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THE GENERALIZED RADEMACHER FUNCTIONS

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Dedicated to the memory of Professor Gottfried Köthe

In [A & G], the authors introuced the so-called generalized Rademacher functions and used them to prove that every continuous multilinear form $A : c_0 \times \ldots \times c_0 \to \mathbb{C}$ has a trace. In this note, we show that these functions are quite useful in obtaining simple proofs of various estimates in several different areas of analysis. We first give a simple proof of a result of I. Zalduendo [Z], which extends the result cited above from c_0 to ℓ_p. Next, we show how generalized Rademacher functions can be used to provide a new proof of a theorem of A. Defant and J. Voigt (see, for example [A & M, 3.10]). Then we exhibit this theorem as a special case of a more general result, which in turn yields other consequences, one new and one old. Next, we use these functions to derive a new polarization formula for symmetric multilinear forms, which yields a new proof of an inequality of Harris. Finally, we provide a simple proof of a theorem of Pelczynski about the continuity of multilinear mappings with respect to a certain sequential topology.

We recall [A & G] that for every natural number $n \geq 2$, the generalized Rademacher functions (s_j) are defined inductively as follows. Let $\alpha_1 = 1, \alpha_2, \ldots, \alpha_n$ be the complex n-th roots of unity. For $j = 1, \ldots, n$ let $I_j = \left(\frac{j-1}{n}, \frac{j}{n}\right)$ and let $I_{j_1 j_2}$ denote the j_2-th open subinterval of length $1/n^2$ of I_{j_1} $(j_1, j_2 = 1, 2, \ldots, n)$. Proceeding like this, it is clear how to define the interval $I_{j_1 j_2, \ldots j_k}$ for any k. Now $s_1 : [0,1] \to \mathbb{C}$ is defined by setting $s_1(t) = \alpha_j$ for $t \in I_j$, where $1 \leq j \leq n$. There is no harm in setting $s_k(t) = 1$ for all endpoints t. We list below the basic properties of the sequence (s_k) of *generalized Rademacher functions* which we will need. The verification of these properties follows exactly the same lines as the corresponding result for the classical Rademacher functions.

Lemma 1. *(1) For every $k = 1, 2, \ldots$ and $t \in [0,1]$, we have $|s_k(t)| = 1$.*
(2) The integral

$$\int_0^1 s_{i_1}(t) \ldots s_{i_n}(t)\, dt = \begin{cases} 1 & \text{if } i_1 = \ldots = i_n \\ 0 & \text{otherwise} \end{cases}$$

(3) If $j_1, \ldots, j_k$ are distinct positive integers, then for $\sigma_j(t) = s_j(t)$ or $\overline{s_j(t)}$,

$$\int_0^1 \sigma_{j_1}^{m_1}(t) \ldots \sigma_{j_k}^{m_k}(t)\, dt = \begin{cases} 1 & \text{if } m_1 \equiv \ldots \equiv m_k \equiv 0 \ (mod\ n) \\ 0 & \text{otherwise.} \end{cases}$$

We observe that, in view of (2) above, it is perhaps surprising that there are standard type Khinchin inequalities for the generalized Rademacher functions. This can be see by copying the usual proof of Khinchin's inequality, as found for example in [L & T, Theorem 2.6.3], and using Lemma 1, (3).

Recall that for a Banach space $E, L({}^nE)$ is the space of continuous n-linear forms $A : Ex\dots xE \to K$, where $K = \mathbb{R}$ or $\mathbb{C}$. $P({}^nE)$ is the Banach space of all continuous n-homogeneous polynomials $P : E \to K$, that is all functions of the form $P(x) = A(x,\dots,x)$ for some $A \in L({}^nE)$, with $||P|| = \sup\{|P(x)| : ||x|| \le 1\}$. The *symmetric* n-linear form A associated with P in this way is uniquely determined by P.

Theorem 2 (Zalduendo). *Let the scalar field be $\mathbb{C}$. For every $p \in [1,\infty)$, every integer $n \le p$, and every $P \in P({}^n\ell_p)$,*

$$||(P(e_j))||_{\frac{p}{p-n}} \le ||P||$$

Proof. Since we agree that $p/0$ is ∞, there is no harm in assuming that $n < p$. For each $j \in \mathbb{N}$, choose $\lambda_j \in \mathbb{C}$ such that

$$\lambda_j^n P(e_j) = |P(e_j)|^{\frac{p}{p-n}}.$$

Let A be the symmetric n-linear form associated to P, and let (s_j) be the sequence of generalized Rademacher functions corresponding to n. Fixing $k \in \mathbb{N}$ and applying Lemma 1,

