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ASPECTS OF THE UNIFORM λ-PROPERTY (*)

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Dedicated to the memory of Professor Gottfried Köthe

ABSTRACT. *If Z is a uniformly convex normed space, the quotient space $\ell_\infty(Z)/c_0(Z)$, which is not strictly convexifiable, is shown to have the uniform λ-property and its λ-function is calculated. An example is given of a Banach space X with a closed linear subspace Y such that Y and X/Y are strictly convex, yet X fails to have the λ-property. Convex sequences which generate B_{ℓ_∞} are characterized.*

Every point in the closed unit ball of a strictly convex space is a convex combination of two extreme points. Thus, every strictly convex space has the uniform λ-property, a geometric property of normed spaces introduced in [1] and which represents one direction that can be followed in generalizing the notion of strict convexity. Because the strictly convex spaces represent such a fundamental class of normed spaces with the uniform λ-property, it is natural to ask whether these classes contain the same spaces, up to isomorphism. That is, does every normed space with the uniform λ-property possess an equivalent strictly convex norm? In this note, we show that the answer is no by proving that $\ell_\infty(Z)/c_0(Z)$ has the uniform λ-property whenever Z is a uniformly convex space. Since $\ell_\infty(Z)/c_0(Z)$ is not strictly convexifiable, we obtain a negative answer to the preceding question. In particular, ℓ_∞/c_0 is an example of a much-studied classical Banach space which has the uniform λ-property but is not strictly convexifiable. We also show that the uniform λ-property is very far from being a three-space property. Namely, we give an example of a Banach space X with a closed linear subspace Y such that Y and X/Y are strictly convex, yet X fails to have the λ-property. In the last section, we examine certain convex sequences, the so-called B_X-generating sequences, which naturally appear in the context of any discussion of a Banach space X with the uniform λ-property. The $B_{\ell_\infty(R)}$-generating sequences are characterized.

0. PRELIMINARIES

Given a normed space X, B_X denotes its closed unit ball and S_X its closed unit sphere. If $x \in B_X$, a triple (e, y, λ) is amenable to x, if $e \in \text{ext}(B_X), y \in B_X, 0 < \lambda \leq 1$ and

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$x = \lambda e + (1 - \lambda) y$. In this case, we define

$$\lambda(x) = \sup\{\lambda : (e, y, \lambda) \text{ is amenable to } x\}.$$

X has the λ-property if each $x \in B_X$ admits an amenable triple. If, in addition $\lambda(X) \equiv \equiv \inf\{\lambda(x) : x \in B_X\} > 0$, then X is said to have the uniform λ-property. General facts concerning these properties appear in [1], [4] and [7]. Discussion of these properties for classical sequence and function spaces can be found in [1], [5], [8]-[10].

If Z is a normed space, $\ell_\infty(Z)$ denotes the normed space of all bounded Z-valued sequences $z = (z_n)$, where $||z|| = \sup_n ||z_n||$. The closed linear subspace of $\ell_\infty(Z)$ consisting of all zose z for which $z_n \to 0$ is denoted by $c_0(Z)$. It is a well-known result of J. Bourgain [3] that $\ell_\infty(Z)/c_0(Z)$ is not strictly convexifiable (i.e., does not admit an equivalent strictly convex norm) in the cases $Z = \mathbf{R}$ or $\mathbb{C}$. In these cases, the latter quotient space is denoted simply by ℓ_∞/c_0. For an arbitrary normed space Z, fix $z \in Z_0 Z$. Then the mapping $T : \ell_\infty/c_0 \to \ell_\infty(Z)/c_0(Z)$, defined by $T((a_n) + c_0) = (a_n z) + c_0(Z)$, is easily seen to be a linear isometry of ℓ_∞/c_0 into $\ell_\infty(Z)/c_0(Z)$. By Bourgain's result, it follows that $\ell_\infty(Z)/c_0(Z)$ is not strictly convexifiable.

