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AN APPLICATION OF SPECTRAL CALCULUS TO THE PROBLEM OF SATURATION IN APPROXIMATION THEORY

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Dedicated to the memory of Professor Gottfried Köthe

Abstract. *Let $\mathscr{L} = (L_\alpha)_\alpha$ be a net of bounded linear operators on the Banach space E converging strongly to the identity on E. For a given complex-valued function f of a fixed type we consider the net $f(\mathscr{L}) := (f(L_\alpha))_\alpha$. Among other things we shall show that under reasonable conditions the saturation space of $\mathscr{L}$ with respect to a given net $\Phi = (\Phi_\alpha)$ of positive real numbers converging to zero is equal to that one of $f(\mathscr{L})$. More generally we consider nets $(f_\alpha(L_\alpha))$ where (f_α) is a net of complex-valued functions and we determine the saturation space of such a net in dependence of the saturation space of $\mathscr{L}$.*

1. INTRODUCTION

Let $\mathscr{L} = (L_\alpha)_{\alpha\in A}$ be a net of boudned linear operators on the Banach space E converging strongly to the identity on E. Then we call $\mathscr{L}$ an approximation processes.

Let $\Phi : A \times E \to \mathbb{R}_+$ be a function satisfying $\lim_\alpha \Phi(\alpha, x) = 0$ for every $x \in E$. Then the saturation space (or Favard-class) of $(\mathscr{L}, \Phi)$ is $S(\mathscr{L}, \Phi) = \{x \in E : \|L_\alpha x - x\| = 0(\Phi(\alpha, x))\}$. $\mathscr{L}$ is saturated with respect to Φ if the following two conditions are satisfied:

(i) If $\|L_\alpha x - x\| = o(\Phi(\alpha, x))$ then $L_\alpha x = x$ for all L_α;

(ii) There exists $x \in S(\mathscr{L}, \Phi)$ such that $L_\alpha x \neq x$ for all α.

For further information see e.g. [1, 4].

Now let L be a closed linear operator from $D(L) \subset E$ into E, and suppose that $D(L)$ is dense in E. Moreover assume that $\lim_\alpha((L_\alpha x - x)/\Phi(\alpha, x)) = Lx$ holds for all $x \in D(L)$.

Nishishiraho [4] proved (under the hypothesis that Φ is independent of x), that then for every fixed positive integer k $\lim_\alpha (L_\alpha^k x - x)/\Phi(\alpha, x) = kLx$ holds for all $x \in D(L)$.

Moreover under certain additional hypotheses he was able to prove

$$S((L_\alpha^k), \Phi) = S(\mathscr{L}, \Phi) = D(L)^E$$

where this latter space is defined as follows:

Let $\|x\|_L := \|x\| + \|Lx\|$ be the graph norm on $D(L)$ and denote by $B_L(0, t)$ the set $B_L(0, t) = \{x \in D(L) : \|x\|_L < t\}$. For a set $A \subset E$ we denote by $\overline{A}$ its closure in E w.r. to the given norm on E.

Then $D(L)^E = \bigcup_{t>0} \overline{B_L(0,t)}$.

In fact this set turns out to be a Banach space with respect to the norm $q(x) = \inf\{t > 0 : x \in \overline{B_L(0,t)}\}$.

It is the main aim of the present note to generalize the results of [4] to the case where the operator function $T \to T^k$ is replaced by a general holomorphic function $f(T)$ or (in the case of self-adjoint operators (L_α)) by a continuous function g differentiable at 1. Moreover we allow that Φ dependes also on x (at least in the basic proposition).

The idea behind our proofs is taken from nonstandard analysis, which enables us to replace a net of operators by one single (internal) operator on a larger space. In such a way it is obvious that one can apply spectral calculus to this single operator.

The paper is organized in the following manner: the second section contains the main additional notions and our main results. In the third section we replace the single function f by a sequence of functions (f_n) and study the rate of convergence of $(f_n(L_n))$. In the fourth section we present the ingredients from nonstandard functional analysis. We refer the reader to [2, 3, 5] for nice introductions into this field. The last two sections contain the proof of the results of sect. 2, sect. 3, resp.

