ON THE $C^\ast$-COMPARISON ALGEBRA OF A CLASS OF SINGULAR STURM-LIOUVILLE EXPRESSIONS ON THE REAL LINE
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Dedicated to the memory of Professor Gottfried Köthe

1. INTRODUCTION

In this article we study a $C^\ast$-comparison algebra in the sense of [C2] with generators related to the ordinary differential expression $H$ on the full real line $\mathbb{R}$ where, with constants $\alpha \geq 0$, $\beta \in \mathbb{R}$,

$$H = -\partial_x (1 + x^2)^\beta \partial_x + (1 + x^2)^\alpha, \quad x \in \mathbb{R}.$$  

(1.1)

More precisely, the algebra, called $A$, is generated by the multiplications $a(M): u(x) \to a(x)u(x)$, by functions $a(x) \in C([-\infty, +\infty])$ and the (singular integral) operators $S_0 = (1 + x^2)^{\alpha/2} \Lambda$, $iS_1 = (1 + x^2)^{\beta/2} \partial_x \Lambda$, and their adjoints. Here $\Lambda = H^{-1/2}$, the inverse positive square root of the unique self-adjoint realization $H$ of the expression (1.1), in the Hilbert space $H = L^2(\mathbb{R})$. (We use the same notation for both, (1.1) and its realization.)

The case of $\beta < \alpha + 1$ was discussed earlier in [Tg1], even for all $n$-dimensional problem. The commutators are compact and the Fredholm properties of operators in $A$ are determined by a complex-valued symbol on a symbol space homeomorphic to that of the usual Laplace comparison algebra on $\mathbb{R}^n$, although the symbol itself is calculated by different formulas.

The algebra, perhaps, is of interest because the singular Sturm-Liouville expression $H$ of (1.1) suggests the existence of a «boundary» at $\pm \infty$, insofar as only finitely many powers $H^m$ are in the limit point case of Weyl - i.e., have a unique self-adjoint realization. Actually, it was shown in [C2], V, Theorem 4.4 that for $\beta \leq 1$ all powers of the minimal operator are essentially self-adjoint, while for large $\beta$ only $H$ itself has a unique self-adjoint realization (cf. also [CA], Theorem 1.6).

Here we focus on the case $\beta \geq \alpha + 1$. In the special case $\beta = 1$, $\alpha = 0$ the algebra proves to be identical with a well known algebra of [CH] (Theorem 2.1). For all other $\beta \geq \alpha + 1$ we prove the following result:

**Theorem 1.1.** The algebra $A$ contains the ideal $K(H)$ of compact operators. Commutators in $A$ are compact; we have $A^\vee = A/K$ commutative.

Moreover, $A^\vee = C(M)$, where the space $M$ is a rectangle with sides $I_{-\infty} = \{-\infty \leq t \leq \infty\}$, $I_{\infty} = \{-\infty \leq t \leq \infty\}$, $I_+ = \{-\infty \leq x \leq \infty\}$, $I_- = \{-\infty \leq x \leq \infty\}$, with endpoints identified as in Fig. 1.1.
The symbol (i.e., the map $\sigma : A \to A^v \to C(M)$) of the generators is given as follows. For $a \in C([-\infty, \infty])$,

$$\sigma_{a(M)} = a(x) \text{ on } I_+ \cup I_-,$$

(1.2) continuous constant $= a(\pm \infty)$ on $I_{\pm \infty}$.

In case of $\alpha = \beta - 1 > 0$ we get

$$\sigma_{S_0} = \frac{\gamma(t)}{\beta - 1}, \sigma_{S_1} = \frac{i + i \beta - t}{2} \gamma(t) \text{ on } I_{\pm \infty},$$

(1.3) continuous constant on $I_{\pm}$, where the function $\gamma(t) \in C^\infty(\mathbb{R}) \cap CO(\mathbb{R})$ is explicitly given by

$$\gamma(t) = \frac{2}{\sqrt{\pi^3}} \int_0^\infty d\sigma \cosh \left(\left(1 + it\right)\frac{\sigma}{2}\right) \int_0^\infty d\tau \frac{\sinh q \tau}{\sinh q + 1 (\sigma + \tau)},$$

(1.4) with $q = \frac{1}{2} \left\{ \frac{1}{\beta - 1} \sqrt{(2\beta - 1)^2 + 4} - 1 \right\}$.

In case of $0 \leq \alpha < \beta - 1$ we get

$$\sigma_{S_0} = 0 \text{ on } \mathbb{M}, \sigma_{S_1} = \frac{t - i}{2} \gamma(t) \text{ on } I_{\pm \infty},$$

(1.5) continuous constant on $I_{\pm}$, with the function $\gamma(t)$ of (1.4), where now $q$ of the second line of (1.4) must be substituted by

$$q = \frac{1}{2} \frac{\beta}{\beta - 1} = \frac{1}{2} (1 + \eta).$$

(1.6)
In each case the symbols $\sigma_{S_j}$ are the complex conjugates of the symbols $\sigma_{S_j}$.

We shall discuss the proof of Theorem 1.1 in section 4, after extensive preparations. The function $\gamma(t)$ of (1.4) really is the Mellin transform of a certain limit of the integral kernel of the Sturm-Liouville transformed operator $S_1$. The appearance of the Mellin transform instead of the Fourier transform indeed seems to parallel the fact that boundary conditions are needed at $\pm \infty$.

We note the series of results of Sohrab [Sr] concerned with similar $C^*$-algebras of singular Sturm-Liouville or Schrödinger operators, where normally the condition $\nabla q = o(q)$ is imposed on the potential $q$. In case of $\beta < \alpha + 1$, discussed in [Tg1], our algebra transforms into an algebra of this type, even for higher dimensions. Our present case seems to be different. Especially, we expect noncompact commutators and an operator-valued symbol as in the algebras discussed by Arsenovic [Ar], Melo [Ml], Plamenewski [Plj], Rabinovic [Rb]. For discussion of other $C^*$-algebras and more general Banach algebras of similar nature see also [Go], [GKj], [Du], [Pw].

Our proof will be accomplished by relating the algebra $A$ to the algebras of [Tg2] and [CA] on $L^2$ of the half-line $\mathbb{R}_+$ by a Sturm-Liouville transform, a perturbation of the comparison expression, and a technique of [C2], VIII, called algebra surgery.