If Z is a uniformly convex normed space and $(u_n), (v_n)$ are sequences in B_Z such that $||u_n + v_n|| \to 2$, then $||u_n - v_n|| \to 0$. If A is a subset of a normed space, $\overline{co}(A)$ denotes the closed convex hull of A.

1. $\ell_\infty(Z)/c_0(Z)$ AND THE UNIFORM λ-PROPERTY

We write $X = \ell_\infty(Z)/c_0(Z)$ and let $Q : \ell_\infty(Z) \to X$ denote the canonical quotient mapping, defined by $Q(z) = z + c_0(Z)$ for all $z \in \ell_\infty(Z)$.

Lemma 1. $Q(B_{\ell_\infty(Z)}) = B_X$ *for any normed space* Z.

Proof. It suffices to show that $B_X \subset Q(B_{\ell_\infty(Z)})$. Let $\hat{z} \in B_X$ and choose $z = (z_n) \in \in \ell_\infty(Z)$ such that $\hat{z} = Q(z)$. Since $||Q(z)|| \leq 1$, the set $\mathbf{N}_k = \{n : ||z_n|| \geq 1 + k^{-1}\}$ is finite for every $k \in \mathbf{N}$. Define $x = (x_n) \in B_{\ell_\infty(Z)}$ as follows:

$$x_n = \begin{cases} z_n, & \text{if } ||z_n|| \leq 1 \\ \dfrac{z_n}{||z_n||}, & \text{if } 1 < ||z_n|| \end{cases}$$

Observe that if $n > \max \mathbf{N}_k$ and $1 < ||z_n||$, then $1 < ||z_n|| < 1 + k^{-1}$. It follows that $||x_n - z_n|| < k^{-1}$ if $n > \max \mathbf{N}_k$. Consequently, we have $x - z \in c_0(Z)$ so that $\hat{z} = Q(z) = Q(x)$. Since $x \in B_{\ell_\infty(Z)}$, the proof is complete.

Lemma 2. *(a) If Z is a strictly convex normed space,* $\mathrm{ext}(B_X) \subset Q(\mathrm{ext}(B_{\ell_\infty(Z)}))$. *(b) If Z is a uniformly convex normed space,* $\mathrm{ext}\,(B_X) = Q(\mathrm{ext}\,(B_{\ell_\infty(Z)}))$.

Proof. (a). If $\hat{e} \in \mathrm{ext}\,(B_X)$, then by Lemma 1, we can write $\hat{e} = Q(x)$, where $x = (x_n) \in B_{\ell_\infty(Z)}$. Then $\|x_n\| \leq 1$ for all n and we claim $\|x_n\| \to 1$. If not, there exists $\varepsilon > 0$ such that $\|x_n\| \leq 1 - \varepsilon$ for an infinite subset $\mathbb{N}_\varepsilon$ of $\mathbb{N}$. Thus, for each $n \in N_\varepsilon$, there exist $u_n, v_n \in B_Z$ such that $1 \geq \|u_n - x_n\|, \|v_n - x_n\| \geq \varepsilon$ and $x_n = \frac{1}{2}(u_n + v_n)$. This implies that there exist $u, v \in B_{\ell_\infty(Z)}$ such that $u - x, v - x \notin C_0(Z)$ and $x = \frac{1}{2}(u + v)$. Hence, $\hat{e} = \frac{1}{2}(Q(u) + Q(v))$, where $Q(u) \neq \hat{e} \neq Q(v)$. This contradiction establishes the claim. Since $\|x_n\| \to 1$, there is no loss of generality in assuming $x_n \neq 0$ for all n. Then $e = \left(\frac{x_n}{\|x_n\|}\right) \in \mathrm{ext}\,(B_{\ell_\infty(Z)})$ and $x - e \in c_0(Z)$, which implies $Q(x) = Q(e)$ and completes the proof.

