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A MAXIMAL EXTENSION OF KÖTHE's HOMOMORPHISM THEOREM

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Dedicated to the memory of Professor Gottfried Köthe

ABSTRACT. *In 1958, Prof. T. Kato gave the following perturbation theorem:* Let E_0 and F_0 be subspaces of Banach spaces E and F, respectively, and let $f : E_0 \to F_0$ be a linear surjective map from E_0 onto F_0 with closed graph in $E \times F$. If $\dim(F/F_0) < \aleph_0$, then f is open and F_0 is closed in F *[4]. Ten years later, Prof. Dr. G. Köthe gave two generalizations [5] which enhanced and were enhanced by considerations of codimension [7], Baire-like (BL) spaces [11a], and quasi-Baire (QB) spaces [11a, 9], and thus, together with a Robertson-Robertson Closed Graph Theorem (cf. [14]), provided significant external impetus for the early study of strong barelledness conditions. Viewed as yet another version of the Kato result, Köthe's Homomorphism Theorem replaces «Banach spaces» with the more general «(LF)-spaces» (cf. 8.4.13 of [6]). Here, again, strong barelledness [12] kindly repays Köthe and allows us to replace «*$< \aleph_0$*» with «*$< c$*». This is, easily, the best possible extension as regards codimension of* F_0.

1. DEFINITIONS

For standard background and terminology, please consult [3].

Our organization of strong barelledness conditions is based on the definition of *db* spaces, and can be found in more detail in [11, 12], where appropriate credit is given to those who provided earlier names for the concepts we restate here. As far as we know, ours is the only attempt at unifying the nomenclature.

A barrelled space E is a *db space*, or is *db*, if, given any increasing sequence of subspaces covering E, at least one of the subspace must be dense and barelled. We define *d spaces* (resp., *b spaces*) as above, with the deletion of «and barrelled» (resp., «dense and»). We define udb, ud, and ub just as we do db, d, and b above, respectively, with «increasing» deleted. [The «u» signals acceptance of unordered covering sequences.]

In the relational picture that emerges, none of the implication arrows can be reversed (fig. 1).

A barrelled space is *totally barrelled* (TB) if, given any covering sequence of subspaces, one must be barrelled and «almost dense»; i.e., its closure must have finite codimension in E. And E is BL (*Blaire-like*) if it is not the union of an increasing sequence of nowhere dense, balanced, convex sets. The TB and BL spaces fit neatly into the picture (fig. 2).

Sequences of subspaces which cover remind one of (LF)-spaces, and, indeed, the (LF)-spaces have been pleasingly partitioned into robust, strong barrelledness-precise subclasses [8-10]. (For example, an (LF)-space is metrizable if and only if it is BL, and there is an

