Note di Matematica Vol. XI, 109-117 (1991)

SURJECTIVE $\mathbb{A}$-COMPACTNESS AND GENERALIZED KOLMOGOROV NUMBERS

A. DEFANT, M.S. RAMANUJAN

Dedicated to the memory of Professor Gottfried Köthe

1. INTRODUCTION

B. Carl and I. Stephani [2, 3] have introduced a refined notion of $\mathbb{A}$-compactness where $\mathbb{A}$ is a given operator ideal; in a more recent paper, I. Stephani [6] has introduced the concept of injectively $\mathbb{A}$-compact operators in view of which the Carl-Stephani notion of $\mathbb{A}$-compactness will be more appropriately one of surjective $\mathbb{A}$-compactness. Carl and Stephani also defined generalized outer entropy numbers in terms of which (surjective) $\mathbb{A}$-compactness is characterized while Stephani characterizes injective $\mathbb{A}$-compactness of operators in terms of their generalized inner entropy numbers and generalized Gelfand numbers.

In this paper we give an appropriate generalization of Kolmogorov numbers of sets and operators with reference to a fixed quasi-normed ideal and characterize surjective $\mathbb{A}$-compactness in terms of these numbers. Also, for a set H of operators we define their measures of equi-$\mathbb{A}$-variation and apply these ideas in studying $\mathbb{A}$-compact sets of $\mathbb{A}$-compact operators and present results analogous to those of K. Vala [7].

2. BASIC DEFINITIONS AND NOTIONS

For definitions and notations not explicitly stated here the reader is referred to A. Pietsch [5] and B. Carl and I. Stephani [2, 3].

Let $[\mathbb{A}, \alpha]$ be a quasi-normed operator ideal in the sense of Pietsch. A subset M of the Banach space E is said to be $\mathbb{A}$-bounded if $M \subset S(B_X)$, where $S \in \mathbb{A}(X, E)$ and B_X is the closed unit ball in X. The surjective hull of $\mathbb{A}$ is denoted $\mathbb{A}^s$ and for $T \in \mathbb{A}^s(E, F)$

$$\alpha_s(T) = \inf \left\{\alpha(S) : T\left(B_E\right) \subset S\left(B_X\right), \quad S \in \mathbb{A}(X, F)\right\}.$$

An $\mathbb{A}$-bounded subset M of E is said to be $\mathbb{A}$-compact (see [2]) if there exists an operator $S \in \mathbb{A}(X, E)$ such that for each $\varepsilon > 0$ there are finitely many elements $y_1, \ldots, y_k$ (depending on ε) such that

$$M \subseteq \bigcup_1^k \left(y_i + \varepsilon S\left(B_X\right)\right).$$

An operator $T \in \mathbb{L}(E, F)$ is said to be a $\mathbb{A}$-compact if it maps every bounded set into an $\mathbb{A}$-compact set. $\mathbb{K}^{\mathbb{A}}$ denotes the set of $\mathbb{A}$-compact maps and Carl and Stephani [2] proved that $\mathbb{K}^{\mathbf{A}} = \mathbf{A}^s \cdot \mathbb{K} = \mathbb{K}^{\mathbf{A}} \cdot \mathbb{K}$ and that $\mathbb{K}^{\mathbf{A}}$ is a surjective operator ideal. They also defined for an $\mathbf{A}$-bounded subset $M \subset E$ its (outer) entropy numbers $e_n(M;\mathbf{A})$ by

$$e_n(M;\mathbf{A}) = \inf \left\{ \alpha(S) : M \subseteq \bigcup_1^{q(n)} (y_i + S(B_X)),\right.$$
$$\left. S \in \mathbf{A}(X,E),\ y_i \in E \text{ and } 1 \leq q(n) \leq 2^{n-1} \right\},$$

and they proved that M is $\mathbf{A}$-compact if and only if $e_n(M;\mathbf{A}) \to 0$.

