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ON SOME CLASSES OF LOTOTSKY-SCHNABL OPERATORS

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Dedicated to the memory of Professor Gottfried Köthe

Abstract. *We study a sequence* $(L_n)_{n\in\mathbb{N}}$ *of positive operators associated with a sequence* $(\gamma_n)_{n\in\mathbb{N}}$ *of real numbers in the unit interval, a lower triangular stochastic matrix* P *and a positive projection* T *acting on the space of all continuous functions defined on a convex compact subset of a locally convex Hausdorff space. These operators are particular cases of the so-called Lototsky-Schnabl operators. Under suitable assumptions on* $(\gamma_n)_{n\in\mathbb{N}}$, P *and* T, *we investigate the asymptotic properties of the sequence* $(L_n)_{n\in\mathbb{N}}$ *and of its iterates in connection with the existence of a* C_0*-semigroup of positive contractions.*

INTRODUCTION

Starting with a positive projection acting on the space of all real-valued continuous functions defined on a metrizable convex compact set, Altomare ([1], [2]) and Campiti ([5], [6]) introduced some sequences of positive operators which generalize the so-called Bernstein-Schnabl operators and Stancu-Mühlbach operators.

These operators furnish examples of approximation processes in general (finite and infinite dimensional) settings and they preserve most of the typical properties of classical Bernstein and Stancu polynomials. In this paper we continue to develop a similar idea and we introduce another sequence of positive operators, by associating them again with a positive projection. These operators are particular cases of the so called Lototsky-Schnabl operators, which were firstly introduced by Schempp ([11]) and subsequently studied by Grossman ([8]) and Nishishiraho ([9], [10]).

We investigate the asymptotic properties of these operators and their iterates and we also give some estimates of the order of convergence.

Finally, we show the existence of a positive contraction semigroup whose generator is obtained as the generator of the C_0-semigroup corresponding to the Bernstein-Schnabl operators multiplied by a suitable positive constant. All these results, together with those obtained in [1], [2], [5] and [6], emphasize an unexpected and harmonious analogy among these operators, Bernstein-Schnabl operators and Stancu-Mühlbach operators.

1. LOTOTSKY-SCHNABL OPERATORS

Let X be a metrizable convex compact Hausdorff subset of some locally convex Hausdorff

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space and let us denote by $C(X,\mathbf{R})$ the Banach lattice of all real continuous functions on X, endowed with the sup-norm topology and the natural order. Let $M^+(X)$ (resp. $M^1(X)$) be the set of all positive (resp. probability) Radon measures on X. Let us consider a linear positive projection $T : C(X,\mathbf{R}) \to C(X,\mathbf{R})$ (i.e. T is a linear positive operator such that $T^2 = T$).

Afer setting

$$H = T(C(X,\mathbf{R})),$$

let us suppose that

$$A(X) \subset H,$$

where $A(X)$ is the space of all real continuous affine functions on X. In addition, let us assume that for every $\overline{x} \in X$, $\lambda \in [0,1]$ and $h \in H$ the function

$$x \in X \to h((1-\lambda)\overline{x} + \lambda x), \tag{1.1}$$

belongs to H.

We refer to [1] for some significant examples of such projections T.

In the sequel we will use the following Korovkin-type theorem which has been obtained by Altomare ([1]), Bauer ([3]), Grossman ([7]) and Schempp ([12]).

Theorem 1.1. *Let $(L_n)_{n\in\mathbf{N}}$ be a net of linear positive operators acting on $C(X,\mathbf{R})$ and satisfying the following conditions*

(1) $\lim_{n\to\infty} L_n(h) = h$ for every $h \in A(X)$ (resp. $\lim_{n\to\infty} L_n(h) = h$ for every $h \in H$);

(2) $\lim_{n\to\infty} L_n(h^2) = h^2$ for every $h \in A(X)$.