$$\begin{aligned}
&\sum_{j=1}^k |P(e_j)|^{\frac{p}{p-n}} = \sum_{j=1}^k \lambda_j^n P(e_j) \\
&= \int_0^1 \sum_{j_1=1}^k \sum_{j_2=1}^k \cdots \sum_{j_n=1}^k \lambda_{j_1}\dots\lambda_{j_n} s_{j_1}(t)s_{j_2}(t)\dots s_{j_n}(t) A(e_{j_1},e_{j_2},\dots,e_{j_n})\,dt \\
&= \int_0^1 A\left(\sum_{j_1=1}^k \lambda_{j_1}s_{j_1}(t)e_{j_1}, \sum_{j_2=1}^k \lambda_{j_2}s_{j_2}(t)e_{j_2},\dots,\sum_{j_n=1}^k \lambda_{j_n}s_{j_n}(t)e_{j_n}\right)dt \\
&= \int_0^1 P\left(\sum_{j=1}^k \lambda_j s_j(t)e_j\right)dt \\
&\le ||P||\cdot||(\lambda_1,\lambda_2,\dots,\lambda_k,0,0,\dots)||_p^n \\
&= ||P||\left(\sum_{j=1}^k |\lambda_j|^p\right)^{n/p}
\end{aligned}$$

Easy arithmetic yields the result.

Note that the proof makes heavy use of the assumption that $K = \mathbb{C}$. An analogous result holds for $K = \mathbf{R}$, namely

$$\|(P(e_j))\|_{\frac{p}{p-n}} \leq c(n,p)\|P\|$$

for every $P \in P({}^n\ell_p)$, and every $n \leq p$. However, $c(n,p)$ is not 1 in general, and in fact the best constants $c(n,p)$ are not known. For example, if $P(x_1, x_2, \ldots) = x_1^2 - x_2^2$ for $(x_1, x_2, \ldots) \in \ell_4$, then $\|(P(e_j))\|_{\frac{4}{4-2}} = \sqrt{2}\|P\|$.

As we now show, the same general argument provides a simple proof of a recent result of A. Defant and J. Voigt concerning absolutely summing multilinear operators. We recall that a sequence (x_i) in a Banach space E is *weakly summable* if for all $\varphi \in E'$, we have $\sum_{i=1}^{\infty} |\varphi(x_i)| < \infty$. The vector space of all weakly summable sequences in E is denoted $\ell_w^1(E)$ and is a Banach space when endowed with the norm

$$\|(x_i)\|_{1,w} = \sup_{\|\varphi\| \leq 1} \sum_{i=1}^{\infty} |\varphi(x_i)|.$$

If $E_1, \ldots, E_n$, and F are Banach spaces, the collection of all continuous n-linear maps $E_1 \times \ldots \times E_n \to F$ is denoted $L(E_1, \ldots, E_n; F)$. A continuous n-linear mapping $A : E_1 \times \ldots \times E_n \to F$ is said to be *absolutely summing* if for all $(x_i^1) \in \ell_w^1(E_1), (x_i^2) \in \ell_w^1(E_2), \ldots, (x_i^n) \in \ell_w^1(E_n)$, the sequence $A(x_i^1, x_i^2, \ldots, x_i^n)$ is in $\ell^1(F)$, that is, $\sum_{i=1}^{\infty} \|A(x_i^1, x_i^2, \ldots, x_i^n)\| < \infty$.

Each such mapping A induces an n-linear map $\hat{A} : \ell_w^1(E_1) \times \ldots \times \ell_w^1(E_n) \to \ell^1(F)$, and it is straightforward that

$$\|\hat{A}((x_i^1), (x_i^2), \ldots (x_i^n))\|_{\ell^1(F)} \leq C\|(x_i^1)\|_{1,w} \cdots \|(x_i^n)\|_{1,w}$$

for some constant C. We agree that the *absolutely summing norm of* A, $\|A\|_{as}$, is the infimum of the constants C for which the inequality above holds. If we restrict each (x_i^j) to be of the form $(x_1^j, 0, 0, \ldots)$ then the above inequality reduces to $\|A(x_1^1, \ldots, x_1^n)\| \leq C\|x_1^1\| \ldots \|x_1^n\|$, so that $\|A\| \leq \|A\|_{as}$. The following result appears in [A & M, 3.10]. We emphasize that the proof offered below simplifies the original proof of A. Defant and J. Voigt, although it works only for the *complex* scalar field.

