Note di Matematica Vol. XII, 145-155 (1992)

THE NUMBER OF POINTS WHERE A LINEAR MAPPING FROM l_2^n INTO l_p^n ATTAINS ITS NORM

WERNER LINDE

Dedicated to the memory of Professor Gottfried Köthe

Abstract. *Let S be a regular $n \times n$-matrix mapping l_2^n onto $l_p^n, 1 \leq p < \infty$, with norm $||S|| = ||S : l_2^n \to l_p^n||$. Then we are interested in the set*

$$C := \{z \in \mathbf{R}^n; ||z||_2 = 1 \qquad \text{and} \qquad ||Sz||_p = ||S||\},$$

i.e. the set of points on the unit sphere where S attains its norm. We prove card(C) $< \infty$ *for $1 \leq p < 2$. This follows from properties of the Taylor expansion of $x \to ||Sx||_p$ near points in C. The case $2 < p < \infty$ remains open. But we show by an example that for $p > 2$ the behaviour of $x \to ||Sx||_p$ may be completely different as for $p < 2$.*

0. INTRODUCTION

As usual we denote by l_p^n the space $\mathbb{R}^n$ equipped with the norm

$$||x||_p := \left\{ \sum_{i=1}^n |x_i|^p \right\}^{1/p}, \quad x = (x_1, \ldots, x_n), \quad 1 \leq p < \infty.$$

We shall write $|x|$ instead of $||x||_2$.

Let S be an $n \times n$-matrix. Then its norm $||S : l_2^n \to l_p^n||$ is defined by $\sup\{||Sx||_p; |x| \leq \leq 1\}$. For simplicity we always assume $||S : l_2^n \to l_p^n|| = 1$. Moreover we always suppose that S is regular, i.e. S maps l_2^n onto l_p^n. The set $C \subseteq \mathbb{R}^n$ of norm attaining points is then defined by

$$C := \{z \in \mathbf{R}^n; |z| = ||Sz||_p = 1\}.$$

The image $S(C)$ may be described geometrically. Define an ellipsoid $\mathscr{E}$ by

$$\mathscr{E} := \{x \in \mathbb{R}^n; \langle R^{-1}x, x\rangle \leq 1\}$$

where $R = SS^*$. By assumption we have

$$\mathscr{E} \subseteq U_p^n := \{x \in \mathbb{R}^n; ||x||_p \leq 1\}.$$

Then $S(C)$ coincides with those points where $\mathscr{E}$ contacts the boundary of U_p^n, i.e.

$$S(C) = \mathscr{E} \cap \partial U_p^n.$$

Hence we have card(C) = card(S((C)) and the problem may be formulated as follows: What is the maximal number of points where an ellipsoid inscribed in U_p^n contacts the boundary of U_p^n? The question is easy to answer if R is diagonal, i.e. if the main axes of $\mathscr{E}$ are the unit vectors in $\mathbb{R}^n$. Here we have card(C) = 2^n for $1 \leq p < 2$ while for $2 < p < \infty$ card (C) coincides with $2k$ and k denotes the multiplicity of the largest eigenvalue of R. Especially we always have card(C) $\leq$ 2n in this case. In a recent paper R. Grząślewicz ([Gr]) proved card(C) $< \infty$ for $p < 2$ and arbitrary S. Unfortunately, his proof is not correct. So our first aim is to give a correct proof of this result. But it remains open whether or not we even have card(C) $\leq 2^n$ in the general case.

The case $2 < p < \infty$ turns out to be much more complicated. It seems to be open if always card(C) $< \infty$ also for those p. We conjecture that this is so at least if p is not an even integer. We show by an example that the method of proof for $p < 2$ does not apply for $p > 2$.

Finally we should mention that there exist estimates for card(C) in the case of special matrices, i.e. for special ellipsoids (John ellipsoids) (cf. [T/P]).

1. DERIVATIVES OF THE l_p^n-NORM

The results of this section are well-known and easy to prove. The summarize them for later use.