Surgery is used to «take apart» a comparison algebra, and examine the closed ideals corresponding to different endpoints of the interval separately; also relate such ideals between algebras living on different manifolds. This was used extensively in [C2], and also in [CDg] and [Tg1]. In all cases it amounts to a standard procedure, repeating the same conclusions.

The perturbation technique also was introduced in [C2] and used in [CDg]. However, in the present case this technique seems possible only by a rather delicate argument, discussed in section 3.

In section 2 we perform a Sturm-Liouville transform, for a modified algebra $S_\alpha$, relating this algebra to an algebra on the half-line. In section 4 the half-line algebra is related to the algebras of [Tg1] and [CA], using the perturbation of section 3. Finally, a surgery argument leads to the full description of the structure of $A$. Also, the case $\beta > \alpha + 1$ is reduced to $\beta = \alpha + 1$.

2. STURM-LIOUVILLE TRANSFORM

We first assume $\alpha = \beta + 1$. Instead of focusing on the expression $H$ of (1.1) we first will deal with a modified expression $M$. Both, $H$ and $M$ are defined on $\mathbb{R}$. For $\alpha = \beta - 1 = 0$ we set $M = H$. For $\alpha > 0$ we let $M = H$ only for $x \geq 0$. For $x \leq -1$ we set

$$M = \frac{4}{5} H = \frac{4}{5}(1 - \partial_x (x)^2 \partial_x),$$

with the standard abbreviation $\langle x \rangle = \sqrt{1 + x^2}$. In the interval $(-1, 0)$ we let $M$ interpolate between both expressions. For example, we set

$$M = -\partial_x p(x) \partial_x + q(x), \quad p = \frac{4}{5} \omega_-(x)^2 + \omega_+(x)^2 \beta^2, \quad q = \frac{4}{5} \omega_- + \omega_+ (x)^{2\beta - 2},$$

(2.1)
with \( \omega_\pm \in C^\infty(\mathbb{R}) \), \( 0 \leq \omega_\pm \leq 1 \), \( \omega_+ + \omega_- = 1 \), \( \omega_\pm = 1 \) near \((0, \infty)\) and \((-\infty, -1)\), resp.

In \( H = L^2(\mathbb{R}) \) again we generate a \( C^* \)-algebra \( \mathcal{S} \), using the same multiplication as in section 1, but replacing \( S_j \), respectively, by

\[
U_0 = \left( \tau \omega_- + \omega_+ \langle x \rangle^\alpha \right) M^{-1/2}, \quad U_1 = -i \left( \tau \omega_- \langle x \rangle + \omega_+ \langle x \rangle^\beta \right) \partial_x M^{-1/2}, \quad \tau = \frac{2}{\sqrt{S}}.
\]

Our point is that the algebras \( \mathcal{A} \) and \( \mathcal{S} \) «coincide» over the subinterval \((0, \infty)\) of \( \mathbb{R} \), by algebra surgery of [C2], VIII. Similarly, for reason of symmetry, \( \mathcal{A} \) «coincides» with \( S_- S S_- \) over \((-\infty, 0)\), where \( S_- u(x) = u(x) \). Thus our task is reduced to the discussion of the structure of the algebra \( \mathcal{S} \), only «over the interval \((0, \infty)\)». We will work out details in section 4.

Generally, for a Sturm-Liouville expression of the form (2.1) one may change both dependent and independent variable by setting

\[
u = \gamma v, \quad \gamma = p^{-1/4}, \quad s(x) = \int_x^\infty \frac{dt}{\sqrt{p(t)}}.
\]

This change (from \( u \) and \( x \) to \( v \) and \( s \), resp.) is meaningful only if the integral exists. For our expression we have \( p = \langle x \rangle^{2\beta}, \quad \frac{1}{\sqrt{p}} = \langle x \rangle^{-\beta} \), as \( x \to 0 \). The latter is integrable at \( \infty \) if \( \beta = \alpha - 1 > 0 \). For the left- over case \( \beta = 1, \quad \alpha = 0 \) we replace the integral in (2.3) by \( \int_0^x \).

For \( \beta > 1 \) the function \( s(x) \) is decreasing from \(+\infty\) to 0, as \(-\infty < x < \infty\). Its inverse function decreases from \( \infty \) to \(-\infty\), providing a diffeomorphism \((-\infty, \infty) \leftrightarrow (0, \infty)\). Then (2.3) provides a unitary map \( U = u(\mathcal{H}) \), where \( U : \mathcal{H} \to \mathcal{H}_+ = L^2(\mathbb{R}_+) \), by \( v(s) = u(x(s))/\gamma(x(s)) \). Indeed, we have \( \int_0^\infty |v(s)|^2 ds = \int_{-\infty}^\infty |u(x)|^2 dx \), by an integral substitution.

For \( \beta = 1 \) we explicitly get \( s(x) = \log(x + \langle x \rangle) \), a diffeomorphism \( \mathbb{R} \leftrightarrow \mathbb{R} \), and \( U \), defined as above, is a unitary map \( \mathcal{H} \to \mathcal{H} \). In either case executing the Sturm-Liouville transform amounts to conjugating all bounded or unbounded operators involved by \( U^* \).

The principal feature of (2.3), called Sturm-Liouville transform, is that the differential expression \( M \) above goes into

\[
M^\Delta = -\partial_s^2 + q^\Delta(s), \quad \text{where} \quad q^\Delta(s) = \left. (H\gamma/\gamma) \right|_{x=x(s)},
\]

as confirmed by a calculation ([C2], V, (5.2)). In particular, speaking in terms of self-adjoint operators, \( M^\Delta = U M U^* \) is the closure and the Friedrichs extension of the minimal operator
of $M^\Delta$ of (2.4), since the minimal domains, self-adjointness and positivity all correspond to each other, under conjugation by $U^\ast$.