Then

$$\lim_{n\to\infty} L_n(f) = f \;\forall\, f \in C(X,\mathbf{R}) \quad (resp.\ \lim_{n\to\infty} L_n(f) = T(f) \forall\, f \in C(X,\mathbf{R})).$$

Consider now an infinite lower triangular stochastic matrix $P = (p_{nj})_{n\geq 1, j\geq 1}$ (i.e. an infinite matrix of positive numbers satisfying $p_{nj} = 0$ whenever $j > n$ and $\sum_{j=1}^{\infty} p_{nj} = \sum_{j=1}^{n} p_{nj} = 1$ for every $n \geq 1$).

Then for every $n \geq 1$ we can define the mapping $\pi_n : X^n \to X$ defined by putting for every $(x_1, \ldots, x_n) \in X^n$:

$$\pi_n(x_1, \ldots, x_n) = \sum_{j=1}^{n} p_{nj} x_j .$$

For every $x \in X$ we shall denote by $\mu_x \in M^1(X)$ the probability Radon measure on X defined as follows:

$$\mu_x(f) = T(f)(x) \quad \text{for every} \quad f \in C(X, \mathbf{R}).$$

We introduce the linear positive operator $B_{n,p_{n1},\dots,p_{nn}} : C(X,\mathbf{R}) \to C(X,\mathbf{R})$ defined by putting for every $f \in C(X,\mathbf{R})$ and $x \in X$

$$B_{n,p_{n1},\dots,p_{nn}}(f)(x) = \int_{X^n} f \circ \pi_n d\left(\bigotimes_{i=1}^{n} \mu_{x,i}\right),$$

where $\mu_{x,i} = \mu_x$ for every $i = 1, \dots, n$.

The linear operator $B_{n,p_{n1},\dots,p_{nn}}$ is called the $n-th$ *Bernstein-Schnabl operator* with respect to the matrix P and the projection T (see [1]). When P denotes the arithmetic mean Toeplitz matrix, i.e. $p_{ni} = \frac{1}{n}$ if $n \geq 1$ and $i = 1, \dots, n$ and $p_{ni} = 0$ if $i > n$, then we shall simply use the symbol B_n.

Moreover, let us recall that B_n is the classical $n-th$ Bernstein operator, provided that X coincides with the standard simplex X_p of $\mathbf{R}^p$ $(p \geq 1)$ (i.e. $X_p = \{(x_1, \dots, x_p) \in \in \mathbf{R}^p | x_i \geq 0,\ \sum_{i=1}^{p} x_i \leq 1\}$).

For other explicit expressions of Bernstein-Schnabl operators we refer to [1], [2] and [4].

Given a sequence $(\lambda_i)_{i \in \mathbf{N}}$ in the unit interval $[0,1]$, for every $n \geq 1$ we define the linear positive operator $L_n : C(X,\mathbf{R}) \to C(X,\mathbf{R})$ by setting for every $f \in C(X,\mathbf{R})$ and $x \in X$

$$L_n(f)(x) = \int_{X^n} f \circ \pi_n d\left(\bigotimes_{i=1}^{n} \nu_{x,i}\right), \tag{1.2}$$

where $\nu_{x,i}$ denotes the positive Radon measure on X defined by putting for every $f \in \in C(X,\mathbf{R})$

$$\nu_{x,i}(f) = \lambda_i \mu_x(f) + \left(1 - \lambda_i\right) f(x).$$

The linear operator L_n will be called the $n-th$ *Lototsky-Schnabl operator* with respect to the matrix P, the projection T and the sequence $(\lambda_i)_{i \in \mathbf{N}}$, according to the definition suggested by Schempp in [11] and Grossman in [8].

In fact, Lototsky-Schnabl operators are defined in a more general manner (see [8], [11], [9], [10]) and our operators L_n are particular cases of them; however the operators defined by (1.2) seem more ductile in order to investigate the limit behaviour of their own iterates.

