THE NUMBER OF POINTS WHERE A LINEAR MAPPING
FROM $l^n_2$ INTO $l^n_p$ ATTAINS ITS NORM

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Dedicated to the memory of Professor Gottfried Köthe

Abstract. Let $S$ be a regular $n \times n$-matrix mapping $l^n_2$ onto $l^n_p$, $1 \leq p < \infty$, with norm
$||S|| = ||S : l^n_2 \rightarrow l^n_p||$. Then we are interested in the set

$$C := \{ z \in \mathbb{R}^n; ||z||_2 = 1 \quad \text{and} \quad ||Sz||_p = ||S||\},$$

i.e. the set of points on the unit sphere where $S$ attains its norm. We prove $\text{card}(C) < \infty$ for
$1 \leq p < 2$. This follows from properties of the Taylor expansion of $x \rightarrow ||Sx||_p$ near points
in $C$. The case $2 < p < \infty$ remains open. But we show by an example that for $p > 2$ the
behaviour of $x \rightarrow ||Sx||_p$ may be completely different as for $p < 2$.

0. INTRODUCTION

As usual we denote by $l^n_p$ the space $\mathbb{R}^n$ equipped with the norm

$$||x||_p := \left\{ \sum_{i=1}^n |x_i|^p \right\}^{1/p}, \quad x = (x_1, \ldots, x_n), \quad 1 \leq p < \infty.$$

We shall write $|x|$ instead of $||x||_2$.

Let $S$ be an $n \times n$-matrix. Then its norm $||S : l^n_2 \rightarrow l^n_p||$ is defined by sup $\{||Sx||_p; |x| \leq 1\}$. For simplicity we always assume $||S : l^n_2 \rightarrow l^n_p|| = 1$. Moreover we always suppose
that $S$ is regular, i.e. $S$ maps $l^n_2$ onto $l^n_p$. The set $C \subseteq \mathbb{R}^n$ of norm attaining points is then
defined by

$$C := \{ z \in \mathbb{R}^n; |z| = ||Sz||_p = 1\}.$$

The image $S(C)$ may be described geometrically. Define an ellipsoid $E$ by

$$E := \{ x \in \mathbb{R}^n; \langle R^{-1}x, x \rangle \leq 1 \}$$

where $R = SS^*$. By assumption we have

$$E \subseteq U^n_p := \{ x \in \mathbb{R}^n; ||x||_p \leq 1\}.$$
Then $S(C)$ coincides with those points where $S$ contacts the boundary of $U_p^n$, i.e.

$$S(C) = S \cap \partial U_p^n.$$ 

Hence we have $\text{card}(C) = \text{card}(S(C))$ and the problem may be formulated as follows: What is the maximal number of points where an ellipsoid inscribed in $U_p^n$ contacts the boundary of $U_p^n$? The question is easy to answer if $R$ is diagonal, i.e. if the main axes of $S$ are the unit vectors in $\mathbb{R}^n$. Here we have $\text{card}(C) = 2^n$ for $1 \leq p < 2$ while for $2 < p < \infty$ card $(C)$ coincides with $2 \tilde{k}$ and $\tilde{k}$ denotes the multiplicity of the largest eigenvalue of $R$. Especially we always have $\text{card}(C) \leq 2n$ in this case. In a recent paper R. Grzaslewicz ([Gr]) proved $\text{card}(C) < \infty$ for $p < 2$ and arbitrary $S$. Unfortunately, his proof is not correct. So our first aim is to give a correct proof of this result. But it remains open whether or not we even have $\text{card}(C) \leq 2^n$ in the general case.

The case $2 < p < \infty$ turns out to be much more complicated. It seems to be open if always $\text{card}(C) < \infty$ also for those $p$. We conjecture that this is so at least if $p$ is not an even integer. We show by an example that the method of proof for $p < 2$ does not apply for $p > 2$.

Finally we should mention that there exist estimates for $\text{card}(C)$ in the case of special matrices, i.e. for special ellipsoids (John ellipsoids) (cf. [T/P]).