For $1 \leq \beta < \infty$ we have $p = \langle x \rangle^{2\beta}$, $q = \langle x \rangle^{2\beta - 2}$, in $x \geq 0$, hence

$$q^\Delta(s(x)) = \left\{ \langle x \rangle^{2\beta - 2} - \langle x \rangle^{\beta/2} \partial_x \left( \langle x \rangle^{2\beta} \partial_x \langle x \rangle^{-\beta/2} \right) \right\}, \quad x \geq 0. \tag{2.5}$$

Formulas (2.5) holds for $\beta = 1$ and $\beta > 1$, but $s(x)$ has different features in either case. Important for us will be the behaviour of $q^\Delta$ near the endpoint 0 (for $\beta > 0$) or $\infty$ (for $\beta = 0$).

In the following let $P(t) = 1 + a_1 t + \ldots$ denote any power series in $t$ convergent near $t = 0$, and with constant term 1, where we will not distinguish between two such series. For example,

$$\langle t \rangle^{-\beta} = (1 + t^2)^{-\beta/2} = t^{-\beta} \left( 1 + \frac{1}{t^2} \right)^{-\beta/2} = t^{-\beta} p \left( \frac{1}{t^2} \right) \quad \text{near } t = \infty. \tag{2.6}$$

Integrating this between $x$ and $\infty$ we get (in case of $\beta > 1$)

$$s(x) = \frac{1}{\beta - 1} x^{1 - \beta} p \left( \frac{1}{x^2} \right), \quad \text{for large } x. \tag{2.7}$$

To invert this relation let $z = 1/x$, $\eta = \frac{1}{\beta - 1}$, so that (2.6) yields $s = \eta z^{1/\eta} P(z^2)$. Or, $\eta^{-\eta} s^n = z P(z^2)$, $z = \eta^{-\eta} s^n P(z^2)$. Or,

$$x(s) = \frac{1}{\varepsilon} s^{-\eta} P(s^2 \eta), \quad \varepsilon = (\beta - 1)^{1/(\beta - 1)}, \quad 0 < s < s_0, \tag{2.7}$$

for small $s_0$.

For the function $q^\Delta$ of (2.5) we get

$$q^\Delta(x(s)) = x^{2\beta - 2} P(x^2) - x^{\beta/2} P(x^2) \partial_x \left( x^{2\beta} P(x^2) \partial_x \left( x^{-\beta/2} P(x^2) \right) \right) = \frac{1}{4} \left( 3 \beta^2 - 2 \beta + 4 \right) x^{2\beta - 2} P(x^2), \tag{2.8}$$

where we used the rule of differentiation $\partial_x(x^\theta P(x^2)) = \theta x^{\theta - 1} P(x^2)$, as $\theta \neq 0$, and that $P(x^2) P(x^2) = P(x^2)^2$. Thus (2.7) and (2.8) imply

$$q^\Delta(s) = \frac{3 \beta^2 - 2 \beta + 4}{4(\beta - 1)^2} \frac{1}{s^2} P(s^2 \eta), \quad \eta = \frac{1}{\beta - 1}, \quad 0 < s \leq s_0, \quad \text{as } \beta > 1. \tag{2.9}$$
In case of $\beta = \alpha + 1 = 1$ we get

$$M^\Delta = H^\Delta = -\partial_s^2 + q^\Delta(s), \quad q^\Delta(s) = \frac{5}{4} P \left( \frac{1}{z(s)^2} \right), \quad |s| > s_0. \tag{2.10}$$

Here $e^{\beta z} = |z| (1 + \sqrt{1 + z^2}) = 2 |z| P(z^2)$. Or, $|z| P(z^2) = 2 e^{-|z|}$, i.e., $z^2 = 4 e^{-2s} P(e^{-2s})$. In other words, with a constant $b$, we get

$$M^\Delta = H^\Delta = -\partial_s^2 + q^\Delta(s), \quad q^\Delta(s) = \frac{5}{4} + b e^{-2s} P(e^{-2s}), \quad s \geq s_0. \tag{2.11}$$

This result at once gives a complete answer, regarding the structure of $A_\beta$, in case of $\alpha = 0$, $\beta = 1$. A calculation shows that

$$US_0 U^* = \Lambda^\Delta = H^{\Delta-1/2}, \quad US_1 U^* = -i (\partial_s + e(s)) \Lambda^\Delta, \tag{2.12}$$

where $e(s) = -\frac{1}{2} \frac{2}{z(s)} \in C([-\infty, \infty])$. Thus $A^\Delta = UA^\Delta U^*$ is generated by $\Lambda^\Delta$, $-i\partial_s \Lambda^\Delta$ (and the multipliers in $C([-\infty, \infty])$) as well.

**Theorem 2.1.** In the case $\alpha = 0$, $\beta = 1$ the algebra $UAU^*$ is identical with the algebra $S$ of [CH], theorem 36, in the special case $\eta = 1$ (cf. also the problems of IV, section 1, where the algebra is called $\mathfrak{S}$).

**Proof.** Cf. [CDg], Theorem 1.1, dealing again with the algebra $S = \mathfrak{S}$, where it is shown that, originally generated by $a(x) \in C([-\infty, \infty])$, $(1 - \partial_x^2)^{-1/2}, \partial_x (1 - \partial_x^2)^{-1/2}$, may also be generated replacing $1 - \partial_x^2$ by a perturbed expression $1 - \partial_x^2 + \tau(x)$. Perturbations allowed there include the above as a special case. (We will discuss a similar more sophisticated result in section 3, below.) Before applying Theorem 1.1, we must conjugate with a suitable dilation $u(x) \to u(\delta x)$, to make the constant term of the expansion of $q^\Delta(s)$ at $\pm \infty$ equal to 1. Such a dilation leaves the algebra $S = \mathfrak{S}$ invariant.