Let us observe that

$$(1.3) \qquad L_n(f)(x) = \sum_{r=0}^{n} a_{n,r} B_{r,p'_{n1},\dots,p'_{nr}} \left(f_{n,r,x}\right)(x),$$

where $B_{r,p'_{n1},\dots,p'_{nr}}$ is the $r-th$ Bernstein-Schnabl operator associated to the matrix P' with entries

$$p'_{nh} = \frac{p_{nh}}{\sum_{i=1}^{r} p_{ni}}, \qquad 1 \le h \le r,$$

$f_{n,r,x} \in C(X, \mathbf{R})$ is obtained from f by putting for every $y \in X$

$$f_{n,r,x}(y) = f\left(\sum_{i=1}^{r} p_{ni}\, y + \sum_{i=r+1}^{n} p_{ni}\, x\right),$$

and, finally, the $a_{n,r}$'s are uniquely determined by the relations

$$\prod_{r=0}^{n} \left(\lambda_r y + 1 - \lambda_r\right) = \sum_{r=0}^{n} a_{nr} y^r, \qquad y \in \mathbf{R}.$$

In particular, if we suppose $\lambda_n = \lambda \in]0,1[$ for every $n \in \mathbf{N}$, then

$$(1.4) \qquad L_n(f)(x) = \sum_{r=0}^{n} \binom{n}{r} \lambda^r (1-\lambda)^{n-r} B_{r,p'_{n1},\dots,p'_{nr}} \left(f_{n,r,x}\right)(x).$$

The previous formulas allow to give an explicit form to the operators L_n, provided that the operators B_r are known (see Example 2.1 of [1], [2], [4]).

When $X = [0,1]$ and P is the arithmetic mean Toeplitz matrix, we have that

$$(1.5) \qquad L_n(f)(x) = \sum_{r=0}^{n} a_{n,r} \sum_{h=0}^{r} \binom{r}{h} f\left(\frac{h}{n} + \left(1 - \frac{r}{n}\right) x\right) x^h (1-x)^{r-h},$$

and, if $\lambda_n = \lambda$ for every $n \ge 1$,

$$(1.6) \quad L_n(f)(x) = \sum_{r=0}^{n} \sum_{h=0}^{r} \binom{n}{r} \binom{r}{h} \lambda^r (1-\lambda)^{n-r} f\left(\frac{h}{n} + \left(1 - \frac{r}{n}\right) x\right) x^h (1-x)^{r-h}.$$

Therefore, for every $x \in X$, we have

$$L_n(h^2)(x) = \sum_{i=1}^{n} p_{ni}^2 \left(\lambda_i T(h^2)(x) + (1-\lambda_i)\, h^2(x)\right) +$$

$$+ 2\left(\sum_{1\le i<j\le n} p_{ni}p_{nj}\right) h^2(x) = \sum_{i=1}^{n} p_{ni}^2 \lambda_i T(h^2)(x) +$$

$$+ \sum_{i=1}^{n} p_{ni}^2 (1-\lambda_i)\, h^2(x) + \left(1 - \sum_{i=1}^{n} p_{ni}^2\right) h^2(x) =$$

$$= \sum_{i=1}^{n} p_{ni}^2 \lambda_i T(h^2)(x) + \left(\sum_{i=1}^{n} p_{ni}^2 - \sum_{i=1}^{n} p_{ni}^2 \lambda_i + 1 - \sum_{i=1}^{n} p_{ni}^2\right) h^2(x) =$$

$$= h^2(x) + \sum_{i=1}^{n} p_{ni}^2 \lambda_i \left(T(h^2)(x) - h^2(x)\right), \text{ i.e..}$$

$$L_n(h^2) = h^2 + \left(\sum_{i=1}^{n} p_{ni}^2 \lambda_i\right)\left(T(h^2) - h^2\right). \tag{2.2}$$

