of course $\mathfrak{A}$ is isomorphic to $\sum/\mathscr{I}$.

Theorem 3H. ***Let*** $\mathscr{I}$ ***be the ideal of subsets of*** $[0,1]^2$ ***generated by the Borel sets*** $E \subseteq [0,1]^2$ ***such that***

$$\{t : t \in [0,1],\ E[\{t\}] \notin \mathscr{N}\}$$

is meagre in $[0,1]$. ***Then*** $\mathscr{I} \equiv [\mathfrak{C}]^{\leq\omega}$.

Proof. By the arguments of [4], Theorems 1.1 and 2.1 there is a family $\langle E_s\rangle_{s\in[0,1]}$ in $\mathscr{I}$ such that $\bigcup_{s\in A} \notin \mathscr{I}$ for any uncountable $A \subseteq [0,1]$. Now $A \mapsto \bigcup_{s\in A} E_s : [[0,1]]^{\leq\omega} \to \mathscr{I}$ is a Tukey function, so $[\mathfrak{C}]^{\leq\omega} \leq \mathscr{I}$. On the other hand, $\mathscr{I}$ is generated by the Borel sets it contains, so $\mathrm{cf}(\mathscr{I}) \leq \mathfrak{C}$ and $\mathscr{I} \leq [\mathfrak{C}]^{\leq\omega}$.

Remark. [3] give a variety of applications of their method which may readily be translated into further results of this kind.

31. Let us turn now to other ideals for which the techniques of this paper can give some information, if not a complete classification.

Theorem. *Let* (X, ρ) *be a separable metric space and* $\mathscr{S}$ *the* σ*-ideal of subsets of* X *of strong measure zero. Then* $\mathscr{S} \leq \mathscr{N}^{\mathfrak{d}}$, *where* $\mathfrak{d} = \mathrm{cf}(\mathbb{N}^{\mathbb{N}})$.

Proof. (a) If $X = \emptyset$ then the result is trivial. Otherwise, let Y be a countable dense subset of X . Fix an w-Tukey function $g : \ell^1(Y \times \mathbb{N}) \to \mathscr{N}$ (using 2G above). Let $D \subseteq \mathbb{N}^{\mathbb{N}}$ be a cofinal subset of cardinal $\mathfrak{d}$.

For each $d \in D$, $S \in \mathscr{S}$ choose a sequence $\langle t_{Sdk}\rangle_{k\in\mathbb{N}}$ in Y such that

$$S \subseteq \bigcap_{m\in\mathbb{N}} \bigcup_{n\geq m} U\left(t_{Sdk}, 2^{-d(k)}\right),$$

where $U(t,\delta) = \{u : \rho(t,u) < \delta\}$. Take $z_{Sd} \in \ell^1(Y \times \mathbb{N})$ such that $z_{Sd}(t_{Sdk}, k) = 2^{-k}$ for each $k \in \mathbb{N}$. Set

$$f(S) = \langle g(z_{Sd})\rangle_{d\in D} \in \mathscr{N}^D$$

for each $S \in \mathscr{S}$.

(b) 1 wish to show that $f : \mathscr{S} \to \mathscr{N}^D$ is a Tukey function. Take $\langle E_d\rangle_{d\in D} \in \mathscr{N}^D$ and set $\mathscr{A} = \{S : S \in \mathscr{S}, f(S) \leq \langle E_d\rangle_{d\in D}\}$, $S_0 = \bigcup\mathscr{A}$. My aim is to show that $S_0 \in \mathscr{S}$.

Let $\langle\varepsilon_i\rangle_{i\in\mathbb{N}}$ be any sequence of strictly positive real numbers. Let $d \in D$ be such that $2^{-d(k)} \leq \varepsilon_i$ whenever $k \in N$ and $i < 2^{k+1}$. Weknow that

$$A = \{x : x \in \ell^1(Y \times \mathbb{N}),\ g(x) \subseteq E_d\}$$

is countably dominated in $\ell^1(Y \times \mathbb{N})$; let $z \in \ell^1(Y \times \mathbb{N})$ be such that $\{(t,k) : x(t,k) > z(t,k)\}$ is finite for every $x \in A$. Let $n \in \mathbb{N}$ be such that

$$\#(\{t : z(t,k) \geq 2^{-k}\}) \leq 2^k \ \forall\, k \geq n.$$

Then there is a sequence $\langle u_i\rangle_{i\in\mathbb{N}}$ in Y such that if $k \geq n$ and $z(t,k) \geq 2^{-k}$ then $t = u_i$ for some i such that $2^k \leq i < 2^{k+1}$.