2. THE RESULTS FOR A SINGLE FUNCTION f

In order to be able to formulate the results in their full generality let us introduce in addition to the notions of section 1 the following one:

Let $(\mathscr{A}, p)$ be a normed algebra of complex-valued functions on the compact subset X of $\mathbb{C}$ containing the polynomials as a dense subalgebra, and assume that point evaluations are continuous. As an example we mention the algebra $A(r)$ of all continuous functions on $K(r) = \{z \in C : |z| \leq r\}$ which are analytic in the interior of $K(r)$. Here we set $p(g) = \sup\{|g(z)| : |z| = r\}$. Another example might be the algebra $C(X)$ of all continuous functions on X. A third example might be the algebra of all C^k-functions on an intervall $X \in \mathbb{R}$.

2.1 Definition. *Let M be an arbitrary positive real number. A bounded linear operator T on E admits $(\mathscr{A}, M)$ spectral calculus if there is an algebra homomorphism φ of $\mathscr{A}$ into the algebra $B(E)$ of all bounded linear operators on E satisfying*

(i) $\varphi(1) = I$ *(identity on E)*

(ii) $\varphi(id_X) = T$

(iii) $\|\varphi(g)\| \leq M \cdot p(g)$

As usual we write $g(T)$ in place of $\varphi(g)$.

2.2 Definition. *The net $\mathscr{L} = (L_\alpha)$ admits $(\mathscr{A}, M)$ spectral calculus if every L_α does. We then write $g(\mathscr{L})$ in place of $(g(L_\alpha))$. (Note that the algebra homomorphisms φ_α depend on α!).*

2.3 Definition. *A function* $f \in \mathscr{A}$ *is* $\mathscr{A}$ *-differentiable at* $z \in X$ *if there exists another function* $g \in \mathscr{A}$ *and a constant* $c := f'(z)$ *satisfying*
(i) $g(z) = 0$
(ii) $f(u) = f(z) + (u - z)(c + g(u))$ *for all* $u \in X$

2.4 Remark. *If* $\mathscr{A} = A(r)$ *and* $|z| < r$ *or if* $\mathscr{A} = C(X)$ *then every function differentiable at* z *in the usual sense is* $\mathscr{A}$ *-differentiable.*

2.5 The main hypotheses:

The following hypotheses shall be satisfied from now on (unless stated otherwise explicitly):

(i) $\mathscr{L} = (L_\alpha)_{\alpha \in A}$ is a uniformly bounded net of linear operators converging strongly to I (the identity on E).

(ii) $\Phi : A \times E \to \mathbb{R}_+$ satisfies $\Phi(\alpha, x) \neq 0$ and $\lim_\alpha \Phi(\alpha, x) = 0$ for all x

(iii) $1 \in X, (\mathscr{A}, p)$ is a normed algebra of functions on X as above, and $\mathscr{L}$ admits $(\mathscr{A}, M)$ - spectral calculus for a certain $M > 0$.

(iv) $(L, D(L))$ is a closed densely defined linear operator in E such that $\lim_\alpha ((L_\alpha x - x))/\Phi(\alpha, x)) = Lx$ holds for all $x \in D(L)$.

(v) $f \in \mathscr{A}$ is a given function which is $\mathscr{A}$ -differentiable at 1.

Our first result generalizes the basic lemma on p. 275 of [4].

2.6 Proposition. *a)* $f(\mathscr{L})$ *converges strongly to* $f(1) \cdot I$.
b) $\lim_\alpha (f(L_\alpha)x - f(1)x)/\Phi(\alpha, x) = f'(1)Lx$ *for all* $x \in D(L)$.

For the sequel we need one further notion.

2.7 Definition. $(\mathscr{L}, L)$ *is called regularizable if there exists a uniformly bounded net* $\mathscr{R} = (R_\beta)$ *of linear operators on* E *satisfying*
(i) (R_β) *converges strongly to* I
(ii) $R_\beta L_\alpha = L_\alpha R_\beta$ *for all* α, β.
(iii) $R_\beta(E) \subset D(L)$
$\mathscr{R}$ *is called a regulator for* $(\mathscr{L}, L)$.