(b) Let $e = (e_n) \in \mathrm{ext}\,(B_{\ell_\infty(Z)})$. Then $\|e_n\| = 1$ for all n and $\|Q(e)\| = 1$. Suppose $Q(e) = \frac{1}{2}(\hat{u} + \hat{v})$, where $\hat{u}, \hat{v} \in B_X$. Then $\|\hat{u}\| = \|\hat{v}\| = 1$ and, by Lemma 1, we can find $u = (u_n), v = (v_n) \in B_{\ell_\infty(Z)}$ such that $\hat{u} = Q(u), \hat{v} = Q(v)$. Since $Q(e) = Q\left(\frac{u+v}{2}\right)$, it follows that $e - \frac{1}{2}(u + v) \in c_0(Z)$; that is, $\left\|e_n - \frac{1}{2}(u_n + v_n)\right\| \to 0$. Uniform convexity of Z and the facts that $\|u_n\|, \|v_n\| \leq 1, \|u_n + v_n\| \to 2$ forces $\|u_n - v_n\| \to 0$. This implies $u - v \in c_0(Z)$ which, in turn, implies $\hat{u} = Q(u) = Q(v) = \hat{v}$. Therefore, $Q(e) = \hat{u} = \hat{v}$, implying $Q(e) \in \mathrm{ext}\,(B_X)$.

Theorem 3. *Let Z be a uniformly convex normed space. The quotient space $X = \ell_\infty(Z)/c_0(Z)$ has the uniform λ-property but is not strictly convexifiable. If $\hat{z} \in B_X$, then*

$$(*) \qquad \lambda(\hat{z}) = \sup\left\{\frac{1}{2}(1 + \inf_n \|z_n\|) : z = (z_n) \in B_{\ell_\infty(Z)} \quad \text{and} \quad Q(z) = \hat{z}\right\}.$$

Proof. Let $\hat{z} \in B_X$ and suppose $\hat{z} = Q(z)$, where $z = (z_n) \in B_{\ell_\infty(Z)}$. By theorem 1.13 of [1], there exist $e \in \mathrm{ext}\,(B_{\ell_\infty(Z)}), y \in B_{\ell_\infty(Z)}$ such that $z = \lambda e + (1 - \lambda) y$, where $\lambda = \frac{1}{2}(1 + \inf_n \|z_n\|)$. By Lemmas 1 and 2, $(Q(e), Q(y), \lambda)$ is amenable to $\hat{z} = Q(z)$. This proves X has the uniform λ-property and that $\lambda(\hat{z})$ is at least as large as the supremum indicated in (*).

On the other hand, given $\varepsilon > 0$, there exists a triple $(\hat{e}, \hat{y}, \lambda)$ amenable to $\hat{z}$ such that $\lambda(\hat{z}) - \varepsilon < \lambda$. By Lemmas 1 and 2, we may assume $\hat{e} = Q(e), \hat{y} = Q(y)$, where $e \in$ $\in \text{ext}\,(B_{\ell_\infty(Z)}), y \in B_{\ell_\infty(Z)}$. Let $z = (z_n) = \lambda e + (1 - \lambda) y$ and observe that $Q(z) = \hat{z}$. By theorem 1.13 of [1], $\lambda \leq \lambda(z) = \frac{1}{2}(1 + \inf_n \|z_n\|)$. Thus,

$$\lambda(\hat{z}) \leq \frac{1}{2}(1 + \inf_n \|z_n\|) + \varepsilon,$$

showing that the supremum indicated in (*) is at least as large as $\lambda(\hat{z})$.

2. THREE-SPACE CONSIDERATIONS

The uniform λ-property is not a three-space property. For example, ℓ_∞ has the uniform λ-property but contains a subspace $Y (= c_0)$ without the λ-property. On the other hand, a classical example of V. Klee [5] shows that ℓ_1 can be given an equivalent strictly convex norm $\|\cdot\|$ such that every seprarable Banach space is isometrically isomorphic to a quotient of $(\ell_1, \|\cdot\|)$. Thus, a quotient of a strictly convex space may fail to ahve the λ- property. Our goal here is to show the existence of a Banach space X with a closed linear subspace Y such that Y and X/Y are both strictly convex, yet X fails to have the λ -property.