Let $y = (y_1, \ldots, y_n)$ be in $\mathbb{R}^n$ with $||y||_p = 1$. Then we define $d_y \in \mathbb{R}^n$ by

$$d_y := (y_1^{p-1}, \ldots, y_n^{p-1})$$

where we write a^α instead of $|a|^\alpha sgn(a)$ for $a, \alpha \in \mathbb{R}$. For $1 < p < \infty$ the vector d_y is uniquely determined by

$$||d_y||_{p'} = 1, \quad 1/p + 1/p' = 1, \quad \text{and} \quad \langle y, d_y \rangle = 1.$$

Let D_y be the diagonal matrix with $(|y_i|^{p-2})_{i=1}^n$ at the diagonal. Of course, we have to assume $y_i \neq 0, 1 \leq i \leq n$, whenever $p < 2$.

Lemma 1. *Let $y = (y_1, \ldots, y_n)$ be in $\mathbb{R}^n$ with $||y||_p = 1$. If $p < 2$ we assume $y_i \neq 0, 1 \leq$ $\leq i \leq n$. The l_p^n-norm is twice differentiable at y and for $v \in \mathbb{R}^n$*

$$||y + v||_p = 1 + \langle v, d_y \rangle + \frac{p-1}{2}[\langle D_y v, v \rangle - |\langle v, d_y \rangle|^2] + o(|v|^2).$$

Corollary 1. *Let $S : l_2^n \to l_p^n$ be as above and suppose that $y = Sz$ satisfies the assumptions of Lemma 1. Then the function $u \to \|Su\|_p$ is twice differentiable at $z \in \mathbb{R}^n$ and for $x \in \mathbb{R}^n$ we have*

$$\|S(z+x)\|_p = 1 + \langle x, S^* d_{Sz}\rangle + \\ + \frac{p-1}{2}[\langle S^* D_{Sz} Sx, x\rangle - |\langle x, S^* d_{Sz}\rangle|^2] + o(|x|^2).$$

Remark. Let $C \subseteq \mathbb{R}^n$ be defined as above, i.e. $z \in C$ iff $|z| = \|Sz\|_p = 1$. As shown in [Li] for those z the image Sz satisfies the assumptions of Lemma 1, i.e. for $p < 2$ all coordinates of Sz are necessarily different of zero. Moreover, since $\|S^* : l_{p'}^n \to l_2^n\| = 1$ and $\|d_{Sz}\|_{p'} = 1$, by

$$\langle z, S^* d_{Sz}\rangle = \langle Sz, d_{Sz}\rangle = 1$$

we obtain $S^* d_{Sz} = z$ in this case. Hence the following is true:

Proposition 1. *Let S be as above and let z be in C. Then we have for all $x \in \mathbb{R}^n$*

$$\|S(z+x)\|_p = 1 + \langle x, z\rangle + \frac{p-1}{2}[\langle S^* D_{Sz} Sx, x\rangle - |\langle x, z\rangle|^2] + o(|x|^2).$$

Especially, for $x \perp z$ it follows

$$\|S(z+x)\|_p = 1 + \frac{p-1}{2}\langle S^* D_{Sz} Sx, x\rangle + o(|x|^2).$$

Remark. If $x \perp z$, then by assumption

$$\|S(z+x)\|_p \leq |z+x| = (1+|x|^2)^{1/2} = 1 + |x|^2/2 + o(|x|^2).$$

Consequently, we necessarily have

$$(p-1)\langle S^* D_{Sz} Sx, x\rangle \leq |x|^2$$

in this case. Let $\lambda_1, \dots, \lambda_n$ be the eigenvalues of $S^* D_{Sz} S$. Because of $S^* D_{Sz} Sz = z$ one, say λ_1, eigenvalue is equal to 1 and the remaining satisfy

$$0 \leq \lambda_i \leq 1/(p-1), \qquad 2 \leq i \leq n.$$

We shall prove later on $\lambda_i < 1/(p-1)$ for $p < 2$ while we may have $\lambda_i = 1/(p-1)$ for some $i \geq 2$ in the case $p > 2$.