1. DERIVATIVES OF THE $l_p^n$-NORM

The results of this section are well-known and easy to prove. The summarize them for later use.

Let $y = (y_1, \ldots, y_n)$ be in $\mathbb{R}^n$ with $\|y\|_p = 1$. Then we define $d_y \in \mathbb{R}^n$ by

$$d_y := (y_1^{p-1}, \ldots, y_n^{p-1})$$

where we write $a^\alpha$ instead of $|a|^{\alpha} \text{sgn}(a)$ for $a, \alpha \in \mathbb{R}$. For $1 < p < \infty$ the vector $d_y$ is uniquely determined by

$$\|d_y\|_{p'} = 1, \quad 1/p + 1/p' = 1, \quad \text{and} \quad \langle y, d_y \rangle = 1.$$ 

Let $D_y$ be the diagonal matrix with $(|y_i|^{p-2})_{i=1}^n$ at the diagonal. Of course, we have to assume $y_i \neq 0, 1 \leq i \leq n$, whenever $p < 2$.

**Lemma 1.** Let $y = (y_1, \ldots, y_n)$ be in $\mathbb{R}^n$ with $\|y\|_p = 1$. If $p < 2$ we assume $y_i \neq 0, 1 \leq i \leq n$. The $l_p^n$-norm is twice differentiable at $y$ and for $v \in \mathbb{R}^n$

$$\|y + v\|_p = 1 + \langle v, d_y \rangle + \frac{p-1}{2} \left[ \langle D_y v, v \rangle - |\langle v, d_y \rangle|^2 \right] + o(|v|^2).$$
Corollary 1. Let \( S : l^n_2 \to l^n_p \) be as above and suppose that \( y = Sz \) satisfies the assumptions of Lemma 1. Then the function \( u \to \|Su\|_p \) is twice differentiable at \( z \in \mathbb{R}^n \) and for \( x \in \mathbb{R}^n \) we have

\[
\|S(z + x)\|_p = 1 + \langle x, S^*d_{Sz} \rangle + \\
\frac{p - 1}{2} \left[ \langle S^*D_{Sz}Sx, x \rangle - |\langle x, S^*d_{Sz} \rangle|^2 \right] + o(|x|^2).
\]

Remark. Let \( C \subseteq \mathbb{R}^n \) be defined as above, i.e. \( z \in C \) iff \( |z| = \|Sz\|_p = 1 \). As shown in [Li] for those \( z \) the image \( Sz \) satisfies the assumptions of Lemma 1, i.e. for \( p < 2 \) all coordinates of \( Sz \) are necessarily different of zero. Moreover, since \( \|S^* : l^n_p \to l^n_2 \| = 1 \) and \( \|d_{Sz}\|_{p'} = 1 \), by

\[
\langle z, S^*d_{Sz} \rangle = \langle Sz, d_{Sz} \rangle = 1
\]

we obtain \( S^*d_{Sz} = z \) in this case. Hence the following is true:

Proposition 1. Let \( S \) be as above and let \( z \) be in \( C \). Then we have for all \( x \in \mathbb{R}^n \)

\[
\|S(z + x)\|_p = 1 + \langle x, z \rangle + \frac{p - 1}{2} \left[ \langle S^*D_{Sz}Sx, x \rangle - |\langle x, z \rangle|^2 \right] + o(|x|^2).
\]

Especially, for \( x \perp z \) it follows

\[
\|S(z + x)\|_p = 1 + \frac{p - 1}{2} \langle S^*D_{Sz}Sx, x \rangle + o(|x|^2).
\]

Remark. If \( x \perp z \), then by assumption

\[
\|S(z + x)\|_p \leq |z + x| = (1 + |x|^2)^{1/2} = 1 + |x|^2/2 + o(|x|^2).
\]

Consequently, we necessarily have

\[
(p - 1) \langle S^*D_{Sz}Sx, x \rangle \leq |x|^2
\]

in this case. Let \( \lambda_1, \ldots, \lambda_n \) be the eigenvalues of \( S^*D_{Sz}S \). Because of \( S^*D_{Sz}S = z \) one, say \( \lambda_1 \), eigenvalue is equal to 1 and the remaining satisfy

\[
0 \leq \lambda_i \leq 1/(p - 1), \quad 2 \leq i \leq n.
\]