1 claim that $S_0 \subseteq \bigcup_{i\in\mathbb{N}} U(u_i, \varepsilon_i)$. For take any $S \in \mathscr{A}$. We have $g(z_{Sd}) \subseteq E_d$ so $z_{Sd} \in A$; let $m \geq n$ be such that $z_{Sd}(t,k) \leq z(t,k)$ whenever $t \in Y$ and $k \geq m$. Then for any $k \geq m$,

$$2^{-k} = z_{Sd}\left(t_{Sdk}, k\right) \leq z\left(t_{Sdk}, k\right),$$

so there is an $i < 2^{k+1}$ such that $t_{Sdk} = u_i$; now $2^{-d(k)} \leq \varepsilon_i$ so that $U(t_{Sdk}, 2^{-d(k)}) \subseteq U(u_i, \varepsilon_i)$. Accordingly

$$S \subseteq \bigcup_{k \geq m} U\left(t_{Sdk}, 2^{-d(k)}\right) \subseteq \bigcup_{i \in \mathbf{N}} U\left(u_i, \varepsilon_i\right).$$

As S is arbitrary, $S_0 \subseteq \bigcup_{i \in \mathbf{N}} U(u_i, \varepsilon_i)$.

Since $\langle \varepsilon_i \rangle_{i \in \mathbf{N}}$ was arbitrary, $S_0 \in \mathscr{S}$ and is an upperbound for $\mathscr{A}$ in $\mathscr{S}$. Since $\langle E_d \rangle_{d \in D}$ was arbitrary, f is a Tukey function, and $\mathscr{S} \leq \mathscr{N}^D \cong \mathscr{N}^{\mathfrak{d}}$, as rquired.

Corollary 3J. ***In the context** of 3I,* add $(\mathscr{S}) \geq$ add $(\mathscr{N})$.

Proof. $\operatorname{add}(\mathscr{N}^{\mathfrak{d}}) = \operatorname{add}(\mathscr{N})$.

Remark. These results may be regarded as descendants of T.J. Carlson's theorem that add $(\mathscr{S}) = \mathfrak{C}$ if $X = \mathbf{R}$ and Martin's axiom is true (see [8], 33B). Note that for any non-empty partially ordered sets P and Q, add$(P) \leq$ add(Q) iff there is some κ such that $Q \leq P^{\kappa}$.

Proposition 3K. ***Let*** $\mathscr{Z}$ ***be the ideal** of subsets of* **N** ***with zero asymptotic density. Then***

(a) $\mathbf{N}^{\mathbf{N}} \leq \mathscr{Z} \leq \ell^1(\mathbf{N})$;

(b) ***if** X **is a** second-countable **space and** μ is **a σ-finite** Borel **measure on** X**, then*** $\mathscr{E}_\mu \leq \mathscr{Z}$*, where* $\mathscr{E}_\mu = \{E : \mu(\overline{E}) = 0\}$*, as in 3E.*

Proof. For $a \subseteq \mathbf{N}$ write

$$d(a) = \lim_{n \to \infty} \frac{1}{n} \#(a \cap n) \text{ if this exists},$$

$$d^*(a) = \limsup_{n \to \infty} \frac{1}{n} \#(a \cap n) \text{ in any case},$$

so that $\mathscr{Z} = \{a : d(a) = 0\} = \{a : d^*(a) = 0\}$.

(a) (i) The map

$$\alpha \mapsto \{2^k i : k \in \mathbf{N},\ i \leq \alpha(k)\}$$

is a Tukey function from $\mathbf{N}^{\mathbf{N}}$ to $\mathscr{Z}$, so $\mathbf{N}^{\mathbf{N}} \leq \mathscr{Z}$.