Proposition 2. *Let z be in C and suppose*

$$(p-1)\langle S^* D_{Sz} Sx, x\rangle < |x|^2$$

for all $x \perp z, x \neq 0$, i.e. we assume $0 \leq \lambda_i < 1/(p-1), 2 \leq i \leq n$, in the above notation. Then there is a neighborhood U of z in $\{u \in \mathbf{R}^n; |u| = 1\}$ such that $\|Su\|_p < 1$ for all $u \in U \backslash \{z\}$.

Proof. The assumption easily implies (use Prop. 1 and the following remark)

$$\|S(z+x)\|_p < (1+|x|^2)^{1/2}$$

for all $x \perp z$ with $0 < |x| < \delta$ for some $\delta > 0$. If $|u| = 1$ and $u \neq z$ we write

$$u = \langle u, z\rangle z + x$$

where $x \perp z$ and $|x|^2 = 1 - |\langle u, z\rangle|^2$.

Hence

$$\begin{aligned} \|Su\|_p &= |\langle u, z\rangle| \|S(z + x/\langle u, z\rangle)\|_p \\ &< |\langle u, z\rangle| (1 + |x|^2 |\langle u, z\rangle|^{-2})^{1/2} = |u| = 1 \end{aligned}$$

provided that

$$0 < (1 - |\langle u, z\rangle|^2)/|\langle u, z\rangle|^2 < \delta^2.$$

Of course, this completes the proof.

Corollary 2. *Let S be as above and suppose we have*

$$(p-1)\langle S^* D_{Sz} Sx, x\rangle < |x|^2$$

for all $z \in C$ and all $x \perp z, x \neq 0$. Then it follows

$$\mathrm{card}(C) < \infty.$$

Proof. This is an easy consequence of Prop. 2 and of the compactness of the unit sphere in $\mathbf{R}^n$.

2. THE CASE $1 \le p \le 2$

Let us first shortly mention the trivial cases $p = 1$ and $p = 2$. As above we define the positive and symmetric matrix R by $R := SS^*$. Recall that we always assume S to be regular and $||S : l_2^n \to l_p^n|| = 1$.

(1) If $p = 2$, then card(C) is finite iff the largest eigenvalue $\lambda_1 (= 1)$ of R has multiplicity 1. Then

$$C = \{\pm S^* x_1\}$$

where x_1 is a normed eigenvector with respect to $\lambda_1 = 1$.

Hence, if $p = 2$, then either card(C) $= 2$ or card(C) $= \infty$ in dependence of the multiplicity of the largest eigenvalue of R.

(2) For $p = 1$ it easily follows that

$$C = \{S^* e; e \in E_n \quad \text{and} \quad \langle Re, e\rangle = 1\}.$$

Here $e \in E_n$ if $e = (\varepsilon_1, \ldots, \varepsilon_n)$ with $\varepsilon_i = \pm 1$. Consequently, for $p = 1$ we always have

$$\text{card(C)} \le 2^n.$$

Moreover, it is not difficult to see that card(C) $= 2^n$ iff R is a diagonal matrix (with trace $(R) = n$).

Next we treat the case $1 < p < 2$. As mentioned above the proof of card(C) $< \infty$ in [Gr] is not correct. The error is in the proof of Lemma 1 in [Gr]. There the author claims (we use his notation) that $||y + \lambda v||_p^p$ is close to $(1 + \lambda^2/(p-1))^{p/2}$ as λ is near to zero. This is indeed true yet doesn't contradict the estimate

$$||y + \lambda v||_p^p \le (1 + \lambda^2)^{p/2}$$

as written in [Gr]. In order to obtain a contradiction one has to have

$$\lambda^{-2}\{||y + \lambda v||_p^p - (1 + \lambda^2/(p-1))^{p/2}\} \to 0$$

as $\lambda \to 0$ which isn't valid in general.

Theorem 1. *Suppose $1 < p < 2$ and let S from l_2^n onto l_p^n be as above. Then the number of points where S attains its norm on the unit sphere is finite* (card(C) $< \infty$).