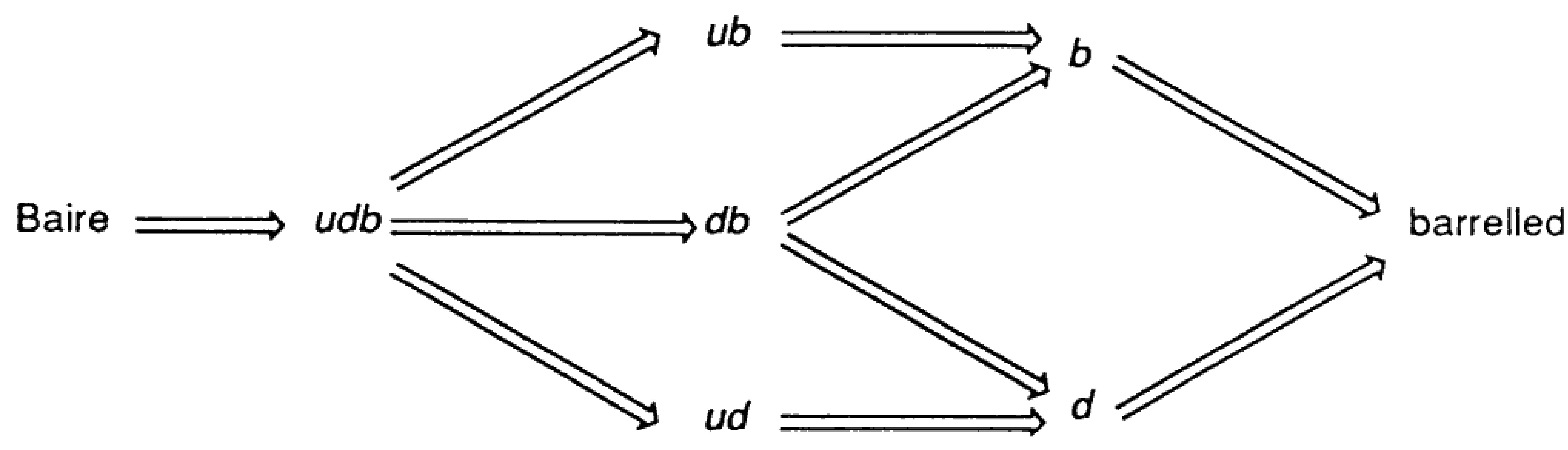

Figure 1. General barrelled spaces.

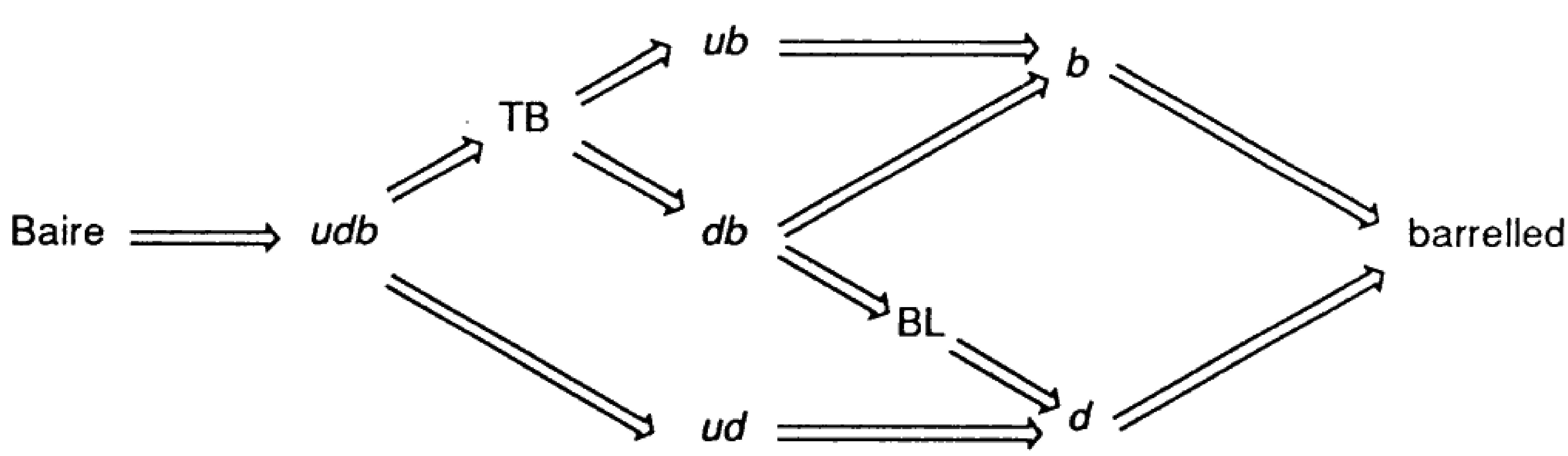

Figure 2. General barrelled spaces.

abundance of such spaces). Grothendieck's original definition of (LF)-space [2] implies the existence of an increasing, covering sequence of dominating Fréchet spaces, and this appears explicitly in the (equivalent) modern definition. There is a useful compromise between the two. (One topological vector space G [*strictly*] *dominates* another, F, if G and F coincide as vector spaces, while the topology of G is [strictly] finer than that of F). Clearly, Grothendieck knew the arguments we now sketch to achieve the compromise.