2.8 Remarks. *(i) If* L *is the infinitesimal generator of the strongly continuous semigroup* $\mathscr{L} = (L_t)_{t \geq 0}$ *then* $(\lambda(\lambda - L)^{-1})_{\lambda > \omega(L)}$ *is a regular for* $(\mathscr{L}, L)$. *Another one is*

$$c_t : x \to c_t x = \frac{1}{t} \int_0^t T_s x ds.$$

(ii) Let (f_α) *be an arbitrary net in* $\mathscr{A}$. *Then any regulartor of* $(\mathscr{L}, L)$ *is such one of* $((f_\alpha(L_\alpha))_\alpha, L)$. *This is obvious if all* f_α *are polynomials. But polynomials are dense in* $\mathscr{A}$, *so the crucial condition (ii) above holds obviously also for* $f_\alpha(L_\alpha)$.

Here is now our main result in this section:

2.9 Theorem. *Assume that* $(\mathscr{L}, L)$ *is regularizable, and moreover assume that* Φ *does not depend on* x. *Then the following three assertions are true:*
a) $\underline{\lim}_\alpha \|(f(L_\alpha)x - f(1)x/\Phi(\alpha) - f'(1)y\| = 0$ *holds iff* $x \in D(L)$ *and* $y = Lx$
b) Assume that $f(1) = 1$ *and* $f'(1) \neq 0$. *Then* $f(\mathscr{L})$ *is an approximation process and*

$$S(f(\mathscr{L}), \Phi) = S(\mathscr{L}, \Phi) = D(L)^E$$

or in other words: the saturation space of $\mathscr{L}$ *and* $f(\mathscr{L})$ *are the same and are equal to* $D(L)^E$.

2.10 Corollary. *Assume that* L *is the infinitesimal generator of the strongly continuous semigroup* $\mathscr{L} = (L_t)_{t\geq 0}$. *Moreover assume that* $f(1) = 1$ *and* $f'(1) \neq 0$. *Then* $f(\mathscr{L})$ *is saturated iff* $L \neq 0$.

A basic ingredient for the proof of this theorem is a slightly specialized version of theorem 1 in [4] which we formulate explicitly for the sake of convenience. The last assertion of our version follows from theorem 2 of [4].

2.11 Theorem. *(cf. [4], thm. 1). Assume that* $\mathscr{L}$ *and* L *satisfy (i) and (iv) of 2.5. Moreover assume that* $(\mathscr{L}, L)$ *is regularizable and that* Φ *does not depend on* x. *Then the following two assertions are true:*
(a) For $x, y \in E \underline{\lim}_\alpha \|(L_\alpha x - x)/\Phi(\alpha) - y\| = 0$ *iff* $x \in D(L)$ *and* $y = Lx$.
(b) $S(\mathscr{L}, \Phi) = D(L)^E$.
Moreover if L *is the infinitesimal generator of the strongly continuous semigroup* $\mathscr{L} = (L_t)_{t\geq 0}$ *then* $\mathscr{L}$ *is saturated with respect to* Φ *iff* $L \neq 0$.

3. SEQUENCES OF FUNCTIONS OF OPERATORS

In the following we adhere to the general hypotheses of 2.5 but in order to facilitate our considerations we restrict ourselves to sequences, i.e. we assume $A = \mathbb{N}$.

Let (f_n) be a sequence of functions in $\mathscr{A}$ which are $\mathscr{A}$-differentiable at 1. We assume that (f_n) satisfies the following two conditions:

(i) $f_n'(1) \neq 0$ and $\lim_n f_n'(1)\Phi(n, x) = 0$ holds for all $x \in E$

(ii) $\lim f_n(1) = 1$.

With these additional hypotheses we obtain:

3.1 Proposition. *(a) For all $x \in D(L)$ we have*

$$\underline{\lim}_{k,n\to\infty} \|(f_k(L_n)x - f_k(1)x)/(f_k'(1)\cdot\Phi(n,x)) - Lx\| = 0 .$$

(b) Consider the decomposition

$$f_k(u) = f_k(1) + f_k'(1)(u-1) + g_k(u)(u-1)$$

Assume that $g_k(u) = h_k(u)(u-1)$ where $h_k \in \mathscr{A}$ and

$$\|h_k(L_n))\| \le C|f_k'(1)| \qquad \text{for all} \qquad k,n.$$

Then for all $x \in D(L)$

$$\lim_{k,n\to\infty} \|(f_k(L_n)x - f_k(1)x)/(f_k'(1)\cdot\Phi(n,x)) - Lx\| = 0 .$$