From now on we consider only the case $\beta > 1$ (right now, $\alpha = \beta - 1$). We then have (2.9) near $s = 0$. For large $s$ we essentially will get the expression $H^\Delta$, with $\alpha = 0$, by our construction of $M$. In detail, for $x < -1$ we have $s(x) = \frac{1}{\tau} \int_x^0 \frac{dz}{\xi(x)} + c_0$, $c_0 = s(-1) - \frac{1}{\tau} \int_{-1}^0 \frac{dz}{\xi(x)}$, $\tau = \frac{2}{\sqrt{s}}$. Or, $s(x) = c_0 + \frac{1}{\tau} \log(|x| + \xi(x))$, $s < -1$. This function may be inverted as above. A calculation yields (with a constant $b_0$

$$q^\Delta(s) = 1 + b_0 e^{-2\tau(s-\alpha)} P(e^{-2\tau(s-\alpha)}), \quad as \ s \geq s_0. \tag{2.13}$$

We summarize:
Proposition 2.2. Let \( 0 < \alpha = \beta - 1 \). Then the algebra \( S^\Delta = USU^* \subset L(H_+) \) is generated by all multiplications with functions \( b(x) \in C([0, \infty)) \), and the operators

\[
V_0 = \left( 1 + \frac{1}{x} \right) M^\Delta_{-1/2}, \quad V_1 = -i \partial_x M^\Delta_{-1/2},
\]

(and their adjoints), where we call the variable \( x \) again and where

\[
M^\Delta = -\partial_x^2 + q^\Delta(x), \quad q^\Delta(x) \in C^\infty(R_+),
\]

with the asymptotic behaviour of \( q^\Delta \) near 0 and \( \infty \) determined by (2.9) and (2.13), respectively.

Indeed, we get

\[
U_0^\Delta = \lambda(s) M^\Delta_{-1/2}, \quad U_1^\Delta = -i \left( \mu(s) \partial_x + \nu(s) \right) M^\Delta_{-1/2}, \quad \text{with}
\]

\[
\lambda(s) = \left( \tau \omega_- + \omega_+ (x)^{\beta-1} \right) |_{x=x(s)}, \quad \mu(s) = -\frac{\tau \omega_- (x) + \omega_+ (x)^{\beta}}{\sqrt{\tau^2 \omega_- (x)^2 + \omega_+ (x)^{2 \beta}}} |_{x=x(s)},
\]

\[
\nu(s) = -\frac{1}{4} (\tau \omega_- (x) + \omega_+ (x)^{\beta}) \left( \frac{p'}{p} \right) |_{x=x(s)}, \quad p(x) \text{ as in (2.1)}.
\]

Clearly \( \lambda(s), -\mu(s) \) are positive and \( C^\infty(R_+) \). We get \( \mu(s) = -1 \) near 0 and \( \mu = -1 \) near \( \infty \), and \( \lambda = \tau \) near \( \infty \). On the other hand,

\[
U(x)^\theta U^* = (x)^\theta |_{x=x(s)} = \varepsilon^{-\theta} s^{-\theta \eta} P(s^{2 \eta}), \quad \varepsilon, \eta \text{ of (2.7)},
\]

applied for \( \theta = \beta - 1 \), shows that we have \( \lambda(s) = \frac{1}{\beta-1} s^{1/2} P(s^{2 \eta}) \) for small \( s \). We have \( \nu(s) = -\frac{\tau}{4} (x)^{\beta} \left( \frac{p'}{p} \right) = -\frac{\tau}{4} \left( \frac{x}{s} \right) \) near \( x = -\infty \), so \( \nu(s) \approx \frac{\tau}{2} \) near \( s = \infty \). Also, \( \nu(s) = -\frac{1}{4} (x)^{\beta} \left( \frac{(1+x^2)^{\beta}}{(1+x^2)^{\beta}} \right) \) near \( x = \infty \).

In view of the fact that all multiplications by functions in \( C([0, \infty)) \) are among the generators, it is then evident that we may replace the generators \( U_j^\Delta \) by the operators (2.15).

We will need later that, near \( x = 0 \), we have

\[
U_0^\Delta = \left( \frac{1}{\beta-1} + o(1) \right) \left( \frac{1}{x} M^\Delta_{-1/2} \right),
\]

\[
U_1^\Delta = (-1 + o(1)) \left( \frac{1}{i} \partial_x M^\Delta_{-1/2} \right) + \left( \frac{\beta}{2} + o(1) \right) \left( \frac{1}{x} M^\Delta_{-1/2} \right),
\]
as was just verified.

We finally indicate the changes to be made in our above discussion, if \(0 \leq \alpha < \beta - 1\). Clearly the Sturm-Liouville transform is independent of the choice of \(\alpha\); it only involves \(\beta\). Hence \(M^\Lambda\) is of the form (2.11) again, on the halfline \(\mathbb{R}_+\). However, the potential \(q^\Lambda\) near \(s = 0\) now is of the form

\[
q^\Lambda(s) = \frac{(3\beta - 2)\beta}{4(\beta - 1)^2} \frac{1}{s^2} P(s^2\eta) + \frac{1}{(\beta - 1)^2 - 2i} \frac{1}{s^2 - 2i} P(s^2\eta), \quad \iota = \frac{\beta - 1 - \alpha}{\beta - 1},
\]

\[
\frac{(3\beta - 2)\beta}{4(\beta - 1)^2} \frac{1}{s^2} \{1 + (bs^2\eta + cs^2i) P(s^2\eta)\},
\]

with constants \(b, c\), while there is no change in \(q^\Lambda\) near \(\infty\).

Also, the generator \(V_0\) of (2.14) now must be replaced by

\[
W_0 = (1 + x^{\iota - 1}) M^{\Lambda - 1/2}, \quad \iota \text{ of (2.19)}.
\]

### 3. A PERTURBATION OF THE COMPARISON EXPRESSION

Let us again work in \(H_+ = L^2(\mathbb{R}_+)\). Consider the two expressions

\[
H = -\partial_x^2 + \frac{\kappa}{x^2} + 1, \quad K = H + p(x), \quad 0 < x < \infty,
\]

where \(p \in C^\infty(\mathbb{R}_+)\) with \(p^{(k)}(x) = x^{k-2} - k \chi_k(x), \chi_k = 0(1), k = 1, 2, \ldots\). Assume that \(\kappa > 0\) is a given constant, and that the «perturbation» \(p(x)\) does not destroy the positivity property of \(H\). That is, we assume that still \(K \geq c_0 > 0\), or,

\[
(\omega, K \omega) \geq c_0 (\omega, \omega), \quad \omega \in C_0^\infty(\mathbb{R}_+).
\]

Denote by \(T_H\) and \(T_K\) the comparison algebras, generated as \(C^\ast\)-subalgebras of \(L(H_+)\) by the multipliers \(a(M), a \in C([0, \infty]),\) and the pair of operators \(\frac{1}{x} H^{-1/2}, -i\partial_x H^{-1/2}\) (for the algebra \(T_H\)), and \(\frac{1}{x} K^{-1/2}, -i\partial_x K^{-1/2}\) (for \(T_K\)), respectively.