(ii) $\mathscr{Z}$ carries a metric ρ defined by

$$\rho(a, b) = \sup_{n \geq 1} \frac{1}{n} \#((a \triangle b) \cap n) \ \forall\, a, b \in \mathscr{Z},$$

which makes $\mathscr{Z}$ a separable complete metric space in which $\cup : \mathscr{Z} \times \mathscr{Z} \to \mathscr{Z}$ is uniformly continuous. So $\mathscr{Z} \leq_{<\omega} C'(\mathbf{N})$ by 2B. As $\mathscr{Z}$ is upwards-directed, $\mathscr{Z} \leq \ell^1(\mathbf{N})$.

(b) By the arguments in 3E, it is enough to consider the case in which p(X) = 1. Let $\langle U_n\rangle_{n\in\mathrm{N}}$ run over a base for the topology $\mathfrak{T}$ of X . For each n ∈ N , let $\mathscr{E}_n$ be the finite subalgebra of $\mathscr{P}X$ generated by $\{U_i : i < n\}$, and let $\mathscr{E}$ be the countable algebra $\bigcup_{n\in\mathrm{N}}\mathscr{E}_n$. Then there is a Boolean homomorphism $\theta : \mathscr{E} \to \mathscr{P}\mathrm{N}$ such that $d(\theta(C))$ exists and is equal to p(C) for every C $\in \mathscr{E}$ (the easiest argument for this is an inductive construction for $\theta|\mathscr{E}_n$, using the fact that if $a \subseteq$ N and d(a) exists and $0 \leq \alpha \leq$ d(a) then there is a $b \subseteq a$ such that $d(b)$ exists = α). Let $\langle k_n\rangle_{n\in\mathrm{N}}$ be a strictly increasing sequence in N \ $\{0\}$ such that

$$k_n^{-1}\#\,\big(\theta(C)\cap k_n\big) \leq \mu(C) + 2^{-n}\ \forall\, C\in\mathscr{E}_n,\ n\in\mathrm{N}.$$

Define $f : \mathscr{E}_\mu \to \mathscr{P}\mathrm{N}$ by setting

$$f(F) = \mathrm{N}\setminus\bigcup\{\theta(G)\setminus k_n : n\in\mathrm{N},\ G\in\mathfrak{T}\cap\mathscr{E}_n,\ G\cap F=\emptyset\}.$$

I claim that f is a Tukey function from $\mathscr{E}_\mu$ to $\mathscr{Z}$. The first thing to check is that $f(F) \in$ $\in \mathscr{Z}$ for $F \in \mathscr{E}_\mu$. But if $F \in \mathscr{E}_\mu$ and $m \in$ N then there is a G $\in \mathfrak{T}\cap\mathscr{E}$ such that $G \subseteq X \setminus F$ and p(G) $\geq 1 - 2^{-m}$; now there is an n $\in$ N such that G $\in \mathscr{E}_n$, and $\theta(G)\setminus k_n \subseteq \mathrm{N}\setminus f(F)$, so that

$$d^*(f(F)) \leq d^*(N \setminus (e(G) \setminus k,)) = d^*(\mathrm{N}\setminus\theta(G)) = d^*(\theta(X\setminus G)) =$$
$$= \mu(X\setminus G) \leq 2^{-m}.$$

As m isarbitrary, $d^*(f(F)) = 0$ and $f(F) \in \mathscr{Z}$.