For $T \in \mathbf{A}^s(F,E)$, define $e_n(T;\mathbf{A}) = e_n(T(B_F);\mathbf{A})$.

Finally, for $T \in \mathbf{A}(E,F)$, its generalized approximation number

$$a_n(T;\mathbf{A}) = \inf \{\alpha(T-S) : \text{rank } S \leq n\}.$$

3. GENERALIZED KOLMOGOROV NUMBERS

For an $\mathbf{A}$-bounded subset M of E, define its generalized n-th Kolmogorov number $\delta_n(M;\mathbf{A})$ by

$$\delta_n(M;\mathbf{A}) = \inf \left\{\alpha(S) : S \in \mathbf{A}(G,E),\ M \subset S(B_G) + \mathbb{L}_n,\right.$$
$$\left.\mathbb{L}_n \text{ a linear subspace of dimension } \leq n\right\}.$$

The existence of such S is guaranteed by M being $\mathbf{A}$-bounded.

By notation $\alpha_s(M) = \delta_0(M;\mathbf{A})$ and for $T \in \mathbf{A}^s(E,F)$ define $\delta_n(T;\mathbf{A}) = \delta_n(T(B_E);\mathbf{A})$.

For M and T as above define $\delta(M;\mathbf{A}) = \lim_n \delta_n(M;\mathbf{A})$ and $\delta(T;\mathbf{A}) = \lim_n \delta_n(T;\mathbf{A})$. Also, we define

$$\Gamma(M;\mathbf{A}) = \inf \left\{\alpha(S) : M \subset T(B_F) + S(B_X),\ T \in \mathbb{K}^{\mathbf{A}}(F,E),\ S \in \mathbf{A}(X,E)\right\}.$$

Observe that when $\mathbf{A} = \mathbb{L}$, $\Gamma(M;\mathbf{A})$ is the standard ball measure of non-compactness of M, as proved by K. Astala ([1], example 3.2(a)).

We shall now establish some basic properties of the generalized Kolmogorov numbers.

Proposition 1. *(i) for each $\mathbf{A}$-bounded set M, $\delta_n(M;\mathbf{A}) = \delta_n(M;\mathbf{A}^s)$, for all n;*

(ii) for $\mathbf{A}$-bounded sets M_1, M_2 (of the Banach space E),

$$\delta_{n+m}(M_1 \cup M_2;\mathbf{A}) \leq \kappa\left(\delta_n(M_1;\mathbf{A}) + \delta_m(M_2;\mathbf{A})\right)$$

(where κ is the constant ≥ 1 occurring in the definition of quasi-normed ideals $(\mathbf{A},\alpha)$).

Proof. (i) Since $\mathbf{A} \subseteq \mathbf{A}^s$ (and the canonical injection is continuous), $\delta_n(M;\mathbf{A}^s) \leq \leq \delta_n(M;\mathbf{A})$.

For the reverse inequality, let n be fixed and $T \in \mathbf{A}^s$ be such that $M \subset T(B_X) + \mathbb{L}_n$, $\dim \mathbb{L}_n \leq n$. Then if Q_X denotes the canonical surjection from X^{sur} to X (see [5]) we have $Q_X(B_{X^{\text{sur}}}) = B_X$ and $TQ_X \in \mathbf{A}(X^{\text{sur}}, E)$; thus

$$M \subset (TQ_X)(B_{X^{\text{sur}}}) + \mathbb{L}_n$$

and $\alpha(TQ_X) = \alpha_s(T)$; thus $\delta_n(M;\mathbf{A}^s) \geq \delta_n(M;\mathbf{A})$.