Proposition. *Let $(E,\mathcal{T})$ be a Hausdorff locally convex space covered by a sequence $(E_n)_n$ of proper subspaces, with each E_n admitting a (unique) Fréchet topology $\mathcal{T}_n$ such that $(E_n,\mathcal{T}_n)$ dominates $(E_n,\mathcal{T})$. If $(F_n)_n$ is an arbitrary sequence of subspaces with each F_n admitting a (unique) Fréchet topology τ_n such that (F_n,τ_n) dominates $(F_n,\mathcal{T})$, then there is a covering, properly increasing subsequence $(E_{n_k})_k$ of $(E_n)_n$ such that $F_1,\ldots,F_k \subset E_{n_k} \subsetneq E_{n_{k+1}}$ for each k, and all of the inclusion maps between the Fréchet spaces are continuous.*

The proof uses the closed graph theorem, the fact that a Fréchet space is udb, and the fact that any subspace $Q = F_1 + \ldots + F_k + E_1 + \ldots E_{n_k}$ of $(E,\mathcal{T})$ has a (unique) finer

Fréchet topology ρ making (Q,ρ) isomorphic to a quotient of the product $P = F_1 \times \ldots \times F_k \times E_1 \times \ldots \times E_{n_k}$ of Fréchet spaces, so that (Q,ρ) will be continuously included in some $(E_{n_{k+1}}, \mathscr{T}_{n_{k+1}})$.

Grothendieck could easily have offered the following alternative

Definition. A Hausdorff locally convex space $(E,\mathscr{T})$ is a *[proper]* $(LF$*-space* if there exists a sequence $((E_n,\mathscr{T}_n))_n$ of Fréchet spaces continuously [and properly] included in $(E,\mathscr{T})$ such that $E = \bigcup_n E_n$ and $\mathscr{T}$ is the finest locally convex topology on E for which each $(E_n,\mathscr{T}_n)$ dominates $(E_n,\mathscr{T})$.

In other words, a Hausdorff locally convex space E is a [proper] (LF)-space if and only if E is the inductive limit of a covering sequence of Fréchet spaces continuously [and properly] included in E. The statement is twice simpler than the usual modern one which explicitly requires a defining sequence of Fréchet spaces that is i) increasing, ii) with each Fréchet space continuously [and properly] included in the next. The Proposition shows the definitions, with and without i) and ii), are equivalent, and proves that there can be at most one (LF)-space dominating a given Hausdorff locally convex space (cf. Grothendieck's Equivalence Theorem [10]). Further, the class of improper (LF)-spaces coincides with the class of Fréchet spaces. [In some previous papers [8-10], «(LF)-space» has meant what we mean here by «proper (LF)-space»].

There are other exchanges between unordered covering sequences and ones that are increasing [12]: *Any Hausdorff barrelled space dominated by an (LF)-space is d, b, or db $\Leftrightarrow$ it is ud, ub, or udb, respectively.* This could save much labor (cf. [11, 12]) is in example of [1]. Also, knowledge of the twice simpler definition would benefit parts of [17]: the first four lines of 1.4.5 (12), p. 84, can be equivalently replaced by «If E is an (LF)-space, then», and 1.6.2 (21), p. 121, is just a special case of the much more general 1.6.2 (7), p. 117.

2. REQUISITE RESULTS

We write $(E,\mathscr{T}) = \varinjlim_n (E_n,\mathscr{T}_n)$ to denote that $(E,\mathscr{T})$ is an (LF)-space and $((E_n,\mathscr{T}_n))_n$ is a covering, increasing sequence of dominating Fréchet spaces [whose inductive limit is necessarily $(E,\mathscr{T})$].

We need the following results.