3.2 Corollary. *Assume that in addition to the general hypotheses of this section the hypothesis of (b) of 3.1 above holds. Moreover assume that $\|f_k(L_n)\| \le D$ for all k,n and some fixed D. Then the following two assertions are true:*
(a) $(f_n(L_n)) = \mathscr{U}$ is an approximation process.
(b) Assume that $(\mathscr{L}, L)$ is regularizable and moreover that Φ does not depend on x. Consider the sequence $\Phi_k^{\sim} := f_k'(1)\Phi_k$. Then we have
(i) For $x,y \in E$ $\lim_n \|(f_n(L_n)x - f_n(1)x)/\Phi_n^{\sim} - y\| = 0$ iff $x \in D(L)$ and $y = Lx$.
(ii) $S(\mathscr{U}, \Phi^{\sim}) = D(L)^{\sim E}$.
(iii) Assume that L is the infinitesimal generator of the strongly continuous semigroup $\mathscr{L} = (L_t)_{t\ge 0}$. Then $(f_n(L_{\Phi_n}))$ is saturated with respect to $\Phi^{\sim}$ iff $L \ne 0$.

3.3 Remarks. *(i): $g_k(u) = h_k(u)(u-1)$ with $h_k \in \mathscr{A}$ (see 3.1 (b) above) holds if $\mathscr{A} = A(r)$ for some $r > 0$ or if the following three conditions are satisfied:*
(a) $1 \in int(X)$ as before
(b) $\mathscr{A} = C(X)$,
(c) each f_k is twice differentiable at 1.

(ii) These two results show that the speed of convergence might be improved by use of $(f_n(L_n))$ in place of (L_n) if $(f_n'(1))$ converges to zero. However a much better result would be of a type where the product might be replaced by $(f_n'(\Phi_n))$. But 3.1 and 3.2 show that this cannot be achieved by means of spectral calculus.

3.4 Example. *The following example is a generalization of [4], sect. 3 (at least for contractions): Let $\mathscr{L}$ be an approximation processes of contractions on E and consider $f_k(u) = u^k$. Then we have $\|f_k L_n\| \le 1$. By 3.1 (a) we obtain*

$$\lim_{k,n\to\infty} \|(L_n^k x - x)/k\cdot\Phi(n)) - Lx\| = 0 \qquad \text{for all} \quad x \quad \text{in} \quad D(L).$$

Moreover

$$g_k(u) = \sum_{j=0}^{k-1} u - k, \quad h_k(u) = \sum_{j=1}^{k-1}\sum_{\ell=0}^{j-1} u^\ell,$$

but $||h_k(L_n)|| = 0(k)$ *uniformly is impossible since* $\mathscr{L}$ *is an approximation process.*

4. THE INGREDIENTS FROM NONSTANDARD ANALYSIS

In the following we adhere to the notions etc. of sect. 2. We consider now a suitable polysaturated enlargement $^*\mathscr{H}$ of the full structure $\mathscr{H}$ built up on $\mathbf{R}$ ($E \cup \mathbf{R}$, resp., if E is not already in $\mathscr{H}$). We refer the reader to [2, 3, 5] for notions not explained here.

Recall that by definition $T \in {}^*B(E)$ is of finite norm if its operator norm is nearstandard, or equivalently if there is a standard positive real number c, say, such that $||T|| < c$ holds. This in turn is equivalent to the assertion: $x \approx y$ implies always $Tx \approx Ty$.

Let $T \in {}^*B(E)$ and let M be a positive real number (standard or not). By the transfer principle the assertion that T admits $(^*\mathscr{A}, M)$-spectral calculus is meaningful.

4.1 Lemma. *Let* $T \in {}^*B(E)$ *of finite norm. Moreover let* T *admit* $(^*\mathscr{A}, M)$*-spectral calculus where* M *is a standard real number. Let* $g \in \mathscr{A}$ *be standard and let* $x \in {}^*E$ *be of finite norm.*

Then $Tx \approx x$ implies $g(T)x \approx g(1)x$.

Proof. Let k be a standard positive integer and assume $Tx \approx x$. Then $T^k - I = \sum_{j=0}^{k-1} T^j$ $(T - I)$, hence $T^k x - x = \sum_{j=0}^{k-1} T^j(Tx - x) \approx 0$ since the sum is of finite norm.