(ii) For $\mathbf{A}$-bounded sets M_1, M_2 in E and for integers n_1, n_2 find, for $i = 1,2$, $S_i \in \in \mathbf{A}(F_i, E)$ with $M_i \subset S_i(B_{F_i}) + \mathbb{L}_i$, with dimension $\mathbb{L}_i \leq n_i$ and with $\alpha(S_i) < < \delta_{n_i}(M_i) + \frac{\varepsilon}{2}$. Let $F = F_1 \oplus F_2$ be equipped with the supremum norm and let $P_i : F \to F_i$, $i = 1,2$ be the canonical projections. Then $\widehat{S}_i = S_i P_i \in \mathbf{A}(F,E)$ and for $S = \widehat{S}_1 + \widehat{S}_2 \in \mathbf{A}(F,E)$ we have

$$\widehat{S}_1\left(B_{F_1}\right) + \widehat{S}_2\left(B_{F_2}\right) \subset S\left(B_F\right) \quad \text{and} \quad \alpha(S) \leq \kappa\left(\alpha\left(\widehat{S}_1\right) + \alpha\left(\widehat{S}_2\right)\right).$$

The rest of the details in the proof are evident.

Proposition 2. *For $T_1 + T_2 \in A^s$ and $\mathbf{A}$-bounded set M we have the following:*

1) $\delta_0(M;\mathbf{A}) \geq \delta_1(M;\mathbf{A}) \geq \ldots \geq \delta_n(M;\mathbf{A}) \geq \ldots$

1') $\alpha(T) \geq \delta_0(T;\mathbf{A}) \geq \ldots \geq \delta_n(M;\mathbf{A}) \geq \ldots$

2) $\delta_{n+m}(T_1 + T_2;\mathbf{A}) \leq \kappa(\delta_n(T_1;\mathbf{A}) + \delta_m(T_2;\mathbf{A}))$

3) for $R \in \mathbb{L}$, $T \in A^s$, $\delta_{n+m}(TR;\mathbf{A}) \leq \delta_n(T;\mathbf{A})\delta_m(R)$

4) for $T \in A^s$, $S \in \mathbb{L}$, $\delta_n(ST) \leq \| S \| \delta_n(T;\mathbf{A})$

5) $\delta_n(TQ_E;\mathbf{A}) = \delta_n(T;\mathbf{A})$ *for* $T \in A^s$.

The proofs are all routine and omitted.

Proposition 3. *If the Banach space E has the lifting property and $T \in \mathbf{A}^s(E,F)$ then $\delta_n(T;\mathbf{A}) = a_n(T;\mathbf{A})$, for all n.*

Proof. For $\varepsilon > 0$, find $S : E \to F$ of rank at most n with $\alpha(T - S) < a_n(T;\mathbf{A}) + \varepsilon$. Then $T(B_E) \subset (T - S)B_E + \mathbb{L}_n$, with $\dim \mathbb{L}_n \leq n$ and thus $\delta_n(T;\mathbf{A}) \leq a_n(T;\mathbf{A})$.

Next, for given $\varepsilon > 0$, find Banach space G and $S \in A(G,F)$ such that $T(B_E) \subset \subset S(B_G) + \mathbb{L}_n$ and $\alpha(S) < \delta_n(T;\mathbf{A}) + \varepsilon$, $\dim \mathbb{L}_n \leq n$. Then $T(B_E) \subset S(B_G) + +P_n(T(B_E))$ where $P_n : F \to \mathbb{L}_n$ is the canonical projection. Then by a basic lemma due to K. Astala ([1], Lemma 2.4.1) there exist $K_1 \in \mathbb{L}(E,G)$, $K_2 \in \mathbb{L}(E,E)$, such that

$$T = SK_1 + P_n TK_2 \quad \text{with} \quad \| K_i \| < 1 + \varepsilon, \quad i = 1,2.$$

Then $\alpha(T - P_n TK_2) = \alpha(SK_1) \leq \alpha(S) \cdot (1 + \varepsilon)(\delta_n(T;\mathbf{A}) + \varepsilon)$ and this gives $a_n(T;\mathbf{A}) \leq \delta_n(T;\mathbf{A})$.