Accordingly

$$L_n^2(h^2) = \sum_{i=1}^{n} p_{ni}^2 \lambda_i T(h^2) + L_n(h^2) - \sum_{i=1}^{n} p_{ni}^2 \lambda_i L_n(h^2) =$$

$$= \sum_{i=1}^{n} p_{ni}^2 \lambda_i T(h^2) + \sum_{i=1}^{n} p_{ni}^2 \lambda_i T(h^2) + \left(1 - \sum_{i=1}^{n} p_{ni}^2 \lambda_i\right) h^2 +$$

$$- \sum_{i=1}^{n} p_{ni}^2 \lambda_i \left\{\sum_{i=1}^{n} p_{ni}^2 \lambda_i T(h^2) + \left(1 - \sum_{i=1}^{n} p_{ni}^2 \lambda_i\right) h^2\right\} =$$

$$= \sum_{i=1}^{n} p_{ni}^2 \lambda_i T(h^2) + \sum_{i=1}^{n} p_{ni}^2 \lambda_i \left(1 - \sum_{i=1}^{n} p_{ni}^2 \lambda_i\right) T(h^2) +$$

$$+ \left(1 - \sum_{i=1}^{n} p_{ni}^2 \lambda_i\right)^2 h^2.$$

More generally, for each $m \geq 1$ we obtain

$$L_n^m\left(h^2\right) =$$

$$= \sum_{i=1}^n p_{ni}^2 \lambda_i \left\{ \sum_{k=0}^{m-1} \left(1 - \sum_{i=1}^n p_{ni}^2 \lambda_i\right)^k T\left(h^2\right) + \left(1 - \sum_{i=1}^n p_{ni}^2 \lambda_i\right)^m h^2 \right\} =$$

$$= \left(1 - \left(1 - \sum_{i=1}^n p_{ni}^2 \lambda_i\right)^m\right) T\left(h^2\right) + \left(1 - \sum_{i=1}^n p_{ni}^2 \lambda_i\right)^m h^2 =$$

$$= h^2 + \left(1 - \left(1 - \sum_{i=1}^n p_{ni}^2 \lambda_i\right)^m\right) \left(T\left(h^2\right) - h^2\right).$$

Since

$$0 < \sum_{i=1}^n p_{ni}^2 \lambda_i \leq \sum_{i=1}^n p_{ni}^2 < 1,$$

it follows that

$$\lim_{m \to \infty} L_n^m\left(h^2\right) = T\left(h^2\right), \tag{2.3}$$

while, if $\lim_{n \to \infty} \sum_{i=1}^n p_{ni}^2 \lambda_i = 0$, then

$$\lim_{n \to \infty} L_n^m\left(h^2\right) = h^2. \tag{2.4}$$

So results (1) and (2) follows from (2.1), (2.3), (2.4) and Theorem 1.1. As regards assertion (3), we can apply formula (2.3). Then, for every $h \in A(X)$ and for every $n \geq 1$ we obtain

$$L_n^{k(n)}\left(h^2\right) = h^2 + \left(1 - \left(1 - \sum_{i=1}^n p_{ni}^2 \lambda_i\right)^{k(n)}\right) \left(T\left(h^2\right) - h^2\right). \tag{2.5}$$

Since

$$\left(1 - \sum_{i=1}^n p_{ni}^2 \lambda_i\right)^{k(n)} =$$

$$= exp \left\{ \frac{-k(n) \left(\sum_{i=1}^n p_{ni}^2 \lambda_i\right) \log \left(1 - \sum_{i=1}^n p_{ni}^2 \lambda_i\right)}{-\sum_{i=1}^n p_{ni}^2 \lambda_i} \right\},$$

we can again conclude by using Theorem 1.1.

Remarks 2.2.

(1) Let us observe that, under the above assumptions, the convergence of L_n toward the identity operator was firstly proved by Grossman in [8].

(2) We can give some quantitative estimates of the convergence of the operators L_n and their iterates by using some general results of Nishishiraho ([9], [10]). More precisely, from [9], Theorem 4, we infer that

$$\| L_n(f) - f \| \leq 2\Omega\left(f, \left(\sum_{i=1}^{n} p_{ni}^2 \lambda_i\right)^{1/2}\right);$$

furthermore, on account of (2.2) and of Corollary 1 of [10] (in particular, see p. 623), we have

$$\| L_n^m(f) - f \| \leq \psi\left(f, \left(1 - \left(1 - \sum_{i=1}^{n} p_{ni}^2 \lambda_i\right)^m\right)^{1/2}\right) \leq$$

$$\leq \psi\left(f, \left(m \sum_{i=1}^{n} p_{ni}^2 \lambda_i\right)^{1/2}\right),$$

and

$$\| L_n^m(f) - T(f) \| \leq \psi\left(f, \left(1 - \sum_{i=1}^{n} p_{ni}^2 \lambda_i\right)^{m/2}\right),$$

where $\Omega(f, \cdot)$ and $\psi(f, \cdot)$ are suitable moduli of continuity of f (see [9] Definition 2 and [10], p. 622).