We shall prove later on \( \lambda_i < 1/(p - 1) \) for \( p < 2 \) while we may have \( \lambda_i = 1/(p - 1) \) for some \( i \geq 2 \) in the case \( p > 2 \).
Proposition 2. Let \( z \) be in \( C \) and suppose

\[
(p - 1) \langle S^* D_{S^* S} x, x \rangle < |x|^2
\]

for all \( x \perp z, x \neq 0 \), i.e. we assume \( 0 \leq \lambda_i < 1/(p - 1) \), \( 2 \leq i \leq n \), in the above notation. Then there is a neighborhood \( U \) of \( z \) in \( \{ u \in \mathbb{R}^n; |u| = 1 \} \) such that \( ||Su||_p < 1 \) for all \( u \in U \setminus \{ z \} \).

Proof. The assumption easily implies (use Prop. 1 and the following remark)

\[
||S(z + x)||_p < (1 + |x|^2)^{1/2}
\]

for all \( x \perp z \) with \( 0 < |x| < \delta \) for some \( \delta > 0 \). If \( |u| = 1 \) and \( u \neq z \) we write

\[
u = \langle u, z \rangle z + x
\]

where \( x \perp z \) and \( |x|^2 = 1 - |\langle u, z \rangle|^2 \).

Hence

\[
||Su||_p = |\langle u, z \rangle||S(z + x/\langle u, z \rangle)||_p
\]

\[
< |\langle u, z \rangle|(1 + |x|^2)(|\langle u, z \rangle|^{-2})^{1/2} = |u| = 1
\]

provided that

\[
0 < (1 - |\langle u, z \rangle|^2) / |\langle u, z \rangle|^2 < \delta^2.
\]

Of course, this completes the proof.

Corollary 2. Let \( S \) be as above and suppose we have

\[
(p - 1) \langle S^* D_{S^* S} x, x \rangle < |x|^2
\]

for all \( z \in C \) and all \( x \perp z, x \neq 0 \). Then it follows

\[
\text{card}(C) < \infty.
\]

Proof. This is an easy consequence of Prop. 2 and of the compactness of the unit sphere in \( \mathbb{R}^n \).
2. **THE CASE** $1 \leq p \leq 2$

Let us first shortly mention the trivial cases $p = 1$ and $p = 2$. As above we define the positive and symmetric matrix $R$ by $R := SS^*$. Recall that we always assume $S$ to be regular and $\|S : l_2^n \to l_p^n\| = 1$.

(1) If $p = 2$, then $\text{card}(C)$ is finite iff the largest eigenvalue $\lambda_1 = 1$ of $R$ has multiplicity 1. Then

$$C = \{ \pm S^*x_1 \}$$

where $x_1$ is a normed eigenvector with respect to $\lambda_1 = 1$.

Hence, if $p = 2$, then either $\text{card}(C) = 2$ or $\text{card}(C) = \infty$ in dependence of the multiplicity of the largest eigenvalue of $R$.

(2) For $p = 1$ it easily follows that

$$C = \{ S^*e; e \in E_n \text{ and } \langle Re, e \rangle = 1 \}.$$

Here $e \in E_n$ if $e = (\varepsilon_1, \ldots, \varepsilon_n)$ with $\varepsilon_i = \pm 1$. Consequently, for $p = 1$ we always have

$$\text{card}(C) \leq 2^n.$$

Moreover, it is not difficult to see that $\text{card}(C) = 2^n$ iff $R$ is a diagonal matrix (with trace $(R) = n$).