To see that f is a Tukey function, let $\mathscr{A}\subseteq\mathscr{E}_\mu$ be a set which is not bounded above; set $H = \bigcup\mathscr{A}$, so that $\mu(H) = \varepsilon > 0$, and write $a = \bigcup\{f(F) : F\in\mathscr{A}\}$. I need to show that $d^*(a) > 0$. But examine $a\cap k_n$, for any $n\in\mathrm{N}$. Because $k_m \geq k_n$ for $m \geq n$,

$$f(F)\cap k_n = k_n\setminus\bigcup\{\theta(G)\setminus k_m : m\leq n, G\in\mathfrak{T}\cap\mathscr{E}_m, G\cap F=\emptyset\} \supseteq$$
$$\supseteq k_n\setminus\bigcup\{\theta(G) : G\in\mathfrak{T}\cap\mathscr{E}_n, G\cap F=\emptyset\}$$

for every $F\in\mathscr{E}_\mu$. If $i\in k_n\setminus a$ consider

$$V = \bigcap\{G : G\in\mathfrak{T}\cap\mathscr{E}_n, i\in\theta(G)\};$$

then $V\in\mathfrak{T}\cap\mathscr{E}_n$ and $i\in\theta(V)$. If $F\in\mathscr{A}$ then $i\notin f(F)$ so there is a $G\in\mathfrak{T}\cap\mathscr{E}_n$ such that $G\cap F=\emptyset$ and $i\in\theta(G)$; now $V\subseteq G$ so $V\cap F=\emptyset$; as F is arbitrary, $V\cap H=\emptyset$. This means that

$$k_n\setminus a\subseteq\bigcup\{\theta(V) : V\in\mathfrak{T}\cap\mathscr{E}_n, V\cap H=\emptyset\} = \theta(W)$$

where $W = \bigcup\{V : V \in \mathfrak{T} \cap \mathscr{E}_n, GV \cap H = \emptyset\} \in \mathscr{E}_n$. Consequently

$$\#\left(k_n \setminus a\right) \leq \#\left(k_n \cap \theta(W)\right) \leq k_n\left(\mu(W) + 2^{-n}\right) \leq$$
$$\leq k_n\left(1 - \mu(H) + 2^{-n}\right) = k_n\left(1 - \varepsilon + 2^{-n}\right),$$

and

$$k_n^{-1}\#\left(a \cap k_n\right) \geq \varepsilon - 2^{-n}.$$

This is true for every $n \in \mathbf{N}$. So

$$d^*(a) \geq \limsup_{n\to\infty} k_n^{-1}\#(a \cap k_n) \geq \varepsilon > 0$$

and $a \notin \mathscr{Z}$. Accordingly f is a Tukey function and $\mathscr{E}_\mu \leq \mathscr{Z}$.

More about measure ideals 3L. (a) The results of § 2 dealt with ideals of negligible sets for Radon measures. For general measures there will be nothing to correspond to them. For instance, if X is any set and $\mathscr{I}$ any σ-ideal of subsets of X, there is a $\{0, 1\}$-valued measure μ on X such that $\mathscr{N}_\mu = \mathscr{I}$. But some of the results do extend to interesting non-Radon measures. in particular, to quasi-Radon measures. Recall that a quasi-Radon measure space, as defined in [7] and [8], is a quadruple $(X, \mathfrak{T}, \Sigma, \mu)$ such that: (i) (X, Σ, μ) is a complete locally determined measure space; (ii) $\mathfrak{T}$ is a topology on X and $\mathfrak{T} \subseteq \Sigma$; (iii) if $\mu(E) > 0$ there is an open set G such that $\mu(G) < \infty$ and $\mu(E \cap G) > 0$; (iv) μ is inner regular for the family of closed subsets of X; (v) if $\mathscr{G}$ is a non-empty upwards-directed family of open sets in X, then $\mu(\bigcup\mathscr{G}) = \sup_{G\in\mathscr{G}}\mu(G)$.