Proof. Choose $z \in C$. In view of Cor. 2 we only have to show

$$(p-1)\langle S^* D_{Sz} Sx, x\rangle < |x|^2$$

for all $x \perp z, x \neq 0$. Let us assume the contrary, i.e. for some $x \perp z, |x| = 1$ we have

$$(p-1)\langle S^* D_{Sz} S x, x\rangle = 1.$$

Defining $y = (y_1, \ldots, y_n)$ by $y := Sz$ and $v = (v_1, \ldots, v_n)$ by $v := Sx$ this yields

$$1 = (p-1)\langle S^* D_{Sz} S x, x\rangle = (p-1)\sum_{i=1}^{n} |y_i|^{p-2}|v_i|^2.$$

Next we define a function f on $\mathbf{R}$ by

$$f(t) := (1+t^2)^{p/2} - \|y + tv\|_p^p$$

which satisfies $f(t) \geq 0$ because of $\|S : l_2^n \to l_p^n\| = 1$.

Since $y_i \neq 0, 1 \leq i \leq n$ (cf. Remark after Cor. 1) f is infinitely often differentiable at zero and its Taylor expansion is given by

$$f(t) = \sum_{k=0}^{\infty} \binom{p/2}{k} t^{2k} - \sum_{k=0}^{\infty} \binom{p}{k} \left[\sum_{i=1}^{\infty} |y_i|^p (v_i/y_i)^k\right] t^k.$$

Recall that $(v_i/y_i)^k = |v_i/y_i|^k sgn(v_i/y_i)$ whenever k is an odd integer.

From the assumption and by

$$\sum_{i=1}^{n} y_i^{p-1} v_i = \langle Sx, d_{Sz}\rangle = \langle x, z\rangle = 0$$

we derive

$$f(t) = \sum_{k=2}^{\infty} \binom{p/2}{k} t^{2k} - \sum_{k=3}^{\infty} \binom{p}{k} \left[\sum_{i=1}^{n} |y_i|^p (v_i/y_i)^k\right] t^k.$$

Because of $f(t) \geq 0$ for all $t \in \mathbf{R}$ the summation of the second term cannot start at $k = 3$, i.e. we also have

$$\sum_{i=1}^{n} |y_i|^p (v_i/y_i)^3 = 0,$$

and, consequently,

$$f(t) = \left[\binom{p/2}{2} - \binom{p}{4}\sum_{i=1}^{n} |y_i|^{p-4}|v_i|^4\right] t^4 + o(t^4).$$

But in view of $1 < p < 2$ we have $\binom{p/2}{2} < 0$ as well as $\binom{p}{4} > 0$ which implies $f(t)/t^4 < 0$ for t near to zero and, of course, contradicts $f(t) \geq 0$ for all real t. This completes the proof of Theorem 1.

Remark. If the proof of Lemma 1 in [Gr] would be correct it would even give a stronger estimate for $\langle S^* D_{Sz} Sx, x\rangle$, namely,

$$\langle S^* D_{Sz} Sx, x\rangle \leq |x|^2$$

for $x \perp z$. Observe that $p-1 < 1$. This estimate is equivalent to $\liminf_{t\to 0} f(t)/t^2 \geq p(2-p)/2 > 0$ where f is defined as in the proof of Th. 1. We do not know if this stronger estimate is true in general. It holds in the case that $R = SS^*$ is diagonal. In this case one always has $S^* D_{Sz} S = I$, I identity, as easy computations show.

3. THE CASE $2 < p < \infty$

It seems to be an open question whether or not a linear mapping from l_2^n onto $l_p^n, p > 2$, may attain its norm at infinitely many points of the unit sphere. The aim of this section is to show that in contrast to the case $p < 2$ an estimate

$$(p-1)\langle S^* D_{Sz} Sx, x\rangle < |x|^2,$$

$x \perp z, x \neq 0$, does not hold for $p > 2$ in general. Thus it may happen that $S^* D_{Sz} S$ possesses an eigenvalue equal $1/(p-1)$. Geometrically this means that an ellipsoid inscribed in $U_p^n, 2 < p < \infty$, may contact the boundary of the l_p^n-ball «very smoothly». Consequently, at least the ideas of the proof of Theorem 1 are not applicable for $p > 2$.