Theorem 2.3. *Under the above assumptions, let us consider the sequence* $(L_n)_{n\in\mathbb{N}}$ *of Lototsky-Schnabl operators associated with the arithmetic mean Toeplitz matrix, a projection* T *and a sequence* $(\lambda_n)_{n\in\mathbb{N}}$ *in the unit interval* $]0,1[$ *such that* $\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} \lambda_i = \lambda$ *where* λ *is a suitable element of* $]0,1[$. *For every* $m \geq 1$ *let us introduce the subspace* A_m *generated by the set*

$$\left\{\prod_{i=1}^{m} h_i | h_i \in A(X), \quad i = 1, \dots, m\right\},$$

and put $A_\infty := \bigcup_{m\in\mathbb{N}} A_m$. *Assume that*

$$(i) \qquad T(A_2) \subset A(X),$$

or alternatively,
(ii) $A(X)$ *is finite dimensional and* $T(A_m) \subset A_m$ *for every* $m \geq 1$.
Then there exists a strongly continuous positive contraction semigroup $(T(t))_{t\geq 0}$ *on* $C(X,\mathbf{R})$ *such that for every* $t \geq 0$ *and for every sequence* $(k(n))_{n\in\mathbf{N}}$ *of positive integers satisfying*

$$\lim_{n\to\infty} \frac{k(n)}{n} = t,$$

we have

$$\lim_{n\to\infty} L_n^{k(n)} = T(t) \quad \textit{strongly on} \quad C(X,\mathbf{R}). \tag{2.6}$$

Moreover,

$$\lim_{t\to+\infty} T(t) = T \quad \textit{strongly on} \quad C(X,\mathbf{R})$$

and the generator of the semigroup $(T(t))_{t\geq 0}$ *is the closure of the linear operator* $Z : D(Z) \to C(X,\mathbf{R})$ *defined by*

$$Z(f) = \lim_{n\to\infty} n\big(L_n(f) - f\big), \qquad (f \in D(Z)),$$

where

$$D(Z) := \left\{ g \in C(X,\mathbf{R}) \mid \lim_{n\to\infty} n\big(L_n(g) - g\big) \quad \textit{exists in} \quad C(X,\mathbf{R}) \right\}.$$

Finally $A_\infty \subset D(Z)$ *and for every* $m \in \mathbf{N}$, $m \geq 1$, *and* $h_1, \ldots, h_m \in A(X)$

$$Z\left(\prod_{i=1}^m h_i\right) = \begin{cases} 0, & \textit{if } m = 1 \\ \lambda\left(T\left(h_1 h_2\right) - h_1 h_2\right) & \textit{if } m = 2 \\ \lambda \sum_{1\leq i<j\leq m} \left(T\left(h_i h_j\right) - h_i h_j\right) \prod_{r\neq i,j\ r=1}^m h_r & \textit{if } m \geq 3. \end{cases}$$

Proof. Here we give only an outline of the proof, and we refer for more details to Theorem 2.6 of [1], where an analogous result was shown for Bernstein-Schnabl operators, i.e. $\lambda_n = 1$ for all $n \geq 1$.

Now, without loss of generality, we can assume $\lambda_1 = \lambda$.