(b) On looking through the results of § 2, we find that the arguments of part (b) of the proof of 2D, some of those of 2H, 2I and 2Jc remain applicable to quasi-Radon measures. Specifically, (i) if (X, μ) is an atomless quasi-Radon probability space of Maharam type $\kappa \geq \omega$, then $\mathscr{N}_\mu \leq \ell^1(\kappa)$ (see 2D); (ii) if (X, μ) is a quasi-Radon measure space, (Y, ν) is a Radon measure space, and their measure algebras are isomorphic (as Boolean algebras), then $\mathscr{N}_\mu \leq \mathscr{N}_\nu$ (see 2H; we need [7], 72B, for the fact that (X, μ) is decomposable, and [8], A7Bk to deal with atoms in X); (iii) if (X, μ) is a quasi-Radon measure space with measure algebra $\mathfrak{A}$, and if $\Sigma_\mu^* = \{E : E \subseteq X, \mu(X \setminus E) > 0\}$, then $\Sigma_\mu^* \equiv \mathfrak{A}^-$ (just as in 2I); (iv) if $bu([\kappa]^{\leq\omega}) \leq \mathrm{add}(\mathscr{N})$ and (X, μ) is a quasi-Radon probability space of Maharam type κ, then $\mathscr{N}_\mu \leq [\kappa]^{\leq\omega} \times \mathscr{N}$.

(c) For examples of quasi-Radon measure spaces see, for instance, [8], 32D. These all have separable L^1 spaces i.e. countable Maharam types. If (X, μ) is a quasi-Radon measure space of countable Maharam type, then $\mathrm{add}(\mathscr{N}_\mu) \geq \mathrm{add}(\mathscr{N})$, $\mathrm{cf}(\mathscr{N}_\mu) \leq \mathrm{cf}(\mathscr{N})$, and $\mathrm{add}_\omega(\Sigma_\mu^*) \geq \mathrm{add}(\mathscr{N})$; compare [8], 32H. Note that these inequalities will also be true if (X, μ) is any σ-finite measure space in which the domain of μ is a countably-generated σ-algebra (see [8], 32Gc).

Two negative results 3M. Consider the five directed sets $\mathbf{N}^{\mathbf{N}}$, $\mathscr{E}_\mu$ (defined as in 3E and 3K, with μ Lebesgue measure on [0,1]), $\mathscr{F}$, $\mathscr{Z}$ (as in 3K) and $\ell^1(\mathbf{N})$. We have

$$(*) \qquad \mathbf{N}^{\mathbf{N}} \le \mathscr{E}_\mu \le \mathscr{F} \le \ell^1(\mathbf{N}), \qquad \mathscr{E}_\mu \le \mathscr{Z} \le \ell^1(\mathbf{N}).$$

The question immediately arises, whether there are any further relations of the type $P \le Q$ among these five sets. Isbell ([13]) showed that $\mathscr{Z} \not\le \mathbf{N}^{\mathbf{N}}$. I can offer the following:

Proposition. (a) $\mathscr{E}_\mu \not\le \mathbf{N}^{\mathbf{N}}$; (b) $\mathscr{Z} \not\le \mathscr{F}$.

Proof. (a) Let $f : \mathscr{E}_\mu \to \mathbf{N}^{\mathbf{N}}$ be any function. Let μ^* be Lebesgue outer measure on [0,1] and choose $\langle \alpha(n) \rangle_{n \in \mathbf{N}}$ inductively in $\mathbf{N}$ so that

$$\mu^*\{t : t \in [0,1],\ f(\{t\})(i) \le \alpha(i)\ \forall\, i \le n\} \ge \frac{1}{2} + 2^{-n-2}$$

for every $n \in \mathbf{N}$. This defines $\alpha \in \mathbf{N}^{\mathbf{N}}$. Set

$$C_n = \overline{\{t :\ f(\{t\})(i) \le \alpha(i)\ \forall\, i \le n\}},\quad C = \bigcap_{n \in \mathbf{N}} C_n$$

so that $\mu(C) \ge \frac{1}{2}$. Let $\langle U_n \rangle_{n \in \mathbf{N}}$ enumerate the set of open intervals with rational endpoints which meet C; then $U_n \cap C_n \ne \emptyset$, so can choose, for each $n \in \mathbf{N}$, a $t_n \in U_n$ such that $f(\{t_n\})(i) \le \alpha(i)$ for every $i \le n$. Examine $A = \{\{t_n\} : n \in \mathbf{N}\}$; then $f[A]$ is bounded above in $\mathbf{N}^{\mathbf{N}}$ (because $\sup_{n \in \mathbf{N}} f(\{t_n\})(i) < \infty$ for every i) but A is not bounded above in $\mathscr{E}_\mu$ (because $\overline{\{t_n : n \in \mathbf{N}\}} \supseteq C$, so $\{t_n : n \in \mathbf{N}\} \notin \mathscr{E}_\mu$). Thus f is not a Tukey function. As f is arbitrary, $\mathscr{E}_\mu \not\le \mathbf{N}^{\mathbf{N}}$.