So the formula is proved to hold for standard polynomials. But by assumption polynomials are dense in $\mathscr{A}$.

Now let $g \in \mathscr{A}$ be standard. Then the formula

$$(*) \qquad \exists Q[p(g - Q) + ||Q(T)x - Q(1)x|| < 1/n]$$

holds for all standard n hence for some N infinitely large. Using the corresponding Q we get

$$\begin{aligned} ||g(T)x - g(1)x|| &\le ||(g(T) - Q(T))x|| + ||Q(T)x - Q(1)x|| + |Q(1) - g(1)| \cdot ||x|| \\ &\le Mp(g - Q)||x|| + 1/N + |Q(1) - g(1)|||x||. \end{aligned}$$

Since point evaluation is continuous on $\mathscr{A}$ this gives $||g(T)x - g(1)x|| \approx 0$.

4.2 Proposition. *Let $T \in {}^*B(E)$ be of finite norm admitting $({}^*\mathscr{A}, M)$-spectral calculus for a certain standard real number M. Let $f \in \mathscr{A}$ be a standard function which is $\mathscr{A}$-differentiable at 1. Finally let $\Phi : {}^*E \to {}^*\mathbf{R}_+$ be an internal function satisfying*

(i) $\Phi(x) \approx 0$ for all standard $x \in E$

(ii) $\Phi(y) \neq 0$ for all $y \in {}^*E$.

Assume that $(Tx - x)/\Phi(x) \approx Lx$ for all standard $x \in D(L)$. Then the following two assertions are true:

(a) $Ty \approx y$ for all standard $y \in E$.

(b) $(f(T)x - f(1)x)/\Phi(x) \approx f'(1)Lx$ for all standard $x \in D(L)$.

Proof. (a) By assumption $||Tx - x|| \approx \Phi(x)||Lx|| \approx 0$ for all standard $x \in D(L)$. Since $D(L)$ is dense in E and T is of finite norm (a) follows by a routine argument.

(b) By assumption $f(z) = f(1)(z-1) + (f'(1) + g(z))(z-1)$ where $g \in \mathscr{A}$ satisfies $g(1) = 0$.

By the transfer principle applied to the notion of $(\mathscr{A}, M)$-spectral calculus we obtain

$$(f(T)x - f(1)x)/\Phi(x) = f'(1)(Tx - x)/\Phi(x) + g(T)(Tx - x)/\Phi(x).$$

Since $(Tx - x)/\Phi(x) \approx Lx$ we get

$$(f(T)x - f(1)x)/\Phi(x) \approx f'(1)Lx + g(T)(Tx - x)/\Phi(x).$$

But $||g(T)|| \leq Mp(g)$ and this latter number is standard, hence $g(T)(Tx - x)/\Phi(x) \approx$ $\approx g(T)Lx$. Now Lx is standard thus $TLx \approx Lx$ by (a). But then 4.1 yields $g(T)Lx \approx$ $\approx g(1)Lx = 0$, and the proposition is proved.

5. PROOF OF THE RESULTS OF SECT. 2

In the following we adhere to the notions and notational conventions of the previous sections.

5.1. *Proof of 2.6.* Choose an arbitrary $\alpha \in {}^*A$ being greater than every standard index γ. Then $T = L_\alpha$ admits $({}^*\mathscr{A}, M)$-spectral calculus by the transfer principle.

Set $\Phi(x) = \Phi(\alpha, x)$ for all $x \in {}^*E$. Since $\mathscr{L}$ is uniformly bounded, T is of finite norm. Moreover $L_\alpha \to I$ strongly implies $Ty \approx y$ for all y standard. Finally assumption (iv) in 2.5 implies $(Tx - x)/\Phi(x) \approx Lx$ for all standard $x \in D(L)$. Hence 4.2 yields $(f(T)x - f(1)x)/\Phi(x) \approx f'(1)Lx$ for all $x \in D(L)$. An equivalent standard formulation (which is possible since α infinitely large was chose arbitrarily) yields 2.6 b and in the same way we obtain 2.6 a from 4.1.

Now we come to the proof of the main result of sect. 2:

5.2. *Proof of 2.9.* I) Let $\mathscr{R} = (R_\beta)$ be a regulator for $(\mathscr{L}, L)$. By 2.8 (ii) $\mathscr{R}$ is also a regulator for $(f(\mathscr{L}), L)$.