Next we treat the case $1 < p < 2$. As mentioned above the proof of $\text{card}(C) < \infty$ in [Gr] is not correct. The error is in the proof of Lemma 1 in [Gr]. There the author claims (we use his notation) that $\|y + \lambda v\|_p^p$ is close to $(1 + \lambda^2/(p - 1))^{p/2}$ as $\lambda$ is near to zero. This is indeed true yet doesn't contradict the estimate

$$\|y + \lambda v\|_p^p \leq (1 + \lambda^2)^{p/2}$$

as written in [Gr]. In order to obtain a contradiction one has to have

$$\lambda^{-2} \{|\|y + \lambda v\|_p^p - (1 + \lambda^2/(p - 1))^{p/2}|\} \to 0$$

as $\lambda \to 0$ which isn't valid in general.

**Theorem 1.** Suppose $1 < p < 2$ and let $S$ from $l_2^n$ onto $l_p^n$ be as above. Then the number of points where $S$ attains its norm on the unit sphere is finite ($\text{card}(C) < \infty$).

**Proof.** Choose $z \in C$. In view of Cor. 2 we only have to show

$$(p - 1)\langle S^*D_{Sz}, x \rangle < |x|^2$$
for all \( x \perp z, x \neq 0 \). Let us assume the contrary, i.e. for some \( x \perp z, |x| = 1 \) we have
\[
(p - 1) \langle S^* D_{sz} Sx, x \rangle = 1.
\]
Defining \( y = (y_1, \ldots, y_n) \) by \( y := Sz \) and \( v = (v_1, \ldots, v_n) \) by \( v := Sx \) this yields
\[
1 = (p - 1) \langle S^* D_{sz} Sx, x \rangle = (p - 1) \sum_{i=1}^{n} |y_i|^{p-2} |v_i|^2.
\]
Next we define a function \( f \) on \( \mathbb{R} \) by
\[
f(t) := (1 + t^2)^{p/2} - ||y + tv||_p^p
\]
which satisfies \( f(t) \geq 0 \) because of \( ||S : l_2^n \rightarrow l_p^n|| = 1 \).

Since \( y_i \neq 0, 1 \leq i \leq n \) (cf. Remark after Cor. 1) \( f \) is infinitely often differentiable at zero and its Taylor expansion is given by
\[
f(t) = \sum_{k=0}^{\infty} \left( \frac{p/2}{k} \right) t^{2k} - \sum_{k=0}^{\infty} \left( \frac{p}{k} \right) \left[ \sum_{i=1}^{n} |y_i|^p (v_i/y_i)^k \right] t^k.
\]
Recall that \( (v_i/y_i)^k = |v_i/y_i|^k \text{sgn}(v_i/y_i) \) whenever \( k \) is an odd integer.

From the assumption and by
\[
\sum_{i=1}^{n} y_i^{p-1} v_i = \langle Sx, d_{sz} \rangle = \langle x, z \rangle = 0
\]
we derive
\[
f(t) = \sum_{k=2}^{\infty} \left( \frac{p/2}{k} \right) t^{2k} - \sum_{k=3}^{\infty} \left( \frac{p}{k} \right) \left[ \sum_{i=1}^{n} |y_i|^p (v_i/y_i)^k \right] t^k.
\]
Because of \( f(t) \geq 0 \) for all \( t \in \mathbb{R} \) the summation of the second term cannot start at \( k = 3 \), i.e. we also have
\[
\sum_{i=1}^{n} |y_i|^p (v_i/y_i)^3 = 0,
\]
and, consequently,
\[
f(t) = \left( \frac{p/2}{2} \right) - \left( \frac{p}{4} \right) \sum_{i=1}^{n} |y_i|^{p-4} |v_i|^4 \right] t^4 + o(t^4).
\]
But in view of \( 1 < p < 2 \) we have \( \left( \frac{p/2}{2} \right) < 0 \) as well as \( \left( \frac{p}{4} \right) > 0 \) which implies \( f(t)/t^4 < 0 \) for \( t \) near to zero and, of course, contradicts \( f(t) \geq 0 \) for all real \( t \). This completes the proof of Theorem 1.
Remark. If the proof of Lemma 1 in [Gr] would be correct it would even give a stronger estimate for $\langle S^* D_{S^*} S x, x \rangle$, namely,

$$\langle S^* D_{S^*} S x, x \rangle \leq |x|^2$$

for $x \perp z$. Observe that $p - 1 < 1$. This estimate is equivalent to $\liminf_{t \to 0} f(t)/t^2 \geq p(2-p)/2 > 0$ where $f$ is defined as in the proof of Th. 1. We do not know if this stronger estimate is true in general. It holds in the case that $R = SS^*$ is diagonal. In this case one always has $S^* D_{S^*} S = I$, $I$ identity, as easy computations show.