(0) Any Hausdorff quotient of an (LF)-space is an (LF)-space.

(1) If f is a linear map from a barrelled space E into a Fréchet space F with closed graph in $E \times F$, then f is continuous.

(2) If G is a subspace of a Fréchet space F with $\dim(F/G) < c$, then G is TB.

(3) If F_0 is a subspace of $(F, \tau) = \varinjlim_n (F_n, \tau_n)$ with $\dim(F/F_0) < c$, then (F_0, τ) is the inductive limit of the sequence $((F_n \cap F_0, \tau_n))_n$.

(4) Let F_0 be a subspace of $(F, \tau) = \varinjlim_n (F_n, \tau_n)$ with $\dim(F/F_0) \leq \aleph_0$. The following are equivalent:

(i) (F_0, τ) is an (LF)-space.

(ii) Each $F_n \cap F_0$ is τ_n-closed.

(iii) F_0 is τ-closed.

We note that (0) follows from Theorem 2 of [10], (1) is a standard closed graph theorem, (3) is proved in [13] in greater generality (where it is also shown that the proof given in [15] and [6] is false), and (4) is proved as in Theorem 10 of the paper [9] on proper (LF)-spaces.

As noted in [7], techniques of Saxon's dissertation prove, without use of the Continuum Hypothesis, that every subspace G of codimension $< c$ in a Fréchet space F is barrelled. Valdivia [16] subsequently obtained the result for more general F. In (2) we claim a stronger conclusion for G when F remains a Fréchet space. Its proof follows from [12], where we interpret and answer a question on (LF)-space dominance (question 13.4.3 of [6]) in terms of strong barrelledness. We provide here a more direct

Proof of (2). Suppose G is not TB. By Proposition 9.3.3 (i) of [6], G is not ub, and so is covered by a sequence $(B_n)_n$ of absolutely convex sets closed in F, with no B_n a neighborhood of 0 in its span. Let $(U_n)_n$ be a countable base of closed absolutely convex neighborhoods of 0 in F with $U_{n+1} + U_{n+1} \subset U_n$ for $n = 1, 2, \ldots$. Let $\{S_n\}_n$ be a partition of $\mathbb{N}$ into denumerably many disjoint infinite sets S_n and define $\sigma : \mathbb{N} \to \mathbb{N}$ by letting $\sigma(n) = p$ if and only if $n \in S_p (n \in \mathbb{N})$. The balanced set $C_1 = \bigcup_n B_n$ is not absorbing in the Baire space F, since F cannot be covered by the nowhere dense sets $mB_n (m, n = 1, 2, \ldots)$. Thus there exists $x_1 \in U_1 \setminus \{0\}$ whose span meets C_1 only at 0. The Bipolar Theorem provides $f_1 \in B^\circ_{\sigma(1)}$ with $f_1(x_1) > 1$. Each $B_{n,1} = B_n + \{a x_1 : |a| \leq 1\}$ is closed and absolutely convex and not a neighborhood of 0 in its span, so the above reasoning applies to $C_2 = \bigcup_n B_{n,1}$ to yield $z_2 \neq 0$ whose span meets C_2 only at 0. For a unique scalar b, f_1 vanishes at the non-zero $y_2 = z_2 + b x_1$, a vector whose span also meets C_2 only at 0, by absolute convexity. There is a positive multiple x_2 of y_2 with $0 \neq x_2 \in U_2$. Thus the span of x_2 meets C_2 only at 0, $f_1(x_2) = 0$, and there exists $f_2 \in [B_{\sigma(2),1}]^\circ$ such that $f_2(x_2) > 2$. We inductively continue to choose the sequences $(x_n)_n$ and $(f_n)_n$ such that $x_n \in U_n, f_n(x_n) > n, f_n(x_q) = 0$ for all $q > n$, and $f_n \in \left[B_{\sigma(n)} + \left\{ \sum_{1 \leq k < n} a_k x_k : \text{each } |a_k| \leq 1 \right\} \right]^\circ$ for $n = 1, 2, \ldots$. Let I be an indexing set of cardinality c. A result of Sierpinski assures that, for each $n \in \mathbf{N}$, there is a collection $\{S_{n,i}\}_{i \in I}$ of infinite subsets of S_n any two of which have a finite intersection. The series $\sum_k x_k$ is absolutely, hence subseries, convergent in the Fréchet space F. For each $i \in I$, let $u_i = \sum \left\{ x_k : k \in \bigcup_n S_{n,i} \right\}$. Suppose L is a non-empty finite subset of I and for each $i \in L, a_i$ is a non-zero scalar. Fix $p \in \mathbf{N}$ arbitrarily. We wish to show that $u = \sum_{i \in L} a_i u_i \notin B_p$. Fix $r \in L$ and choose M such that $|a_r| \cdot M > \left(\sum_{i \in L} |a_i| \right) + 1$. Since $S_{p,r} \setminus \bigcup_{\substack{i \in L \\ i \neq r}} S_{p,i}$ is infinite, it contains some $m > M$. Clearly, $u = \sum_{k \geq 1} b_k x_k$ for appropriate scalars b_k satisfying $|b_k| \leq \sum_{i \in L} |a_i| (k = 1, 2, \ldots)$ and with $b_m = a_r$. Thus