Corollary 4. $\delta_n(T;\mathbb{A}) = a_n(TQ_E;\mathbb{A})$.

Remark. When $\mathbf{A} = \mathbb{L}$ the above equality if often taken as the definition of $\delta_n(T)$ (see [5], 11.6.1); indeed the generalized (injective) Gelfand numbers $c_n(T;\mathbf{A})$, for $T \in \mathbf{A}^{\mathbf{I}}$, are defined by I. Stephani [6] in a dual manner via $c_n(T;\mathbf{A}) = a_n(J_F;\ \mathbf{A})$, $J_F : F \to F^{inj}$ being the canonical injection.

Our next result is a comparison between $\delta(M;\mathbf{A})$, $e(M;\mathbf{A})$ and $\Gamma(M;\mathbf{A})$.

Proposition 5. *For an* $\mathbb{A}$*-bounded set* $D \subset E$,

$$\Gamma(D;\mathbf{A}) \leq \delta(D;\mathbf{A}) \leq e(D;\mathbf{A}) \leq \kappa\Gamma(D;\mathbf{A}).$$

Proof. From our definition of $\Gamma(D;\mathbf{A})$ it follows that for given $\varepsilon > 0$, there are $T \in$ $\in \mathbb{K}^{\mathbf{A}}(F,E)$, $S \in \mathbf{A}(X,E)$ such that $\alpha(S) < \Gamma(D;\mathbf{A}) + \varepsilon$ and $D \subset T(B_F) + S(B_X)$; since $T \in \mathbb{K}^{\mathbf{A}}$, $T(B_F)$ is $\mathbf{A}$-compact and therefore $T(B_F) \subset \bigcup_1^k(y_i + R(B_Z))$ with $\alpha(R) < \varepsilon$, for suitable k and R. Thus $D \subset \bigcup_1^k(y_i + R(B_Z) + S(B_X))$ and then as in the proof of Proposition 1 we see

$$e_n(D;\mathbf{A}) \leq \kappa(\alpha(R) + \alpha(S)) \quad \text{for} \quad 2^n \geq k$$

and then $e(D;\mathbf{A}) \leq \kappa(\Gamma(D;\mathbf{A}))$.

From the definitions of δ_n and e_n and taking limits we have $\delta(D;\mathbf{A}) \leq e(D;\mathbf{A})$.

Finally, let $r = \delta(D;\mathbf{A})$. Then, for given $\varepsilon > 0$, there is a suitable N such that $\delta_N(D;\mathbf{A}) < r+\varepsilon$, and then there exists $S \in \mathbf{A}(F,E)$ with $\alpha(S) < r+\varepsilon$ and $D \subset S(B_F)+$ $+\mathbb{L}_N = S(B_F) + M_N$ where M_N is a bounded subset of $\mathbb{L}_N$. If P_N is the natural projection of E onto $\mathbb{L}_N$ then $P_N \in \mathbf{A}(E,E)$ and $D \subset S(B_F) + T(B_X)$ where $D \subset R(B_X)$, $R \in \mathbf{A}(X,E)$ and $T = P_N R$ is a finite rank map, and therefore is $\mathbf{A}$-compact. Thus $\Gamma(D;\mathbf{A}) \leq \delta(D;\mathbf{A})$.

Corollary 6. *D is $\mathbf{A}$-compact* $\Longleftrightarrow \delta_n(D;\mathbf{A}) \to 0 \Longleftrightarrow e_n(D;\mathbf{A}) \to 0$.

4. $\mathbf{A}$-COMPACT SETS OF $\mathbf{A}$-COMPACT OPERATORS

In this section we turn our attention to $\mathbf{A}$-compact subsets of $(\mathbb{K}^{\mathbf{A}}, \alpha_s)$. Recall that $(\mathbb{K}^{\mathbf{A}}, \alpha_s)$ has been shown to be a complete quasi-normed ideal by B. Carl and I. Stephani [2].