3. THE CASE $2 < p < \infty$

It seems to be an open question whether or not a linear mapping from $l_2^n$ onto $l_p^n$, $p > 2$, may attain its norm at infinitely many points of the unit sphere. The aim of this section is to show that in contrast to the case $p < 2$ an estimate

$$(p - 1) \langle S^* D_{S^*} S x, x \rangle < |x|^2,$$

$x \perp z, x \neq 0$, does not hold for $p > 2$ in general. Thus it may happen that $S^* D_{S^*} S$ possesses an eigenvalue equal $1/(p - 1)$. Geometrically this means that an ellipsoid inscribed in $U^n_p, 2 < p < \infty$, may contact the boundary of the $l_p^n$-ball «very smoothly». Consequently, at least the ideas of the proof of Theorem 1 are not applicable for $p > 2$.

We start with an interesting inequality due to W. Beckner ([Be]). A proof can be found in [L/T], p. 75.

**Proposition 3.** For $2 \leq p < \infty$ and $t \in \mathbb{R}$ we have

$$\langle |1 + t|^p + |1 - t|^p \rangle / 2 \leq (1 + (p - 1)t^2)^{p/2}.$$

We define now a $2 \times 2$-matrix as follows:

$$S := 2^{-1/p} \begin{pmatrix} 1 & (p - 1)^{-1/2} \\ 1 & -(p - 1)^{-1/2} \end{pmatrix}.$$

**Proposition 4.** For $2 < p < \infty$ this matrix $S$ satisfies

$$\|S : l_2^2 \to l_p^2\| = 1.$$

**Proof.** This is an easy consequence of Prop. 3. Observe that $S$ attains its norm at $\pm e_1$ with $e_1 := (1, 0)$. 


Corollary 3. If $e_2 := (0, 1)$, then we have

$$(p - 1) (S^* D_{S e_1} S e_2, e_2) = 1,$$

i.e. it holds

$$(1 + t^2)^{p/2} - ||S(e_1 + t e_2)||_p^p = o(t^2).$$

Proof. It is easy to see that

$$S^* D_{S e_1} S = \begin{pmatrix} 1 & 0 \\ 0 & (p - 1)^{-1} \end{pmatrix}$$

which proves the Corollary.

Remark. (1) We should mention that in this example

$$(1 + t^2)^{p/2} ||S(e_1 + t e_2)||_p^p = \frac{p^2(p - 2)}{12(p - 1)} t^4 + o(t^4).$$

Consequently, also in this example $e_1$ is an isolated maximum of $u \to ||S u||_p$ on the unit sphere.

(2) The corresponding ellipsoid $\mathcal{E}$ in $\mathbb{R}^2$ is generated by

$$R = S S^* = \frac{2^{-2/p}}{p - 1} \begin{pmatrix} p & p - 2 \\ p - 2 & p \end{pmatrix},$$

i.e.

$$\mathcal{E} = \{ (x_1, x_2); |x_1 + x_2|^2 + (p - 1)|x_1 - x_2|^2 \leq 2^{2/p'} \}.$$ 

It contacts $\partial U_p^2$ at the points $\pm 2^{-1/p}(1, 1)$ «very smoothly».

We shall prove now that up to a unitary mapping the matrix $S$ defined above is the only example in $\mathbb{R}^2$ where $S^* D_{S z} S$ possesses the eigenvalue $(p - 1)^{-1}$ for some $z \in C$.