$$|f_m(u)| \geq |f_m(b_m x_m)| - \left| f_m \left(\sum_{1 \leq k < m} b_k x_k \right) \right| > |a_r| \cdot M - \sum_{i \in L} |a_i| > 1.$$

But $m \in S_{p,r} \subset S_p \Rightarrow \sigma(m) = p$, so that $f_m \in B_p^\circ$, and therefore $u \notin B_p$. Since p was arbitrary, we see that $\{u_i\}_{i \in I}$ is a linearly independent set of c vectors whose span meets $\bigcup_n B_n$ and thus G, only at 0. But this forces $\dim(F/G) \geq c$, a contradiction. Therefore it must be that G is TB, after all. ∎

3. MAIN RESULT

Theorem. *Let E_0 and F_0 be subspaces of (LF)-spaces E and F, respectively, and let $f : E_0 \to F_0$ be a linear surjective map from E_0 onto F_0 with closed graph in $E \times F$. If $\dim(F/F_0) < c$, then f is open and F_0 is closed in F; and then, in fact, $\dim(F/F_0) \leq \aleph_0$ and F_0 is an (LF)-space.*

According to 8.4.13 of [6], Köthe's homomorphism theorem is the above but with $E_0 = E$, « $< c$» replaced by «$< \aleph_0$ », and the deletion of «and F_0 is closed ... (LF)-space». However, the Abstract makes the case for more generously crediting Prof. Köthe, allowing E_0 to remain arbitrary and deleting only from the semicolon on, keeping «$< \aleph_0$ », of course. In any case, our principal contribution is to extend the codimension allowed F_0, and here our work cannot be improved: Let $F = \varinjlim_n (F_n, \tau_n)$ where each (F_n, τ_n) is an infinite-dimensional separable Fréchet space, so that $\dim(F) = c$. By Theorem 2 of [8] (cf. [10], Theorem 6 (i) $\Rightarrow$ (vi)), there is a dense subspace H of (F_1, τ_1) which is dominated by a metrizable proper (LF)-space G. The improper (F_1, τ_1) cannot be dominated by G, from uniqueness of domination, so there exists some $x \in F_1 \backslash H$. Then, viewing $F_0 = H + sp(\{x\})$ as either a subspace of (F_1, τ_1) or of F, H is a dense subspace of F_0, but G is a closed, non-dense subspace of the proper (LF)-space $E_0 = G \oplus sp(\{x\})$, so that E_0 *strictly* dominates F_0. Set $E = E_0$ and let f be the continuous inclusion map from E_0 onto F_0. Now, the codimension of F_0 is $\leq c$ (in fact, $= c$), but all the other hypotheses of the Theorem are satisfied; and yet, not one of the four conclusions holds.