If $f \in A_1(X) = A(X)$, then $L_n(f) = f$ for every $n \geq 1$ and $\lim_{n\to\infty} n(L_n(f) - f) = 0$. If $m \geq 2$ and $f = \prod_{i=1}^m h_i$, with $h_1, \ldots, h_m \in A(X)$, for every $n \geq m$ we can write

$$f \circ \pi_n = \frac{1}{n^m} \left(\sum_{s_1+\ldots+s_n=m} \sum \left(\prod_{u=1}^{s_1} h_{j_1}\right) \circ pr_1 \ldots \left(\prod_{u=1}^{s_n} h_{j_u^n}\right) \circ pr_n \right),$$

and

$$f = \frac{1}{n^m} \sum_{s_1+\ldots+s_n=m} \sum \left(\prod_{u=1}^{s_1} h_{j_u^1} \right) \cdots \left(\prod_{u=1}^{s_n} h_{j_u^n} \right),$$

where in both formulas the last sum is extended to all subsets of integers $j_1^1, \ldots, j_{s_1}^1, \ldots, j_1^n, \ldots, j_{s_n}^n \in [1, m]$ such that $\{j_1^1, \ldots, j_{s_1}^1\} \cap \ldots \cap \{j_1^n, \ldots, j_{s_n}^n\} = \emptyset$; moreover, we will use the convention that

$$\left\{ j_1^k, \ldots, j_{s_k}^k \right\} = \emptyset \quad \text{and} \quad \prod_{u=1}^{s_k} h_{j_u^k} = 1,$$

whenever some s_k is equal to zero.

Now, denote by I the identity operator on $C(X, \mathbf{R})$ and define

$$S_i = \lambda_i T + (1 - \lambda_i) I, \qquad i = 1, \ldots, n.$$

Let us observe that every S_i is a positive operators such that

$$S_i(f)(x) = \nu_{x,i}(f),$$

for all $f \in C(X, \mathbf{R})$, $x \in X$ and $i = 1, \ldots, n$. Then, we can apply a reasoning analogous to that of Altomare in [1] provided that the operator T is replaced by S_i for every $i = 1, \ldots, n$.

As a consequence, we obtain

$$L_n(f) - f = \frac{1}{n^m} \sum_{s_1+\ldots+s_n=m} \sum \cdot$$

$$\cdot \left\{ S_1 \left(\prod_{u=1}^{s_1} h_{j_u^1} \right) \ldots S_n \left(\prod_{u=1}^{s_n} h_{j_u^n} \right) - \left(\prod_{u=1}^{s_1} h_{j_u^1} \right) \cdots \left(\prod_{u=1}^{s_n} h_{j_u^n} \right) \right\},$$

where the integers $j_u^\nu (\nu = 1, \ldots, n)$ vary as above.

In particular, for $m = 2$ and $n \geq m$ we deduce that

$$n(L_n(f) - f) = n\left(\frac{2}{n^2} \binom{n}{2} f + \frac{1}{n^2} \sum_{i=1}^{n} (\lambda_i T(f) + (1 - \lambda_i) f) - f \right) =$$

$$= n\left(-\frac{1}{n} f + \frac{1}{n^2} \sum_{i=1}^{n} (\lambda_i T(f) + (1 - \lambda_i) f) \right) =$$

$$= -f + \frac{1}{n} \sum_{i=1}^{n} (\lambda_i T(f) + (1 - \lambda_i) f) =$$

$$= \frac{1}{n} \sum_{i=1}^{n} \lambda_i (T(f) - f).$$

Therefore

$$\lim_{n\to\infty} n\left(L_n(f) - f\right) = \lambda(T(f) - f).$$

For $m \geq 3$, let us observe that

$$\lim_{n\to\infty} n^{k-m+1} = \begin{cases} 1 & \text{if } s = 1 \text{ and } k = m-1, \\ 0 & \text{otherwise}. \end{cases}$$

Then, repeating the combinatorial arguments used in [1], we deduce that the following equalities hold

$$\lim_{n\to\infty} n\left(L_n(f) - f\right) = \sum_{1\leq i<j\leq m} \left(S_1\left(h_i h_j\right) - h_i h_j\right) \prod_{r\neq i,j,r=1}^{m} h_r =$$

$$= \lambda_1 \sum_{1\leq i<j\leq m} \left(T\left(h_i h_j\right) - h_i h_j\right) \prod_{r\neq i,j,r=1}^{m} h_r =$$

$$= \lambda \sum_{1\leq i<j\leq m} \left(T\left(h_i h_j\right) - h_i h_j\right) \prod_{r\neq i,j,r=1}^{m} h_r .$$

So the inclusion $A_\infty \subset D(Z)$ is true; but, as a consequence of Weierstrass-Stone theorem, the subalgebra A_∞ is dense in $C(X, \mathbf{R})$ and hence $D(Z)$ is dense in $C(X, \mathbf{R})$ too. Now, the proof can be concluded applying a result of Trotter ([14], Theorem 5.3) under the assumption (i) and a result of Schnabl ([13], Satz 4) under the assumption (ii).