Theorem 3 (A. Defant and J. Voigt). *Let* $\mathbf{K}$ *be the real or complex scalar field. Then for all* $A \in L(E_1, \ldots, E_n; \mathbf{K})$ *we have* $\|A\| = \|A\|_{as}$. *In particular, every continuous* n*-linear scalar valued mapping* $A : E_1 \times \ldots \times E_n \to K$ *is absolutely summing.*

Proof. As mentioned above, our proof of the isometry works only for $K = \mathbb{C}$. Given $A : E_1 \times \ldots \times E_n \to \mathbb{C}$ and $(x_i^1, \ldots, x_i^n) \in E_1 \times \ldots \times E_n, i = 1, \ldots, m$, choose $\lambda_i \in$

$\mathbb{C}$, $|\lambda_i| = 1$ such that $|A(x_i^1, \dots, x_i^n)| = \lambda_i^n A(x_i^1, \dots, x_i^n)$. Let (s_j) be the sequence of generalized Rademacher functions corresponding to n. Then

$$\sum_{i=1}^m |A(x_i^1, \dots, x_i^n)| = \sum_{i=1}^m \lambda_i^n A(x_i^1, \dots, x_i^n)$$

$$= \int_0^1 \sum_{i_1, \dots, i_n = 1}^m \lambda_{i_1} \dots \lambda_{i_n} s_{i_1}(t) \dots s_{i_n}(t) A(x_{i_1}^1, \dots, x_{i_n}^n)\, dt$$

$$= \int_0^1 A\left(\sum_{i_1=1}^m \lambda_{i_1} s_{i_1}(t) x_{i_1}^1, \dots, \sum_{i_n=1}^m \lambda_{i_n} s_{i_n}(t) x_{i_n}^n \right) dt$$

$$\leq ||A|| \max_{0 \leq t \leq 1} \left\{ \left\| \sum_{i_1=1}^m \lambda_{i_1} s_{i_1}(t) x_{i_1}^1 \right\| \dots \left\| \sum_{i_n=1}^m \lambda_{i_n} s_{i_n}(t) x_{i_n}^n \right\| \right\}$$

$$\leq ||A|| \cdot ||(x_1^1, x_2^1, \dots, x_m^1, 0, 0, \dots)||_{1,w} \cdots ||(x_1^n, x_2^n, \dots, x_m^n, 0, 0, \dots)||_{1,w}.$$

Since m was arbitrary, it follows that $||A||_{as} \leq ||A||$, and the proof is complete.

Note that there is apparently no hope that the above argument can be modified to obtain the isometry in case $K = \mathbb{R}$. However, it is possible to view Theorem 3 as a special case of a more general result, which also admits another theorem of Alencar and Matos [A & M] as a special case, and which allows us to establish a multilinear form of Grothendieck's inequality.

We need some terminology. Partition the integers $\{1, \dots, n\}$ into two disjoint blocks $\{a_1, \dots, a_r\}$ and $\{b_1, \dots b_s\}$, where of course $r + s = n$. Consider a continuous multilinear map $A : E_1 \times \dots \times E_n \to F$. When we select $x_{a_j} \in E_{a_j}, (1 \leq j \leq r)$ we can define a «projection»

$$A_{x_{a_1}, \dots x_{a_r}} : E_{b_1} \times \dots \times E_{b_s} \to F,$$

by

$$A_{x_{a_1}, \dots x_{a_r}}(x_{b_1}, \dots, x_{b_s}) = A(x_1, \dots, x_n).$$

It is quite clear that

$$(*) \qquad ||A_{x_{a_1}, \dots, x_{a_r}}|| \leq ||A||\, ||x_{a_1}|| \dots ||x_{a_r}||.$$

Control of the absolutely summing norms of the projection yields a surprising amount of information.

Theorem 4. *Let the scalar field be* $\mathbb{C}$. *For some subset* $\{a_1, \ldots, a_r\}$ *of* $\{1, \ldots, n\}$, *let* **all** *the projections* $A_{x_{a_1}, \ldots x_{a_r}}$ *of the* n*-linear map* $A \in L(E_1, \ldots, E_n; F)$ *be absolutely summing and satisfy*

$$(**) \qquad \|A_{x_{a_1}, \ldots, x_{a_r}}\|_{as} \leq K\|x_{a_1}\| \ldots \|x_{a_r}\|.$$

Then A *is also absolutely summing with* $\|A\|_{as} \leq K$.