To this end, let Y be a real Banach space having two equivalent norms, $\|\cdot\|_1$ and $\|\cdot\|_2$, such that $(Y, \|\cdot\|_1)$ is strictly convex and the closed unit ball of $(Y, \|\cdot\|_2)$ fails to have an extreme point. For example $Y = c_0$ is such a space. Let B_i denote the closed unit ball of $(Y, \|\cdot\|_i), i = 1, 2$. We may assume that $2B_2 \subset B_1$. In $X = Y \times \mathbb{R}$, let

$$B = \overline{co}((B_1 \times \{0\}) \cup (B_2 \times \{-1, 1\})).$$

If $\|\cdot\|$ denotes the gauge functional of B in X, then $\|\cdot\|$ is a norm on X, $(X, \|\cdot\|)$ is a Banach space and B is closed unit ball of $(X, \|\cdot\|)$. Routine calculations show that $(Y, \|\cdot\|_1)$ is isometrically isomorphic to the subspace $Y \times \{0\}$ of $(X, \|\cdot\|)$, so that $Y \times \{0\}$ is strictly convex. Obviously, the one-dimensional space $X/(Y \times \{0\})$ is strictly convex.

Assume that $(X, \|\cdot\|)$ has the λ-property and define $f \in X^*$ by $f(x, t) = t$. Then $f(0, 1) = \|f\| = 1$. Since $(X, \|\cdot\|)$ has the λ-property, f must attain its maximum on B at a member (x, t) of ext (B) (see Theorem 3.3 of [1]). This forces $t = 1$ and hence, by the definition of B, we obtain $x \in B_2$. Since B_2 does not contain an extreme point, it follows that $(x, 1) \notin ext(B)$, a contradiction.

Remark 4. The preceding example also shows that a Banach space without the λ-property can contain a closed, one-codimensional subspace that is strictly convex.

3. B_X-GENERATING SEQUENCES

A sequence (λ_k) of positive real numbers will be called a convex sequence in case $\sum_{k=1}^{\infty} \lambda_k = 1$. The following result has recently been shown in [2]:

Theorem 5. *A Banach space X has the λ-property if and only if B_X has the convex series representation property; that is, for each $x \in B_X$, there exists a convex sequence (λ_k) and a sequence $(e_k) \subset \operatorname{ext}(B_X)$ such that $x = \sum_{k=1}^{\infty} \lambda_k e_k$.*

On the other hand, it is well-known from [1] that

Theorem 6. *A Banach space X has the uniform λ-property if and only if there exists a convex sequence (λ_k) such that for each B_X, there exists a sequence $(e_k) \subset \operatorname{ext}(B_X)$ satisfying $x = \sum_{k=1}^{\infty} \lambda_k e_k$.*

A sequence (λ_k) satisfying the condition of Theorem 6 will be called a B_X-generating sequence. We see that Theorems 5 and 6 mark a clear distinction between Banach spaces with the λ-property and Banach spaces with the uniform λ-property. In the case of the λ-property, the sequence (λ_k) of Theorem 5 depends on $x \in B_X$. In the case of the uniform λ-property, the sequence (λ_k) is fixed and changing the extreme points e_k in the sums $\sum_{k=1}^{\infty} \lambda_k e_k$ is sufficient to produce all the members of B_X. Hence, a Banach space X has the uniform λ-property if and only if there is a B_X-generating sequence. For such a space X, it would be of interest to determine all the B_X-generating sequences. If this were possible, then given Banach spaces X, Y with the uniform λ-property, one might be able to distinguish certain geometric or quantitative differences between B_X and B_Y in terms of differences between the collections of B_X-generating and B_Y-generating sequences.

The problem of characterizing the B_X-generating sequences for a Banach space X with the uniform λ-property has only recently been considered. In this section, however, we settle this question for $X = \mathbb{R}$ and $X = \ell_\infty(\mathbb{R})$. Recall from [8] that a Banach space X has the uniform λ-property if and only if $\ell_\infty(X)$ has the uniform λ-property.