The following two notions are crucial for our discussion.

Let H be a set of continuous, linear maps between Banach spaces E and F. We define H to be of *equi-$\mathbf{A}$-variation* if for each $\varepsilon > 0$ there exists a partition $P_1, \ldots, P_m$ of B_E and a suitable $S \in \mathbf{A}(X,F)$ with $\alpha(S) < \varepsilon$ such that for $u, v \in$ (same) P_i, $h(u) - h(v) \in$

$\in S(B_X)$ holds for each $h \in H$. Trivially, equi-$\mathbf{A}$–variation $\Longleftarrow$ equi-variation in the sense of K. Vala [7].

Next we define a Banach ideal $(\mathbf{A}, \alpha)$ to have the property (V) if for each $S \in \mathbf{A}(X, \mathbb{K}^{\mathbf{A}}(E,F))$ with $[S(B_X)](B_E) \subseteq B_F$ there exists a $T \in \mathbf{A}(Z,F)$ such that

$$[S(B_X)](B_E) \subseteq T(B_Z) \quad \text{and} \quad \alpha(T) \le \alpha_s(S).$$

The Banach ideal of all bounded linear operators has this property. We shall also show that the ideals $(\mathbf{N}, \nu)$ of nuclear operators with the nuclear norm and $(\mathbb{I}, \imath)$ of integral operators have property (V).

Notation. *For* $H \subset \mathbb{L}(E,F)$, $H(e) = \{h(e) : h \in H\}$.

Proposition 7. *Suppose* $H \subset (\mathbb{K}^{\mathbf{A}}(E,F), \alpha_s)$ *is* $\mathbf{A}$*-compact,* $(\mathbf{A}, \alpha)$ *assumed to be a Banach ideal. Then*

(i) for each $e \in E$, $H(e)$ *is an* $\mathbf{A}$*-compact set in* F*;*

(ii) if $(\mathbf{A}, \alpha)$ *has property (V) then* H *has equi-*$\mathbf{A}$*-variation.*

Proof. (i) Since H is $\mathbf{A}$-compact, we have for given $\varepsilon > 0$

$$H \subset \bigcup_{1}^{k} (T_i + S(B_Z)), \quad S \in \mathbf{A}\left(Z, \mathbb{K}^{\mathbf{A}}(E,F)\right), \quad \alpha_s(S) < \varepsilon$$

and without loss of generality we assume each $T_i \in H$. For each $e \in E$, the evaluation map $\phi_e : \mathbb{L}(E,F) \to F$ defined by $\phi_e(T) = T(e)$ is continuous and $\| \phi_e \| = 1$; since $(\mathbb{K}^{\mathbf{A}}(E,F), \alpha_s)$ is continuously imbedded in $(\mathbb{L}(E,F), \| \, \|)$ $\phi_e : \mathbb{K}^{\mathbf{A}} \to F$ is continuous.

For each $e \in B_E$, $H(e) \subset \bigcup_1^k (T_i + (\phi_e \circ S)(B_Z))$ and $\phi_e \circ S \in \mathbf{A}(Z,F)$ with $\alpha_s(\phi_e \circ S) \le \| \phi_e \| \, \alpha_s(S) < \epsilon$ and the proof of (i) is complete.

(ii) As before $H \subset \bigcup_{i=1}^k (T_i + S(B_Z))$, $\alpha_s(S) < \varepsilon$. Since each T_i is $\mathbf{A}$-compact $T_i(B_E) \subset \bigcup_{j=1}^{n(i)} (f_{i,j} + C_i(B_{Y(i)}))$ with $C_i \in \mathbf{A}(Y(i), F)$ and $\alpha(C_i) < \varepsilon$. Let $P_{i,j} = \{e \in B_E : T_i(e) \in f_{i,j} + C_i(B_{Y(i)})\}$, $j = 1, 2, \ldots, n(i)$. Then the sets $\{P_{i,j} : i$, fixed and $j = 1, 2, \ldots, n(i)\}$ give a partition of B_E.