**Theorem 3.1.** Assume that \(\nu^2 = \kappa + \frac{1}{4} > 1\). Then we have \(T_H = T_K\). Moreover, the operators \(S = K^{1/2} H^{-1/2}\) and \(S^{-1} = H^{1/2} K^{-1/2}\) are in this algebra \(T\), and are both of the form \(1 + C, C \in K(H_+)\).

**Proof.** Let us first give a survey of this proof. We start with examining the operator \(K H^{-1} = 1 + p H^{-1}\), and (i) will show that \(p H^{-1} \in K(H_+)\). It is trivial that

\[
\|K^{1/2} u\|^2 = (u, K u) \leq c (u, H u) = c \|H^{1/2} u\|^2, \quad u \in C_0^\infty,
\]
which implies that \( S = K^{1/2} H^{-1/2} \in L(H_+) \). In fact, the well known result of E. Heinz and K. Loewner (cf. [C2], 1.5) implies that \( K^s H^{-s} \in L(H_+) \) for all \( 0 \leq s \leq \frac{1}{2} \). We then will show that (ii) \( K^{1/2} H^{-1/2} \) is hermitian mod \( K(H_+) \), and (iii) \( (K^{1/4} H^{-1/4})^2 = K^{1/2} H^{-1/2} \) (mod \( K \)), where \( K^{1/4} H^{-1/4} \) also is hermitian (mod \( K \)), so that the coset mod \( K \) of \( K^{1/2} H^{-1/2} \) is positive hermitian. Finally (iv) we show that \( S^2 = (K^{1/2} H^{-1/2})^2 = KH^{-1} \) (mod \( K \)). All together it then follows that the coset \( S^V = S + K \) must be the positive square root of \( (KH^{-1})^V = I^V \). By uniqueness of the positive square root it then follows that \( S = 1 + C, C \in K \). Next (v) we conclude that \( S^{-1} \) exists. Then, of course, it also must be of the form \( 1 + C, C \in K \). However, then it follows that the generators of \( T_H \) are contained in \( T_K \) (and vice versa), since every comparison algebra contains \( K(H) \) and since \( \frac{1}{2} H^{-1/2} = \frac{1}{2} K^{-1/2} S = (\frac{1}{2} K^{-1/2})(1 + C), C \in K \), etc. Q.E.D.

To complete this program we start with (v): This follows if the converse of (3.3) can be established. However, (3.2) implies \( (N + 1)(u, Ku) \geq Nc_0(u, u) + \| u' \|^2 + +\kappa \| \frac{u}{x} \|^2 - (|p|u, u) \), where we may use the estimate \( |p| \leq \frac{d}{x^2} + c(\delta) \), valid for all \( \delta > 0 \) with suitable \( c(\delta) \), for \( (N + 1)(u, Ku) \geq (Nc_0 - c(\kappa + \frac{1}{4})) \| u \|^2 \geq \| u \|^2 \) for \( N = (1 + c(\kappa + \frac{1}{4})) / c_0 \). Thus,

\[
(3.4) \quad (u, Hu) \leq c(u, Ku), \text{ for all } u \in C_0^\infty(\mathbb{R}_+), \quad c = \frac{1}{N + 1},
\]

and this implies existence of \( S^{-1} = H^{1/2} K^{-1/2} \). Next we prove (i).

**Proposition 3.2.** The operators \( U = \frac{1}{x^2} H^{-1} \) and \( V = \frac{1}{x} \partial_x H^{-1} \) are bounded. Moreover, the same is true for \( \frac{1}{x^2} R(\lambda) \) and \( \frac{1}{x} \partial_x R(\lambda) \) whenever \( \lambda > 0 \), where we have set \( R(\lambda) = \lambda^{-1} \), and then we have the estimates

\[
(3.5) \quad \| \frac{1}{x^2} R(\lambda) \| \leq c, \quad \| \frac{1}{x} \partial_x R(\lambda) \| \leq c, \quad 0 \leq \lambda < \infty,
\]

with \( c \) independent of \( \lambda \).

Also, the operators \( b(M)U \) and \( b(M)V \) are compact whenever \( b \in C(\mathbb{R}_+) \), \( b(0) = b(\infty) = 0 \).

**Proof.** We recall from [CA], (3.7), (3.8) that the resolvent \( R(1) = (H + \lambda)^{-1} \) is an integral operator with kernel \( G_\lambda(x, y) \) expressible in terms of Bessel functions, i.e.,

\[
(3.6) \quad G_\lambda(x, y) = G_\lambda(y, x) = -\sqrt{xy}K_\nu(x\sqrt{1+\lambda})I_\nu(y\sqrt{1+\lambda}), \text{ as } y < x.
\]
First set $\lambda = 0$. It suffices to show boundedness of the integral operator with kernel
\[
\frac{1}{x^2} G_0(x, y) \in L^2((0, 1)).
\]
Indeed, for any partition $1 = \chi + \omega$, $\chi \in C_0^\infty([0, \infty))$, $\chi = 1$ near 0, write $Z = \frac{1}{x^2} R(\lambda) = \chi Z \chi + \omega Z \omega + \chi \omega^0 Z \omega + \omega Z \omega^0 \chi + C$, $C \in K(H)$, where $\omega^0 = \omega$, $\omega^0 = 0$ near 0, and where we used Proposition 1.2 to commute (mod $K(H)$) $\omega^0$ and $R(\lambda)$. Only the first term $\chi Z \chi$ needs consideration, since all others are trivially bounded, the functions $\frac{\omega}{x}$ or $\omega^0/x^2$ being bounded at 0. Since $\chi$ has compact support, the operator $\chi Z \chi$ only involves a bounded interval $[0, a]$, where we may assume $a = 1$.