We start with an interesting inequality due to W. Beckner ([Be]). A proof can be found in [L/T], p. 75.

Proposition 3. *For $2 \leq p < \infty$ and $t \in \mathbf{R}$ we have*

$$(|1+t|^p + |1-t|^p)/2 \leq (1 + (p-1)t^2)^{p/2}.$$

We define now a 2×2-matrix as follows:

$$S := 2^{-1/p} \begin{pmatrix} 1 & (p-1)^{-1/2} \\ 1 & -(p-1)^{-1/2} \end{pmatrix}.$$

Proposition 4. *For $2 < p < \infty$ this matrix S satisfies*

$$\|S : l_2^2 \to l_p^2\| = 1.$$

Proof. This is an easy consequence of Prop. 3. Observe that S attains its norm at $\pm e_1$ with $e_1 := (1,0)$.

Corollary 3. *If* $e_2 := (0, 1)$, *then we have*

$$(p-1)\langle S^* D_{Se_1} Se_2, e_2\rangle = 1,$$

i.e. it holds

$$(1+t^2)^{p/2} - \|S(e_1 + te_2)\|_p^p = o(t^2).$$

Proof. It is easy to see that

$$S^* D_{Se_1} S = \begin{pmatrix} 1 & 0 \\ 0 & (p-1)^{-1} \end{pmatrix}$$

which proves the Corollary.

Remark. (1) We should mention that in this example

$$(1+t^2)^{p/2} \ \| S(e_1 + te_2) \|_p^p = \frac{p^2(p-2)}{12(p-1)} t^4 + o(t^4).$$

Consequently, also in this example e_1 is an isolated maximum of $u \to \|Su\|_p$ on the unit sphere.

(2) The corresponding ellipsoid $\mathscr{E}$ in $\mathbf{R}^2$ is generated by

$$R = SS^* = \frac{2^{-2/p}}{p-1} \begin{pmatrix} p & p-2 \\ p-2 & p \end{pmatrix},$$

i.e.

$$\mathscr{E} = \{(x_1, x_2); |x_1 + x_2|^2 + (p-1)|x_1 - x_2|^2 \le 2^{2/p'}\}.$$

It contacts ∂U_p^2 at the points $\pm 2^{-1/p}(1, 1)$ «very smoothly».

We shall prove now that up to a unitary mapping the matrix S defined above is the only example in $\mathbf{R}^2$ where $S^* D_{Sz} S$ possesses the eigenvalue $(p-1)^{-1}$ for some $z \in C$.

Proposition 5. *Let* S *be an arbitrary mapping from* l_2^2 *onto* l_p^2, $2 < p < \infty$, *with* $\|S\| = 1$. *Suppose that* $\|Sz\|_p = |z| = 1$ *and*

$$(p-1)\langle S^* D_{Sz} Sx, x\rangle = 1$$

for some $x \perp z, |x| = 1$. *Then either*

$$Sz = \pm 2^{-1/p}(1,1) \qquad \textit{and} \qquad Sx = \pm \frac{2^{-1/p}}{(p-1)^{1/2}}(1,-1)$$

or

$$Sz = \pm 2^{-1/p}(1,-1) \qquad \textit{and} \qquad Sx = \pm \frac{2^{-1/p}}{(p-1)^{1/2}}(1,1).$$

Proof. Set $Sz = (y_1, y_2)$ and $Sx = (v_1, v_2)$. Using the same argument as in the proof of Theorem 1 we obtain the following four equations:

$$\begin{aligned} &|y_1|^p + |y_2|^p = 1, \\ &y_1^{p-1} v_1 + y_2^{p-1} v_2 = 0, \\ &|y_1|^{p-2} v_1^2 + |y_2|^{p-2} v_2^2 = (p-1)^{-1} \qquad \text{and} \\ &y_1^{p-3} v_1^3 + y_2^{p-3} v_2^3 = 0. \end{aligned}$$

The only solutions are $|y_1| = |y_2| = 2^{-1/p}, |v_1| = |v_2| = 2^{-1/p}(p-1)^{-1/2}$ and $(y_1, y_2) \perp$ $\perp (v_1, v_2)$. This completes the proof.