(b) Let $f : \mathscr{Z} \to \mathscr{F}$ be any function. Let $\langle U_n \rangle_{n \in \mathbf{N}}$ enumerate a base $\mathscr{U}$ for the topology of $\mathbf{N}^{\mathbf{N}}$ which contains $\emptyset$ and is closed under finite unions. For each $n \in \mathbf{N}$, set

$$a_n = \{i : i \in \mathbf{N},\ f(a) \cap U_n \ne \emptyset \ \text{ whenever } \ i \in a \in \mathscr{Z}\}.$$

Take $a \in \mathscr{Z}$ such that $a \cap a_n \ne \emptyset$ whenever $n \in \mathbf{N}$ and a_n is infinite. Set $K = \{n : n \in \mathbf{N}, U_n \cap f(a) = \emptyset\}$, so that $\bigcup_{n \in K} U_n = \mathbf{N}^{\mathbf{N}} \setminus \overline{f(a)}$ is dense, while a_n is finite for every $n \in K$ (since otherwise there is an $i \in a \cap a_n$ and $f(a) \cap U_n \ne \emptyset$). For $n \in \mathbf{N}$, $\bigcup_{m \in K, m \le n} U_m \in \mathscr{U}$; say $\bigcup_{m \in K, m \le n} U_m = U_{r(n)}$. Then $r(n) \in K$ for every n. Take a strictly increasing sequence $\langle k_n \rangle_{n \in \mathbf{N}}$ in $\mathbf{N}$ such that $\sup a_{r(n)} \le k_n$ for each $n \in \mathbf{N}$.

For $i \le k_0$ set $b_i = \{i\}$; for $k_n < i \le k_{n+1}$ choose $b_i \in \mathscr{Z}$ such that $i \in b_i$ and $f(b_i) \cap U_{r(n)} = 0$ (such exists because $i \notin a_{r(n)}$). Now examine

$$E = \bigcup_{i\in\mathbf{N}} f(b_i) \subseteq \mathbf{N}^{\mathbf{N}}.$$

If $m \in K$ then $U_m \subseteq U_{r(n)}$ forevery $n \ge m$ so $U_m \cap f(b_i) = 0$ forevery $i > k_m$ and $U_m \cap E = \bigcup\{f(b_i) \cap E : i \le k_m\}$ is nowhere dense. As $\bigcup_{m\in K} U_m$ is a dense open set, E is nowhere dense, and $\{f(b_i) : i \in \mathbf{N}\}$ is bounded above in $\mathscr{F}$. But $\bigcup_{i\in\mathbf{N}} b_i = N$ so $\{b_i : i \in \mathbf{N}\}$ is not bounded above in $\mathscr{Z}$, and f is not a Tukey function. As f is arbitrary, $\mathscr{Z} \not\le \mathscr{F}$. For more about the relationship between $\mathscr{F}$ and $\mathscr{Z}$ see [12], Prop. 23, and [14].

Problems 3N. Innumerable questions are left open by the work above, besides those mentioned at the end of § 2. It seems possible that the following may lead somewhere.

(a) Taking $\mathscr{Z}$, to be the ideal of sets of zero asymptotic density as usual, is $\mathscr{F} \not\le \mathscr{Z}$? Is $\mathscr{Z} \equiv \ell^1(\mathbf{N})$? (These are the questions left over from 3M above.) I have been able to prove that $\mathrm{cf}(\mathscr{Z}) = \mathrm{cf}(\ell^1(\mathbf{N}))$, $\mathrm{add}_\omega(\mathscr{Z}) = \mathrm{add}_\omega(\ell^1(\mathbf{N}))$ and that $\ell^1(\mathbf{N}) \le \mathscr{Z}^\delta$.