In $0 < x \leq 1$ we may estimate

\[
K_\nu(x) = o \left( |x|^{-\nu} \right), \quad I_\nu(x) = o \left( |x|^{\nu} \right),
\]

hence

\[
\frac{1}{x^2} G_0(x, y) = \frac{\sqrt{y}}{\sqrt{x^3}} O \left( \min \left\{ \left| \frac{y}{x} \right|, \left| \frac{x}{y} \right| \right\} \right).
\]

Thus we get $\| \frac{1}{x^2} \int_0^1 dy G_0(x, y) u(y) dy \|_{L^2((0, 1))}^2 \leq c \| T_\psi u \|^2$, with the Mellin convolution

\[
T_\psi u(x) = \frac{1}{2\sqrt{\pi}} \int_0^\infty \psi \left( \frac{x}{y} \right) u(y) \frac{dy}{y}, \quad \psi(t) = t^{-3/2} \min \left\{ t^\nu, t^{-\nu} \right\}.
\]

The function $\psi$ is $L^1(\mathbb{R}_+, \frac{dx}{x^2})$, as $\nu > 1$, since $\int_0^\infty \frac{dt}{\sqrt{t}} \psi(t) = \int_0^1 t^{\nu-2} dt + \int_1^\infty t^{-2-\nu} dt < \infty$. Thus we indeed have $\frac{1}{x^2} H^{-1} = \frac{1}{x^2} R(0) \in L(H)$, as $\nu > 1$.

**Remark.** It is interesting to note that this operator is no longer bounded for $\nu \leq 1$.

The estimates (3.7) remain true if differentiated, assuming $\nu > 1$ again. Therefore the same conclusion leads to boundedness of the operator $\frac{1}{x^2} \partial_x H^{-1}$.

For general $\lambda$ we conclude a bound independent of $\lambda$ by a scaling argument: For $\eta > 0$ define the (unitary) dilation operator $J_\eta : H_* \to H_*$ by setting $J_\eta u(x) = \sqrt{\eta} u(\eta x)$. From (3.1) we conclude that

\[
J_\eta^* H J_\eta u = \eta^2 (H + \lambda) u, \quad \lambda = \frac{1}{\eta^2} - 1, \quad u \in C_0^\infty([0, \infty)),
\]

and this relation holds for our realization $H$ as well. For any $\lambda \geq 0$ and $\eta = \frac{1}{\sqrt{\lambda + 1}}$ we thus get

\[
R(\lambda) = \frac{1}{\lambda + 1} J_\eta^* R(0) J_\eta, \quad \lambda \geq 0.
\]
On the other hand, we have \( J_\eta \frac{1}{x} J_\eta = \frac{1}{\lambda + 1} \frac{1}{x^2} \), \( J_\eta \frac{1}{x} \partial_x J_\eta = \frac{1}{\lambda + 1} \frac{1}{x} \partial_x \), so that

\[
\frac{1}{x^2} R(\lambda) = J_\eta \frac{1}{x^2} R(0) J_\eta,
\]
\[
\frac{1}{x} \partial_x R(\lambda) = J_\eta \frac{1}{x} \partial_x R(0) J_\eta.
\]

Since \( J_\eta \) is unitary and we have proven boundedness of \( \frac{1}{x^2} H^{-1} \), and \( \frac{1}{x} \partial_x H^{-1} \) the uniform boundedness (3.5) follows.

Finally, regarding compactness of \( \frac{b(x)}{x^2} H^{-1} = Y \), write \( Y = b(x) (\frac{1}{x^2} H^{-1}) \), where the second factor is bounded, as just seen, while \( b(x) \) vanishes near 0 and \( \infty \), hence is uniform limit of a sequence \( b_j(x) \in C_0^\infty((0, \infty)) \). Notice that \( b_j(\frac{1}{x^2} H^{-1}) = (b_j/x^2) H^{-1} \) is compact, since \( b_j/x^2 \in C_0 \). Also \( \| Y - b_j(\frac{1}{x^2} H^{-1}) \| \to 0 \), so that compactness of \( Y \) follows. Similarly one proves compactness of \( \frac{b(x)}{x} \partial_x H^{-1} \). Q.E.D.

Note that (i) is proven as well.

We are left with proving (ii), (iii) and (iv). These involve the operators \( V_s = K^s H^{-s} \) and their cosets \( U_s = V_s + K(H) \), for various \( s \), notably \( s = \frac{1}{2} \) and \( s = \frac{1}{4} \). We need \( U_s^* = U_s \), and the semi-group property \( U_s^2 = U_{2s} \), for \( s = \frac{1}{4} \) and \( s = \frac{1}{2} \). The first amounts to compactness of

\[
(3.13) \quad K^s H^{-s} = H^{-s} K^s = K^s [H^{-s}, K^{-s}] K^s.
\]

The second amounts to compactness of

\[
(3.14) \quad K^{2s} H^{-2s} - K^s H^{-s} K^s H^{-s} = \{ K^{2s} [H^{-s}, K^{-s}] \} (K^s H^{-s}).
\]

To control the commutator \([H^{-s}, K^{-s}]\) we involve the resolvent integrals

\[
(3.15) \quad H^{-s} = \frac{\sin \pi \sigma}{\pi} \int_0^\infty \frac{d \lambda}{\lambda^s} R(\lambda), \quad K^{-s} = \frac{\sin \pi \sigma}{\pi} \int_0^\infty \frac{d \lambda}{\lambda^s} S(\lambda), \quad S(\lambda) = (K + \lambda)^{-1}.
\]

We get

\[
(3.16) \quad [H^{-s}, K^{-s}] = \frac{\sin^2 \pi \sigma}{\pi^2} \int_0^\infty \int_0^\infty \frac{d \lambda d u}{(\lambda \mu)^s} S(\mu) R(\lambda) [H, K] R(\lambda) S(\mu).
\]

Actually, we first verify (3.15) for all \( u \in \tilde{D} = H_+ \cap C_0(R_+) \), noting the fact that \( \tilde{D} \) is dense in \( H_+ \) while \( f = R(\lambda) S(\mu) u \in \tilde{D} \) has \( f \in \text{dom } H = \text{dom } K, H f \in \text{dom } K = \)
dom $H$, so that the expression $[H, K]f$ is well defined as an element in $\tilde{D}$ again. The fact that $f = RSu \in C^\infty$ again is a matter of hypo-ellipticity of $(K + \mu)(H + \lambda)$. It of course may be proven directly, using the Greens function (36) of $H + \lambda$ (which turns out to be a parametrix of $K + \lambda$ as well).