Assume that $f(1) = 1$ holds. then by 2.6 $f(\mathscr{L})$ is an approximation process which satisfies (i) and (iv) of 2.5 (where $\Phi(\alpha, x)$ is replaced by $\Phi^\sim(\alpha, x) = f'(1) \cdot \Phi(\alpha, x))$. So we can apply 2.11 replacing $\mathscr{L}$ by $f(\mathscr{L})$ and Φ by $\Phi^\sim$. This gives 2.9b.

II) So it remains to prove (a). To this end assume first of all that $f(1) = 1$. Then (a) follows from 2.6 and 2.11 (cf. Part (I) above). Assume then $0 \neq f(1) \neq 1$. Then (a) holds for $f^\sim = f/f(1)$, and an easy calculation shows that (a) holds also for f.

Finally assume $f(1) = 0$ and consider $f^\sim = f + 1$. Then (a) holds for $f^\sim$. But

$$(f^\sim(L_\alpha)x - f^\sim(1)x)/(f'^\sim(1)\Phi(\alpha)) = f(L_\alpha)x/f'(1)$$

and the assertion follows.

Note that Corollary 2.10 follows obviously from 2.11 and 2.9.

6. PROOF OF THE RESULTS OF SECTION 3

6.1. *Proof of 3.1*: I) Choose an $n \in {}^*\mathbb{N} \setminus \mathbb{N}$.

Then for $T = L_n$ and $\Phi = \phi(n, x)$ we have $(Tx - x)/\Phi \approx Lx$ for all standard $x \in D(L)$ by hypothesis.

Consider the set

$$B := \{k \in {}^*\mathbb{N} : \|(f_k(T)x - f_k(1)x)/(f_k'(1) \cdot \Phi) - Lx\| < 1/k\}$$

B is internal and contains all standard k by 4.2. Hence B contains an infinitely large k. Since n above was arbitrary in ${}^*\mathbb{N} \setminus \mathbb{N}$ the first assertion follows by a routine nonstandard formulation of the notion of $\underline{\lim}$.

(II) Let x, T, Φ be as above and let $k \in N$ be arbitrary. Set $f = f_k$. Then

$$\frac{(f(T) - f(1))x}{f'(1)\Phi} - Lx = \frac{Tx - x}{\Phi} - Lx + \frac{h(T)}{f'(1)}(T - I)\frac{Tx - x}{\Phi}$$

By hypothesis $(Tx - x)/\Phi \approx Lx$, and $TLx \approx Lx$, hence $\|(T - I)(Tx - x)/\Phi\| \approx 0$. Since $h(T)/f'(1)$ is of finite norm by assumption the second summand on the righthand side is infinitesimally small. Hence

$$\|(f(T)x - f(1)x)/(f'(1)\Phi) - Lx\| \approx 0.$$

Since $T = L_n$ for an arbitrary $n \approx \infty$ and $f = f_k$ for an arbitrary $k \approx \infty$ the assertion (b) follows.

6.2 *Proof of 3.2.* (a) Choose $n \approx \infty$. Then for $x \in D(L)$ standard we have $(f_n(L_n) - f_n(1))x \approx f_n'(1)\Phi(n,x)Lx \approx 0$ by 3.1b). But $f_n(1) \approx 1$ by assumption. Moreover $\|f_n(L_n)\| \leq D$, and $D(L)$ is dense in E. Hence $f_n(L_n)y \approx y$ for all standard $y \in E$ (cf. the proof of 4.2).

(b) Consider $f_n^{\sim} := f_n / f_n(1)$. By (a) $(f_n^{\sim}(L_n))$ is also an approximation process. Moreover every regulator of $(\mathscr{L}, L)$ is also a regulator of $((f_n^{\sim}(L_n)), L)$ (cf. 2.8 (ii)). Then again (a) implies that 2.11 is applicable to $\mathscr{U}^{\sim} := (f_n^{\sim}(L_n))$, L, and $\Phi^{\sim}$, where $\Phi^{\sim}$ is given by $\Phi_n^{\sim} = f_n'^{\sim}(1)\Phi_n$. Thus (i)-(iii) holds for $\mathscr{U}^{\sim}$ in place of $\mathscr{U}$. But an easy calculation shows that this implies (b).

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