(b) Still working with the five sets $\mathbf{N}^{\mathbf{N}}$, $\mathscr{E}_\mu$, $\mathscr{F}$, $\mathscr{Z}$ and $\ell^1(\mathbf{N})$ of 3M, there are consistent relations of the form $\le_\omega$ which are not consequences of (*) in 3M (for instance, under CH we have $\mathbf{N}^{\mathbf{N}} \equiv_\omega \omega_1 \equiv_\omega \ell^1(\mathbf{N})$). But are there any theorems of ZFC of the form $P \le_\omega Q$, or $\mathrm{add}_\omega(P) \le \mathrm{add}_\omega(Q)$, or $\mathrm{cf}(P) \le \mathrm{cf}(Q)$, where P and Q are taken from these five sets, which are not consequences of (*)? (Several cases are ruled out by the results of [20].)

(c) Suppose that P and Q are partially ordered sets with Polish topologies such that their orderings $\le_P, \le_Q$ are Borel sets in P^2, Q^2 respectively. Suppose that $P \le Q$. Does it follow that there is a Tukey function from P to Q which is Borel measurable?

(The point of this question is that (i) the Tukey functions actually constructed in such theorems as 2B, 3B, 3K are generally not complicated according to the criteria of descriptive set theory; (ii) an affirmative answer would imply absoluteness results relevant to such questions as (a) above.)

(d) For a topological space X, let $\mathscr{K}_X$ be the ideal of relatively compact subsets of X . What types can $\mathscr{K}_X$ have? I discuss these spaces at length (concentrating on separable metric X) in [12]. For instance, if $X \subseteq \mathbf{R}$, then (i) $\mathscr{K}_X \equiv \{0\}$ iff X is compact; (ii) $\mathscr{K}_X \equiv \mathbf{N}$ iff X is locally compact not compact; (iii) $\mathscr{K}_X \equiv \mathbf{N}^{\mathbf{N}}$ iff X is G_δ, not locally compact; and $\mathscr{K}_X \equiv \mathscr{K}_{\mathbf{Q}}$ iff X is coanalytic, not G_δ ([12], Theorem 15). Conceming $\mathscr{K}_{\mathbf{Q}}$, I find that $\mathscr{K}_{\mathbf{Q}} \not\le_\omega \omega_1 \times \mathbf{N}^{\mathbf{N}}$ and $\omega_1 \times \mathbf{N}^{\mathbf{N}} \le \mathscr{K}_{\mathbf{Q}}$ but $\mathscr{K}_{\mathbf{Q}} \not\le \omega_1 \times \mathbf{N}^{\mathbf{N}}$ (see [12], Theorem 16); $\mathscr{Z} \not\le \mathscr{K}_{\mathbf{Q}}$ (in fact, $\mathscr{Z} \not\le \mathscr{K}_X$ for any separable metric X) and $\mathscr{K}_{\mathbf{Q}} \le \ell^1$ but it is undecidable whether $\mathscr{E}_\mu \mathscr{K}_{\mathbf{Q}}$ or $\mathscr{F} \le \mathscr{K}_{\mathbf{Q}}$ ([12], Proposition 23). I do not know whether it is relatively consistent with ZFC to suppose that there is an analytic non-Borel set $X \subseteq \mathbf{R}$

such that $\mathscr{K}_X \leq \mathscr{K}_{\mathbb{Q}}$ (this is surely inconsistent with the axiom of projective determinacy; see [12], Theorem 18).

(e) For a topological space X, let $\mathscr{F}_X$ be the ideal of nowhere dense subsests of X, and $\mathscr{M}_X$ the σ-ideal of meagre sets. Is there a coherent classification of these, in terms of $\leq$ and $\leq_\omega$ and topological properties of X? What if X is known to be a compact Hausdorff space?

Note added in proof. T. Bartoszyiski and S. Shelah [2b] have shown that $\mathrm{add}_\omega(\mathscr{E}_\mu) = \mathrm{add}(\mathscr{M})$, $\mathrm{cf}(\mathscr{E}_\mu) = \mathrm{cf}(\mathscr{M})$. Further results may be found in [29].

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