Lemma 7. *Let X be a Banach space with the uniform λ-property. Then the families of B_X-generating sequences and $B_{\ell_\infty(X)}$-generating sequences are equal.*

Proof. Let (λ_k) be a B_X-generating sequence and let $x = (x_n) \in B_{\ell_\infty(X)}$. For each n, there exist a sequence $(e_k^{(n)})_{k=1}^{\infty} \subset \mathrm{ext}\,(B_X)$ such that

$$x_n = \sum_{k=1}^{\infty} \lambda_k e_k^{(n)}, \qquad n = 1, 2, \ldots$$

If we define $e_k = (e_k^{(n)})_{n=1}^{\infty}$, then $e_k \in \mathrm{ext}\,(B_{\ell_\infty(X)})$ and for each m, we have

$$\left\| x - \sum_{k=1}^{m} \lambda_k e_k \right\| = \sup_n \left\| x_n - \sum_{k=1}^{m} \lambda_k e_k^{(n)} \right\| \leq \sum_{k=m+1}^{\infty} \lambda_k.$$

Consequently, $x = \sum_{k=1}^{\infty} \lambda_k e_k$, proving that (λ_k) is a $B_{\ell_\infty(X)}$-generating sequence. The proof of the converse is equally simple.

Theorem 8. *Let (λ_k) be a convex sequence. The following are equivalent:*
(a) (λ_k) is $B_{\ell_\infty(R)}$-generating.
(b) (λ_k) is B_R-generating.
(c) For each $t \in [0,1]$, there is a sequence (a_k) of 0's and 1's such that $t = \sum_{k=1}^{\infty} \lambda_k a_k$.
(d) If $(\hat{\lambda}_k)$ is the nonincreasing rearrangement of (λ_k), then for all n we have $\hat{\lambda}_n \leq$ $\leq \sum_{k=n+1}^{\infty} \hat{\lambda}_k$.

Proof. That (a) $\Leftrightarrow$ (b) follows from Lemma 7.

(b) $\Rightarrow$ (c). Let $t \in [0,1]$. Since $\mathrm{ext}(B_R) = \{-1, 1\}$, there is a sequence (b_k) of $\pm$ $\pm$ 1's such that $2t - 1 = \sum_{k=1}^{\infty} \lambda_k b_k$. Then $t = \sum_{k=1}^{\infty} \lambda_k (b_k + 1)/2$, and, for each k, $(b_k + 1)/2$ equals 0 or 1.

(c) $\Rightarrow$ (b). If $-1 \leq x \leq 1$, write $x = 2t - 1$, where $0 \leq t \leq 1$. By hypothesis, there is a sequence (a_k) of 0's and 1's such that $t = \sum_{k=1}^{\infty} \lambda_k a_k$. It follows that $x = \sum_{k=1}^{\infty} \lambda_k (2a_k - 1)$ and, for each k, $2a_k - 1$ is either -1 or 1. Consequently (λ_k) is a B_R-generating sequence.

(c) $\Rightarrow$ (d). Since $(\hat{\lambda}_k)$ also satisfies statement (c), there is no loss of generality in assuming (λ_k) is nonincreasing. Assume, to the contrary, that $\lambda_n > \sum_{k=n+1}^{\infty} \lambda_k$ for some n. Given a sequence (a_k) of 0's and 1's, write $t = \sum_{k=1}^{\infty} \lambda_k a_k$. If $a_k = 0$ for $1 \leq k \leq n$, then $t \leq$

$\leq \sum_{k=n+1}^{\infty} \lambda_n$. On the other hand, if $a_{k_0} = 1$ for some k_0 between 1 and n, then $t \geq \lambda_{k_0} \geq \lambda_n$. This shows that $t \notin \left(\sum_{k=n+1}^{\infty} \lambda_k, \lambda_n \right)$ and contradicts the fact that (λ_k) satisfies (c).