Proof of Theorem. Since f has closed graph in $E \times F$, $N = f^{-1}(\{0\})$ is closed in E. Thus, E_0/N is a subspace of E/N, the associated injection $g = f^-$ has closed graph in $E/N \times F$, and, by (0), we can write $E/N = \varinjlim_n (G_n, \mathscr{T}_n)$. We also have $F = \varinjlim_n (F_n, \tau_n)$; fix $n \in N$. From (2) we conclude that $(F_n \cap F_0, \tau_n)$ is TB, hence db, so for some p, $H = F_n \cap F_0 \cap g(G_p \cap E_0/N)$ is a dense, barrelled subspace of $(F_n \cap F_0, \tau_n)$. Certainly, $g^{-1}|_H : (H, \tau_n) \to (G_p, \mathscr{T}_p)$ has closed graph in $(H, \tau_n) \times (G_p, \mathscr{T}_p)$, since the same is true with coarser topologies on H and G_p, and by (1), $g^{-1}|_H$ is continuous. Therefore, if $(y_m)_m$ is a sequence in H converging to some y in (F_n, τ_n), then $(g^{-1}(y_m))_m$ is a Cauchy sequence which must converge to some x in $(G_p, \mathscr{T}_p)$. Again, the graph of $g^{-1}|_{F_n \cap F_0}$ is closed in $(F_n, \tau_n) \times (G_p, \mathscr{T}_p)$, requiring $g^{-1}(y) = x$, so that $y \in H$. Thus $H = F_n \cap F_0$ is closed and of codimension $< c$ in the Fréchet space (F_n, τ_n), and then, indeed, must be of finite codimension in F_n. By (3), $F_0 = \varinjlim_n (F_n \cap F_0, \tau_n)$ is an (LF)-space, and is countable-codimensional in F, since the deficiency in each of the countably many steps of F is finite. Now, by (4), F_0 is also closed in F. Since each $g^{-1}|_{F_n \cap F_0}$ is continuous on $(F_n \cap F_0, \tau_n)$, g^{-1} is continuous on the inductive limit F_0, and $g : E_0/N \to F_0$ is open; thus so is $f : E_0 \to F_0$. ■

4. APPLICATION

We can maximally extend (4) to obtain

(4'). *(Cf. [13].) Let F_0 be a subspace of $(F, \tau) = \varinjlim_n (F_n, \tau_n)$ with $\dim(F/F_0) < c$. The following are equivalent:*
(i) (F_0, τ) is an (LF)-space
(ii) Each $F_n \cap F_0$ is closed in (F_n, τ_n)
(iii) F_0 is closed in (F, τ).
Moreover, if (i), (ii), and/or (iii) holds, then $\dim(F/F_0) \leq \aleph_0$.

Proof. If (i) holds, in the Theorem set $E = E_0 = (F_0, \tau)$ and let f be the inclusion map to obtain $\dim(F/F_0) \leq \aleph_0$ and (iii), from which (ii) follows. If (ii) holds, so does (i), by (3).■

To see that (4') can fail if $\dim(F/F_0) = c$ is allowed, choose

$$(F, \tau) = \lim_{\overrightarrow{n}} \underbrace{\omega \times \ldots \times \omega}_{n \text{factors}} \times \ell_2 \times \ell_2 \times \ldots \quad \text{and}$$

$$(F, \mathscr{T}) = \lim_{\overrightarrow{n}} \underbrace{\omega \times \ldots \times \omega}_{n \text{factors}} \times \ell_1 \times \ell_1 \times \ldots$$

(cf. Example 3 of [10]), so that $(F_0, \mathscr{T}) = (F_0, \tau)$ is an (LF)-space [(i) holds], yet neither (ii) nor (iii) holds. (Also, see [13]).

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