Let us look at the commutator

$$
(3.17) \quad [H, K] = [H, (H + p)] = [H, p] = -[\partial_p^2, p] = -2p'\partial_p - p''.
$$

Examining $p'$ and $p''$, for $p = \chi(x) x^{\varepsilon - 2}$, we find that expressions of the form $\chi^{(x)}_{x^{\varepsilon - 2}}$ and $\chi^{(2)}_x \partial_x$ occur, where $\chi$ has the properties of $\chi$ above, but need not be the same function. In view of Proposition 3.2 it is clear that the expression $R(\lambda) [H, K] R(\lambda) = I(\lambda)$ is not only bounded, but even compact, as a map $H_+ \rightarrow H_+$. For (3.13) and (3.14) we need compactness of $K^S(\mu) I(\lambda) S(\mu) K^S = X$ and $K^{2S}(\mu) I(\lambda) S(\mu) = Y$ respectively. Also the functions $X(\lambda, \mu), Y(\lambda, \mu)$ should be seen norm continuous and integrable (after division with $(\lambda \mu)^{-\varepsilon}$).

Clearly $S(\mu)$ and $K^S(\mu)$ and $K^{2S}(\mu)$ are bounded so that $X$ and $Y$ are compact, due to compactness of $I(\lambda)$. Norm continuity of the two functions of $\lambda$ and $\mu$ is no problem: write $X(\lambda, \mu) = (K^{2S}(\mu)) (H R(\lambda)) (H^{-1}[H, K] H^{-1}) (H R(\lambda)) (K^S(\mu))$ each of these factors is either bounded and constant or norm continuous, in view of the resolvent formula $H R(\lambda) - H R(\lambda') = (\lambda' - \lambda) H R(\lambda) R(\lambda')$. Similarly for $Y(\lambda, \mu)$.

Regarding integrability of $Y(\lambda, \mu)$: we need only $s = \frac{1}{2}$ and $s = \frac{1}{2}$. For $s = \frac{1}{2}$ write $Y = (K S(\mu)) (R(\lambda) [H, K] R(\lambda) R(\mu)) (1 + p R(\mu))^{-1} = F_1 F_2 F_3$. Here $F_1 = O(1)$, and $F_3 = O(1)$ as well. Indeed, we get $\| p R(\lambda) u \| = \| x^{\varepsilon - 2} \chi(x) R(\lambda) u \| \leq \frac{\varepsilon}{(1 + \lambda)^3} \| J^*_f (x^\varepsilon \chi(x)) \frac{1}{\partial x} H^{-1} J_f u \| \leq \frac{\varepsilon}{(1 + \lambda)^3} \| u \|$, with $\zeta = \varepsilon/2$, hence $\| p R(\lambda) \| \rightarrow 0$, as $\lambda \rightarrow \infty$. Or,

$$
(3.18) \quad \| p R(\lambda) \| = o\left((1 + \lambda)^{-\varepsilon/2}\right), \quad as \quad 0 \leq \lambda < \infty.
$$

In particular we get $\| p R(\lambda) \| < 1$ for large $\lambda$ so that the inverse $(1 + p R(\mu))^{-1}$ exists and is bounded for $\mu \geq \mu_0$, for some $\mu_0$. Thus $F_3 = o(1)$, since we may restrict the argument to large $\mu$. Regarding the factor $F_2$ with (3.18) we now get

$$
(3.19) \quad F_2 = O\left(\text{Min}\left\{\frac{(1 + \lambda)^{-\zeta}}{1 + \mu}, \frac{(1 + \mu)^{-\zeta}}{1 + \lambda}\right\}\right).
$$

One checks easily that $\frac{\lambda}{(\lambda \mu)^{2\varepsilon}}$ is integrable on $\mathbb{R}_+ \times \mathbb{R}_+$. 

For $s = \frac{1}{4}$ we get $F_1 = O(1 + \mu)^{-1/2}$ and the same estimates as before for $F_2$ and $F_3$. It follows that

$$\frac{Y}{(\lambda \mu)^{1/4}} = \frac{1}{(\lambda \mu)^{1/4}} \frac{1}{1 + \mu} O \left( \min \left\{ \frac{(1 + \lambda)^{-\zeta}}{1 + \mu}, \frac{(1 + \mu)^{-\zeta}}{1 + \lambda} \right\} \right),$$

which is integrable as well.

Now we write

$$X = G_1 G_2 G_3 G_4 G_5, \quad \text{with} \quad G_1 = G_5 = H^s R(\mu)^{1/2},$$

$$G_2 = G_4 = R(\mu)^{-1/2} S(\mu) R(\mu)^{-1/2}, \quad G_3 = R(\mu)^{1/2} R(\lambda) [H, K] R(\lambda) R(\mu)^{1/2}.$$

Clearly $G_1 = G_2$ are bounded, and $G_2 = G_4$ is bounded as well: combining (3.3) and (3.4) we get $c(u, Hu) \leq (u, Ku) \leq C(u, Hu)$, where one may assume $c \leq 1$, $C \leq 1$, so that also

$$c(u, (H + \lambda)u) \leq (u, (K + \lambda)u) \leq C(u, (H + \lambda)u), \quad u \in C_0^\infty (\mathbb{R}_+).$$

This implies $(H + \lambda)^{1/2} (K + \lambda)^{-1/2} = o(1)$, $(K + \lambda)^{1/2} (H + \lambda)^{-1/2} = o(1)$, so that $G_2 = G_4$ are bounded.

To control $G_3$ we write $(H + \lambda)(H + \mu) = L^2 + (2 + \lambda + \mu)L + (1 + \lambda)(1 + \mu) \geq (L + \sqrt{(1 + \lambda)(1 + \mu)})^2$, where $L = H - 1 = -\frac{\partial^2}{\partial x^2} + \frac{\zeta}{x}$, $\zeta \geq 0$. It follows that

$$G_3 = O \left( \min \left\{ \frac{(1 + \lambda)(1 + \mu)^{-\zeta}}{1 + \lambda}, \frac{(1 + \lambda)^{-\zeta}}{1 + \mu} \right\} \right).$$

Substituting this we again get convergence of the integral for $s = \frac{1}{4}$, and for $s = \frac{1}{4}$ as well if we still observe that then $G_1 = G_2 = O((1 + \mu)^{-1/4})$.