(d) $\Rightarrow$ (c). Without loss of generality, (λ_k) is nonincreasing. Define $\lambda_0 = 1$ and let $0 < t < 1$. Let n_1 be the smallest nonnegative integer such that $\lambda_{n_1+1} \leq t$. If $\sum_{k=n_1+1}^{m} \lambda_k \leq t$ for all $m \geq n_1 + 1$, then we obtain $\lambda_{n_1} \leq \sum_{k=n_1+1}^{\infty} \lambda_k \leq t$, which is impossible if $n_1 = 0$ and contradicts the definition of n_1 if $n_1 \geq 1$. Consequently, there is an integer $m_1 \geq n_1 + 1$ such that

$$(1) \qquad \sum_{k=n_1+1}^{m_1} \lambda_k \leq t < \sum_{k=n_1+1}^{m_1+1} \lambda_k$$

If $t = \sum_{k=n_1+1}^{m_1} \lambda_k$, then $t = \sum_{k=1}^{\infty} \lambda_k a_k$ for the obvious sequence (a_k) of 0's and 1's. Thus, we may assume strict inequality holds in (1).

Assume that $0 \leq n_1 < m_1 < \ldots < n_j < m_j$ have been chosen so that

$$(2) \qquad \sum_{i=1}^{j} \left(\sum_{k=n_i+1}^{m_i} \lambda_k \right) < t < \sum_{i=1}^{j-1} \left(\sum_{k=n_i+1}^{m_i} \lambda_k \right) + \sum_{k=n_j+1}^{m_j+1} \lambda_k$$

If s_j denotes the left-hand side of (2), then (2) becomes

$$s_j < t < s_j + \lambda_{m_j+1},$$

which implies $0 < t - s_j < \lambda_{m_j+1}$. There is a largest integer $n_{j+1} \geq m_j + 1$ such that

$$(3) \qquad t < s_j + \lambda_{n_{j+1}}$$

This implies

$$(4) \qquad s_j + \lambda_{n_{j+1}+1} \leq t$$

If $s_j + \sum_{k=n_{j+1}+1}^{m} \lambda_k \leq t$ for all $m \geq n_{j+1} + 1$, we obtain

$$s_j + \lambda_{n_{j+1}} \leq s_j + \sum_{k=n_{j+1}+1}^{\infty} \lambda_k \leq t,$$

which contradicts (3). Therefore, there exists $m \geq n_{j+1} + 1$ such that $t < s_j + \sum_{k=n_{j+1}+1}^{m} \lambda_k$.
This fact, together with (4), implies that there is an integer $m_{j+1} \geq n_{j+1} + 1$ such that

$$(5) \qquad s_j + \sum_{k=n_{j+1}+1}^{m_{j+1}} \lambda_k \leq t < s_j + \sum_{k=n_{j+1}+1}^{m_{j+1}+1} \lambda_k$$

If equality holds in the left-hand side of (5), then $t = s_{j+1}$. If the left-hand side of (5) is a strict inequality, then $0 < t - s_{j+1} < \lambda_{m_{j+1}+1}$. By induction, either $t = s_j$ for some j or $t = \lim_{j \to \infty} s_j$. In either case, $t = \sum_{h=1}^{\infty} \lambda_k a_k$ for a sequence (a_k) of 0's and 1's.

Remark 9. In order to illustrate an application of these result, let $0 < \lambda < 1$ and define $\lambda_k = \lambda(1-\lambda)^{k-1}$. Then (λ_k) a nonincreasing sequence of positive numbers satisfying $\sum_{k=1}^{\infty} \lambda_k = 1$. For each positive integer n, we have $\sum_{k=n+1}^{\infty} \lambda(1-\lambda)^{k-1} = (1-\lambda)^n$. It follows that (λ_k) is a $B_{\ell_\infty(R)}$-generating sequence if and only if

$$\lambda(1-\lambda)^{n-1} \leq (1-\lambda)^n$$

for each n. This occurs if and only if $\lambda \leq \frac{1}{2}$.

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