Proposition 5. Let $S$ be an arbitrary mapping from $l^2_2$ onto $l^2_p$, $2 < p < \infty$, with $||S|| = 1$. Suppose that $||S z||_p = |z| = 1$ and

$$(p - 1) (S^* D_{S z} S z, z) = 1$$
for some $x \perp z, |x| = 1$. Then either

$$Sz = \pm 2^{-1/p}(1, 1) \quad \text{and} \quad Sx = \pm \frac{2^{-1/p}}{(p - 1)^{1/2}} (1, -1)$$

or

$$Sz = \pm 2^{-1/p}(1, -1) \quad \text{and} \quad Sx = \pm \frac{2^{-1/p}}{(p - 1)^{1/2}} (1, 1).$$

Proof. Set $Sz = (y_1, y_2)$ and $Sx = (v_1, v_2)$. Using the same argument as in the proof of Theorem 1 we obtain the following four equations:

$$|y_1|^p + |y_2|^p = 1,$$

$$y_1^{p-1}v_1 + y_2^{p-1}v_2 = 0,$$

$$|y_1|^{p-2}v_1^2 + |y_2|^{p-2}v_2^2 = (p - 1)^{-1} \quad \text{and}$$

$$y_1^{p-3}v_1^3 + y_2^{p-3}v_2^3 = 0.$$

The only solutions are $|y_1| = |y_2| = 2^{-1/p}, |v_1| = |v_2| = 2^{-1/p}(p - 1)^{-1/2}$ and $(y_1, y_2) \perp \perp (v_1, v_2)$. This completes the proof.

Remark. Prop. 5 tells us that only at the points $\pm 2^{-1/p}(1, 1)$ and $\pm 2^{-1/p}(1, -1)$ of $\partial U_p$ an ellipsoid may contact the boundary «very smoothly». Moreover, there are exactly two possible ellipsoids.

One may ask now if such examples also exist in higher dimensions. The answer is affirmative by trivial reasons. For $n \geq 2$ we fix two integers $k, l \in \{1, \ldots, n\}$, $k \neq l$, and define

$$S_0e_1 := 2^{-1/p}(e_k + e_\ell)$$

and

$$S_0e_2 := \frac{2^{-1/p}}{(p - 1)^{1/2}}(e_k - e_\ell).$$

Here $e_1, \ldots, e_n$ denotes the sequence of unit vectors in $\mathbb{R}^n$. If $S_1$ maps $l_p^{n-2}$ onto $l_p^{n-2}$ with $||S_1|| \leq 1$ we define it as a mapping from span $\{e_3, \ldots, e_n\}$ onto span $\{e_j; j \neq k, \ell\}$ in canonical way. Then $S$ from $l_2^n$ onto $l_p^n$ with

$$Sx := S_0(x_1, x_2) + S_1(x_3, \ldots, x_n),$$
\[ x = (x_1, \ldots, x_n), \text{satisfies} \]
\[
||Sx||_p = (||S_0(x_1, x_2)||_p^p + ||S_1(x_3, \ldots, x_n)||_p^p)^{1/p} 
\leq \left( (|x_1|^2 + |x_2|^2)^{p/2} + \left( \sum_{i=3}^{n} |x_i|^2 \right)^{p/2} \right)^{1/p} \leq |x|
\]
as well as
\[
\langle S^* D_{Se_1} S e_2, e_2 \rangle = (p - 1)^{-1}.
\]

**Problems.** (1) In the preceding example the matrix \( S^* D_S S \) \((z = e_1)\) had exactly one eigenvalue equal to \((p - 1)^{-1}\). What is the maximal multiplicity of this eigenvalue? Equivalently, in how many orthogonal directions may an ellipsoid contact \( \partial U_p^n \) «very smoothly»?

(2) In our example the «exceptional» contact points are of the form \( 2^{-1/p} (y_1, \ldots, y_n) \) with \( y_i \in \{-1, 0, 1\} \) and \( \text{card} \{i; y_i \neq 0\} = 2 \). Are this the only points of \( \partial U_p^n \) where an ellipsoid may contact the \( l_p^n \)-sphere «very smoothly»?

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REFERENCES


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