Remark. Prop. 5 tells us that only at the points $\pm 2^{-1/p}(1,1)$ and $\pm 2^{-1/p}(1,-1)$ of ∂U_p^2 an ellipsoid may contact the boundary «very smoothly». Moreover, there are exactly two possible ellipsoids.

One may ask now if such examples also exist in higher dimensions. The answer is affirmative by trivial reasons. For $n \geq 2$ we fix two integers $k, l \in \{1, \ldots, n\}, k < \ell$, and define

$$S_0 e_1 := 2^{-1/p}(e_k + e_\ell)$$

and

$$S_0 e_2 := \frac{2^{-1/p}}{(p-1)^{1/2}}(e_k - e_\ell).$$

Here $e_1, \ldots, e_n$ denotes the sequence of unit vectors in $\mathbb{R}^n$. If S_1 maps l_p^{n-2} onto l_p^{n-2} with $\|S_1\| \leq 1$ we define it as a mapping from span $\{e_3, \ldots, e_n\}$ onto span $\{e_j; j \neq k, l\}$ in canonical way. Then S from l_2^n onto l_p^n with

$$Sx := S_0(x_1, x_2) + S_1(x_3, \ldots, x_n),$$

$x = (x_1, \dots, x_n)$, satisfies

$$\|Sx\|_p = (\|S_0(x_1, x_2)\|_p^p + \|S_1(x_3, \dots, x_n)\|_p^p)^{1/p}$$
$$\leq \left((|x_1|^2 + |x_2|^2)^{p/2} + \left(\sum_{i=3}^{n} |x_i|^2 \right)^{p/2} \right)^{1/p} \leq |x|$$

as well as

$$\langle S^* D_{Se_1} Se_2, e_2 \rangle = (p-1)^{-1}.$$

Problems. *(1) In the preceding example the matrix $S^* D_{Sz} S$ ($z = e_1$) had exactly one eigenvalue equal to $(p-1)^{-1}$. What is the maximal multiplicity of this eigenvalue? Equivalently, in how many orthogonal directions may an ellipsoid contact ∂U_p^n «very smoothly»?*

(2) In our example the «exceptional» contact points are of the form $2^{-1/p}(y_1, \dots, y_n)$ with $y_i \in \{-1, 0, 1\}$ and $\text{card}\{i; y_i \neq 0\} = 2$. Are this the only points of ∂U_p^n where an ellipsoid may contact the l_p^n-sphere «very smoothly»?

ACKNOWLEDGEMENT

Part of this paper has been written during our visit at the universities in Lecce and in Roma II. The author is grateful for the hospitality during his stay at these universities. Especially we want to thank G. Metafune (Roma II) for several helpful discussions about the subject of this paper.

REFERENCES

[Be] W. BECKNER, *Inequalities in Fourier analysis*, Ann. of Math., **102** (1975), pp. 159-182.

[Gr] R. GRZAŚLEWICZ, *Extremal structure of* $L(l_m^2, l_n^p)$, Lin. Multilin. Alg., **24** (1989), pp. 117-125.

[Li] W. LINDE, *Gaussian measures of large balls in* R^n, Progress in Probability, **25** (1991), pp. 1-26.

[L/T] J. LINDENSTRAUSS, L. TZAFRIRI, *Classical Banach spaces II*, Springer-Verlag, Berlin-Heidelberg-New York, 1979.

[T/P] N. TOMCZAK-JAEGERMAN, A. PEŁCZYNSKI, *On the length of faithful nuclear representations of finite rank operators*, Matematika, **35** (1988), pp. 126-143.

Received November 6, 1990
W. Linde
Matematische Fakultät
Universität Jena
Universitätshochhaus 17.0G
0-6900 Jena, Germany