4. $S^\Delta$ and the half-line algebra; proof of Theorem 1.1

In this section we first will prove
Theorem 4.1. For every \( \beta = \alpha + 1 > 1 \) the algebra \( S^\Delta = USU^* \) coincides with the half-line’s Laplace comparison algebra \( P \) of [C1], V, 4, i.e., with the algebra \( C \) of [Tg2], or with \( A \) of [CA], section 3. Moreover, in the representation of the symbol space \( M(P) \) used in [CA] and described in detail in [C1], V, 7 (fig. 7.5) the symbols of the generators \( b(x) \), and the operators (2.14) are given by

\[
\sigma_{b(x)} = b(x) \text{ on } I_3 \cup I_4, \quad \text{constant } = b(0) \text{ on } I_1, \quad b(\infty) \text{ on } I_2;
\]

\[
\sigma_{\gamma(t)} = \gamma(t) \text{ on } I_1, \quad \text{constant continuous on } I_3 \cup I_4,
\]

where

\[
\gamma(t) = \frac{2}{\sqrt{\pi^3}} \int_0^\infty d\sigma \cosh \left( (1 + it) \frac{\sigma}{2} \right) \int_0^\infty d\tau \frac{\sinh q\tau}{\sinh q^{1/2}(\sigma + \tau)}.
\]

Here \( q = \sqrt{\kappa + 1/4} - \frac{1}{2} = \frac{1}{2} \{ \frac{1}{\beta - 1} \sqrt{(2\beta - 1)^2 + 4 - 1} \}. \) The values of \( \sigma_{\gamma(t)} \) at \( I_4 \) are unimportant in the following (cf. [CA], (5.10)).

The proof of Theorem 4.1 is an immediate consequence of Theorem 3.1, combined with the statements of [CA], (5.10) concerning symbols. Just note that \( M^\Delta \) of section 2 can be written as

\[
M^\Delta = K = -\partial_x^2 + \frac{\kappa}{x^2} + 1 + p(x), \quad p(x) = q^\Delta(x) - 1 - \frac{\kappa}{x^2},
\]

where we let \( \kappa = \frac{3\beta^2 - 2\beta + 4}{4(\beta - 1)^2} \), and where \( p(x) \) satisfies the assumptions of (3.1) (we denote \( s \) by \( x \) again). Indeed, near \( x = 0 \) we use (2.9) for \( p(x) = \frac{\kappa}{x^2} \left\{ (P(x^2\eta) - 1) - \frac{1}{\kappa} x^2 \right\} = bx^2\eta - 2 P(x^2\eta) - 1 \), implying the estimates of (3.1) with \( \varepsilon = \min \{ 2\eta, 2 \} \). Near \( x = \infty \) we use (2.13) for \( p(x) = bx^{\varepsilon} P(e^{-2\varepsilon(x-c)}) - \frac{\kappa}{x^2} \). Thus we again get (3.1), for any \( \varepsilon > 0 \).

Thus the assumptions of Theorem 3.1 hold. (Observe that also \( \kappa + \frac{1}{4} = \frac{4\beta^2 - 4\beta + 5}{4(\beta - 1)^2} > 1 \). As a consequence of Theorem 3.1 and of [CA], Theorem 5.7 (together with the remark at the end of [CA]) it follows that \( P \) equals the algebra with generators \( \frac{1}{2} M^\Delta - 1/2, \ -i\partial_x M^\Delta - 1/2 \) (and the multipliers). It also follows at once that \( \frac{1}{2} M^\Delta - 1/2 \) may be replaced by \( (1 + \frac{1}{2} M^\Delta - 1/2, \) since the algebra clearly contains \( (1 + \frac{1}{2} M^\Delta - 1/2, \) and by inspection of symbols. This proves Theorem 4.1.

The proof of Theorem 1.1 now follows as a straight application of [C2], VIII, Theorem 3.3. Indeed, first of all, if we conjugate \( S^\Delta = P \) with the unitary map \( U \) of section 2 we get \( S = U^*PU \). Both algebras \( A \) and \( S \) live on \( R \), in \( H = L^2(R) \). The generating differential expressions \( U_0, \ U_1 \) of (2.2) coincides with those of \( A \) of section 1 near the subinterval \( [0, \infty) \subset R \). Thus by [C2], VIII, Theorem 3.3 symbol space and symbol of
both algebras agree over \([0, \infty)\). Similarly for the other interval \([-\infty, 0)\), as indicated in section 2. This established Theorem 1.1, as far as the case \(\beta = \alpha + 1 > 1\) is concerned. In the case \(\beta > \alpha + 1\) we will follow a similar course. Note first that now the asymptotic behaviour of \(q^A\) near \(x = 0\) is slightly different, insofar as the first term at right of (2.5) now reads \(\langle x \rangle^{2\alpha} = \langle x \rangle^{2(\alpha + 1 - \beta)} \langle x \rangle^{2\beta - 2}\), where the exponent \(\beta - 1 - \alpha > 0\). This term now is of lower order, compared to the other term. This leads to formula (2.19) instead of (2.13), near 0, accounting for the amended \(\gamma(x)\) of (1.5), (1.6). Secondly, we get \(S_0 = \langle x \rangle^{\alpha - 1 - \beta} (\langle x \rangle^{\beta - 1} H^{-1/2})\), so we must deal with a multiplication \(\langle x \rangle^{\alpha - 1 - \beta} \in C([0, \infty))\). As a consequence, the algebra \(A\) now is a subalgebra of \(A\) with parameters \(\alpha_0\), \(\beta_0 = \alpha_0 + 1\), where \(\beta_0\) is chosen such that the factors \(\kappa\) in (3.1) coincide. The generators of this subalgebra are \(a(M) : a \in C([-\infty, \infty])\), \(S_1\) and \(\langle x \rangle^{\alpha - 1 - \beta} S_0\). However, the subalgebra coincides with the entire algebra. To see this we require an application of the Stone-Weierstrass theorem similar to that at the end of [CA] which we will not discuss in detail.
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