Next consider all sets of the form

$$P_{1,j(1)} \cap P_{2,j(2)} \cap \ldots \cap P_{k,j(k)}$$

and they too yield a partition of B_E. Consider $e, e' \in B_E$ belonging the same set of the above type; for arbitrary $h \in H$ we have $h = T_i + S(z)$ for a suitable T_i and $z \in B_Z$. Then

$$h(e) - h(e') = T_i(e) + S(z)(e) - T_i(e') - S(z)(e') \in 2C_i\left(B_{Y(i)}\right) + 2S\left(B_Z\right)\left(B_E\right).$$

Let now $C : \bigoplus_1^k Y(i) \to F$ be defined by $C(y_1, \ldots, y_k) = C_1(y_1) + \ldots + C_k(y_k)$; then $C \in \mathbf{A}$ and $\alpha(C) < k\varepsilon$. Since $(\mathbf{A}, \alpha)$ has property (V), we can find $R \in \mathbf{A}(X, F)$ with $S(B_Z)(B_E) \subseteq R(B_X)$ with $\alpha(R) \leq \alpha_s(S) < \varepsilon$. Thus

$$h(e) - h(e') \in 2C\left(B_Y\right) + R\left(B_X\right), \quad Y = \bigoplus_1^k Y(i)$$

and this essentially completes the proof of (ii).

The next proposition gives (partial) converses of the preceding result.

Proposition 8. *Suppose (i) $H \subset (\mathbb{K}^{\mathbf{A}}(E,F), \alpha_s)$ is of equi-$\mathbf{A}$-variation and (ii) for each $e \in E$, $H(e)$ is $\mathbf{A}$-compact in F. Then*

(a) $H(B_E)$ is $\mathbf{A}$-compact in F;

(b) H is compact in $(\mathbb{K}^{\mathbf{A}}(E,F), \alpha_s)$.

Proof. By hypothesis (i) above, for a given $\varepsilon > 0$ there exist a partition $P_1, P_2, \ldots, P_n$ of B_E and $S \in \mathbf{A}(X, F)$ with $\alpha(S) < \varepsilon$ such that for each $h \in H$ and $e, e' \in$ (same) P_i,

$$h(e) - h(e') \in S\left(B_X\right).$$

Now in each P_k pick an x_k and fix it. Since by hypothesis (ii), $H(x_k)$ is $\mathbf{A}$-compact we have

$$H\left(x_k\right) \subset \bigcup_{i=1}^{n(k)} \left(h_{i,k}\left(x_k\right) + A_k\left(B_{Y(k)}\right)\right)$$

with $A_k \in \mathbf{A}(Y(k), F)$ and $\alpha(A_k) < \varepsilon/2n$ and $h_{i,k} \in H$. For each $k = 1, 2, \ldots, n$ let

$$H_{i,k} = \left\{h \in H : h\left(x_k\right) \in h_{i,k}\left(x_k\right) + A_k\left(B_{Y(k)}\right)\right\}, \quad i = 1, 2, \ldots, n(k).$$

Then, for arbitrary $h \in H$ and $e \in B_E$ we have $e \in P_k$ for a suitable k and for this k, $h \in H_{i,k}$ for a suitable i. Then $h(e) - h(x_k) \in S(B_X)$ and $h(x_k) \in h_{i,k}(x_k) + A_k(B_{Y(k)})$. Thus $h(e) \in h_{i,k}(x_k) + A_k(B_{Y(k)}) + S(B_X)$ and

$$H\left(B_E\right) \subset \bigcup_{k=1}^{n} \bigcup_{i=1}^{n(k)} h_{i,k}\left(x_k\right) + A_k\left(B_Y\right) + S\left(B_X\right)$$

where, as before $A : \bigoplus_1^k Y(i) \to F$, $A(y(1) + y(2) + \ldots + y(n)) = \sum_1^n A_k(y(k))$ with the obvious norm. Then $A \in \mathbf{A}(Y, F)$ and $\alpha(A) < \varepsilon$. This proves (a).