3. AN EXAMPLE

We firstly note that if A denotes the generator of the semigroup indicated in Theorem 2.3, then $A = \lambda A_0$ where A_0 generates the semigroup coming from the Bernstein-Schnabl operators associated with the projection T (see [1], Theorem 2.6). Hence the abstract Cauchy problem

$$\begin{cases} \dfrac{du}{dt}(t) = \lambda A_0 u(t), \\ u(0) = u_0 \in D\left(A_0\right), \end{cases}$$

has a unique solution given by

$$u(t) = T(t) u_0 = \lim_{n\to\infty} L_n^{[nt]} u_0 ,$$

where $[nt]$ denotes the integer part of nt (this follows from (2.6) and from the fact that $\lim_{n\to\infty} \frac{[nt]}{n} = t$). In many concrete situations when $X \subset \mathbf{R}^p$, $p \geq 1$, A_0 is an elliptic differential operator, as it was proved in [1], section 3 (see also [2], [4]). If $X = [0,1]$, then $D(A_0) = \{f \in \mathscr{C}([0,1]) \cap \mathscr{C}^2(]0,1[) \mid \lim_{x\to 0^+} x(1-x)f''(x) = \lim_{x\to 1^-} x(1-x)f''(x) = 0\}$ and for every $f \in D(A_0)$ and $x \in [0,1]$

$$A_0 f(x) = \begin{cases} \dfrac{x(1-x)}{2} f''(x), & \text{if } 0 < x < 1, \\ 0 & \text{if } x = 0,1. \end{cases}$$

In this case the operators L_n are defined as in (1.5) or in (1.6).

If $X = B(x_0, \delta) = \{x \in \mathbf{R}^p \mid \| x - x_0 \| \leq \delta\}$, then A_0 is the closure of the differential operator defined on $\mathscr{C}^2(x)$ by

$$Bf(x) = \frac{\delta^2 - \| x - x_0 \|^2}{2p} \Delta f(x),$$

where Δ denotes the Laplacian on X.

In this case, the n-th Lototsky-Schnabl operator is defined by putting for every $f \in C(X, \mathbf{R})$ and $x \in X$

$$L_n f(x) = \begin{cases} \sum_{r=0}^{n} a_{nr} \left(\dfrac{\delta^2 - \| x - x_0 \|^2}{\delta \sigma_p} \right)^r \displaystyle\int_{\partial X} \cdots \int_{\partial X} \cdot \\ \cdot \dfrac{f\left(\frac{x_1 + \ldots + x_r}{n} + \left(1 - \frac{r}{n}\right) x\right)}{\| x_1 - x \|^p \ldots \| x_r - x \|^p} d\sigma(x_1) \ldots d\sigma(x_r) & \text{if } \| x - x_0 \| < \delta \\ f(x) & \text{if } \| x - x_0 \| = \delta \end{cases}$$

where σ_p denotes the surface area of the unit sphere of $\mathbb{R}^p$ and σ is the surface measure on ∂X.

Note added in proof. The main results of this paper have been considerably extended in some recent papers by Altomare (see, for istance, Mh. Math. **114**, 1992, 1-13, or the paper «On some approximation processes and their associated parabolic problems», which will appear in Conf. Sem. Mat. Fis. Univ. Milano, 1994).

Moreover, a complete survey concerning the positive approximation processes associated with positive projections can be also found in the forthcoming monograph «Korovkin-type Approximation Theory and Applications» by F. Altomare and M. Campiti.

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