To prove (b), we consider sets of the form

$$H_{i(1),1} \cap H_{i(2),2} \cap \ldots \cap H_{i(n),n} .$$

Then, for $h, \widehat{h}$ belonging to one and the same set of the above type, we have

$$\begin{aligned} h(e) - \widehat{h}(e) &= h(e) - h(x_k) + h(x_k) - \widehat{h}(x_k) + \widehat{h}(x_k) - \widehat{h}(e) \in \\ &\in S(B_X) + 2A_k(B_{Y(k)}) + S(B_X) \subset 2S(B_X) + 2A(B_Y) \subset \\ &\subset T(B_Z), \end{aligned}$$

where $Z = X \oplus Y$ and $T = 2S \oplus 2A$.

This shows that the (metric) diameter of each of the above sets in the normed space $(\mathbb{K}^{\mathbf{A}}(E,F), \alpha_s)$ is less than 4ε and H is the disjoint, finite, union of sets of diameter less than 4ε. Thus H is compact in $(\mathbb{K}^{\mathbf{A}}(E,F), \alpha_s)$.

Remark. The result in (b) is stronger than the claim that H is compact in $\mathbb{K}^{\mathbf{A}}$ with the uniform norm (of $\mathbb{L}(E,F)$) and is weaker than the (desirable) conclusion of $\mathbf{A}$-compactness of H in $(\mathbb{K}^{\mathbf{A}}, \alpha_s)$.

Finally we prove

Proposition 9. *The Banach operator ideals* $(\mathbf{N}, \nu)$ *and* $(\mathbf{I}, i)$ *have property (V).*

Proof. Some standard facts about the projective norm π and the injective norm ε on tensor products of Banach spaces and their relation to $\mathbb{I}$ and $\mathbf{N}$ are used in the sequel and the reader is referred to A. Defant and K. Floret [4] for these details.

First we shall consider the integral operators $(\mathbb{I}, i)$. For three Banach space X, Y and Z the linear map

$$(*) \qquad \begin{cases} \mathbb{I}(X, \mathbb{I}(Y, Z'')) \to \mathbb{I}\left(X \widehat{\otimes}_\varepsilon Y, Z''\right), & \text{defined by} \\ \qquad S \to (x \otimes y \to (Sx)(y)) \end{cases}$$

is a metric surjection since, via natural identifications

$$\begin{aligned} \mathbb{I}(X, \mathbb{I}(Y, Z'')) &= \mathbb{I}\left(X, \left(Y \widehat{\otimes}_\varepsilon Z'\right)\right)' = \\ &= \left(X \widehat{\otimes}_\varepsilon Y \widehat{\otimes}_\varepsilon Z'\right)' = \left(\left(X \widehat{\otimes}_\varepsilon Y\right) \widehat{\otimes}_\varepsilon Z'\right)' = \mathbb{I}\left(X \widehat{\otimes}_\varepsilon Y, Z''\right). \end{aligned}$$

Let now $S \in \mathbb{I}(X, \mathbb{K}^{\mathbf{I}}(E,F))$ with $S(B_X)(B_E) \subset B_F$. Since

$$S \in \mathbb{I}^s\left(X, \mathbb{K}^{\mathbf{I}}(E,F)\right) \subset \mathbb{I}\left(X, I^s(E,F)\right)$$

and $\mathbb{I}^s(E^{sur}, F'') = \mathbb{I}(E^{sur}, F'')$ holds isometrically, we have from (*) that

$$T_0 : X\widehat{\otimes}_\varepsilon E^{sur} \to F'' \quad \text{defined by}$$

$$x \otimes e \to (Sx)\left(Q_E e\right)$$

is integral and $i(T_0) \leq i(S)$. But the integral operators are regular and therefore the astriction T of T_0 to F must be integral as well and $i(T) = i(T_0) \leq i(S)$.

Since $S(B_X)(B_E) = S(B_X(Q_E(B_{E^{sur}}))) = T(B_X \otimes B_{E^{sur}}) \subset T(B_{X\widehat{\otimes}_\varepsilon E^{sur}})$, the proof that $(\mathbb{I}, i)$ has property (V) is complete.

In the case of nuclear operators $(\mathbf{N}, \nu)$ the proof runs along the same lines. It suffices to show that the map

$$\mathbf{N}(X, \mathbf{N}(Y,Z)) \to \mathbf{N}\left(X\widehat{\otimes}_\varepsilon Y, Z\right) \quad \text{defined by}$$

$$S \to (x \otimes y \to (Sx)(y))$$

is well-defined and has norm ≤ 1. The natural mappings

$$Y'\widehat{\otimes}_\pi Z \to \mathbf{N}(Y,Z)$$

$$X'\widehat{\otimes}_\pi \mathbf{N}(Y,Z) \to \mathbf{N}(X, \mathbf{N}(Y,Z))$$

$$\left(X\widehat{\otimes}_\varepsilon Y\right)'\widehat{\otimes}_\pi Z \to \mathbf{N}\left(X\widehat{\otimes}_\varepsilon Y, Z\right)$$

are metric surjections; in particular

$$\left(X\widehat{\otimes}_\pi Y'\right)\widehat{\otimes}_\pi Z = X'\widehat{\otimes}_\pi\left(Y'\widehat{\otimes}_\pi Z\right) \to \mathbf{N}(X, \mathbf{N}(Y,Z)$$

is a metric surjection. Since the natural map $X'\widehat{\otimes}_\pi Y' \to (X\widehat{\otimes}_\varepsilon Y)'$ has norm ≤ 1 the following commuting diagram gives the desired conclusion:

$$\begin{cases} \mathbf{N}(X, \mathbf{N}(Y,Z) & \to & \mathbf{N}\left(X\widehat{\otimes}_\varepsilon Y, Z\right) \\ \uparrow & & \uparrow \\ \left(X'\widehat{\otimes}_\pi Y'\right)\widehat{\otimes}_\pi Z & \to & \left(X\widehat{\otimes}_\varepsilon Y\right)'\widehat{\otimes}_\pi Z. \end{cases}$$

REFERENCES

[1] K. Astala, *On measures of noncompactness and ideal variations in Banach spaces*, Ann. Acad. Sci. Fenn. Ser. AI, **29** (1980), pp. 1-42.

[2] B. Carl, I. Stephani, *On $\mathbf{A}$ − compact operators, generalized entropy numbers and entropy ideals*, Math. Nachr., **119** (1984), pp. 77-95.

[3] B. Carl, I. Stephani, *Estimating compactness properties of operators by the aid of generalized entropy numbers*, J. Reine Angew. Math., **350** (1984), pp. 117-136.

[4] A. Defant, K. Floret, *Tensor norms and operator ideals*, North-Holland, 1993.

[5] A. Pietsch, *Operator ideals*, North-Holland, 1980.

[6] I. Stephani, *Injectively $\mathbf{A}$ − compact operators, generalized inner entropy numbers and Gelfand numbers*, Math. Nachr., **133** (1987), pp. 247-272.

[7] K. Vala, *On compact sets of compact operators*, Ann. Acad. Sci. Fenn. Ser. AI, **351** (1964), pp. 1-9.

Received January 11, 1991

A. Defant
Fachbereich Mathematik
Universität Oldenburg
D-2900 Oldenburg (Oldb.)
Germany

M.S. Ramanujan
Department of Mathematics
University of Michigan
Ann Arbor, MI 48109
U.S.A.