Proof. Choose $x_1^j, \ldots, x_m^j$ in $E_j (1 \leq j \leq n)$ and use the Hahn-Banach theorem to select $\varphi_i \in F_i'$ with $\|\varphi_i\| = 1$ and

$$\|A(x_i^1, \ldots, x_i^n)\| = \varphi_i(A(x_i^1, \ldots, x_i^n)).$$

If $\{b_1, \ldots, b_s\} = \{1, \ldots, n\} - \{a_1, \ldots, a_r\}$, it will be convenient to write

$$\tilde{A}(y^{a_1}, \ldots, y^{a_r}, y^{b_1}, \ldots, y^{b_s}) = A(y^1, \ldots, y^n).$$

When $s_1, s_2, \ldots$ is the sequence of generalized Rademacher functions corresponding to $r+1$, we have

$$\sum_{i=1}^{m} \|A(x_i^1, \ldots, x_i^n)\| = \sum_{i=1}^{m} \varphi_i(A(x_i^1, \ldots, x_i^n))$$

$$= \int_0^1 \sum_{i, i_1, \ldots i_r = 1}^{m} s_i(t)\varphi_i(\tilde{A}(s_{i_1}(t)x_{i_1}^{a_1}, \ldots, s_{i_r}(t)x_{i_r}^{a_r}, x_i^{b_1}, \ldots, x_i^{b_s}))\,dt$$

$$= \int_0^1 \sum_{i=1}^{m} s_i(t)\varphi_i\left(\tilde{A}\left(\sum_{i_1=1}^{m} s_{i_1}(t)x_{i_1}^{a_1}, \ldots, \sum_{i_r=1}^{m} s_{i_r}(t)x_{i_r}^{a_r}, x_i^{b_1}, \ldots, x_i^{b_s}\right)\right) dt.$$

Now, if we write $X_{a_j}(t) = \sum_{i_j=1}^{m} s_{i_j}(t)x_{i_j}^{a_j} \quad (1 \leq j \leq r)$, we obtain

$$\sum_{i=1}^{m} \|A(x_i^1, \ldots, x_i^m)\| \leq \int_0^1 \sum_{i=1}^{m} \|A_{X_{a_1}(t), \ldots, X_{a_r}(t)}(x_i^{b_1}, \ldots, x_i^{b_s})\| dt$$

$$\leq \int_0^1 \|A_{X_{a_1}(t), \ldots, X_{a_r}(t)}\|_{as} \left(\prod_{j=1}^{s} \|(x_1^{b_j}, \ldots, x_m^{b_j}, 0, \ldots)\|_{1,w}\right) dt$$

$$\leq K \left(\prod_{j=1}^{s} \|x_1^{b_j}, \ldots, x_m^{b_j}, 0, \ldots)\|_{1,w}\right) \int_0^1 \|X_{a_1}(t)\| \ldots \|X_{a_r}(t)\| dt$$

$$\leq K\|(x_1^1, \ldots, x_m^1, 0, \ldots)\|_{1,w} \ldots \|(x_1^n, \ldots, x_m^n, 0, \ldots)\|_{1,w}.$$

This is just what we needed to find. Q.E.D.

Corollaries flow thick and fast, but first we should note that although the proof of theorem 4 has no chance of working when the scalar field is real, the result is still true and may be proved by incorporating the same sort of ideas as those contained in the proof of theorem 3 of Defant and Voigt.

To see how their theorem follows from our Theorem 4, first observe that any scalar valued linear map is, by the very definition, absolutely summing with absolutely summing norm equal to the operator norm. Thus, if we have an n-linear scalar valued map A, inequality (*) conspires with (**) to ensure that

$$||A||_{as} \leq ||A||.$$

The next corollary concerns Banach spaces with the Orlicz property, that is spaces which have the property that for every unconditionally summable sequence (x_j) we have

$$\sum ||x_j||^2 < \infty.$$

Orlicz himself showed that $L_p[0,1]$ has the Orlicz property when $1 \leq p \leq 2$. Alencar and Matos [A & M. proposition 3.8] demonstrated that when $n \geq 2$ and $E_1, \dots, E_n$ are Banach spaces with the Orlicz property, all bounded multilinear mappings $E_1 \times \dots \times E_n \to F$ into an arbitrary Banach space F are absolutely summing. The restriction $n \geq 2$ is necessary because of the Dvoretzky-Rogers theorem. Theorem 4 allows us to go a little further.

Corollary 5. *If $n \geq 2$ and at least 2 of the Banach spaces $E_1, \dots, E_n$ have the Orlicz property, then every continuous multilinear map $E_1 \times \dots \times E_n \to F$ is absolutely summing, regardless of the Banach space F.*

Proof. There is no loss of generality if we assume that E_1 and E_2 have the Orlicz property. Now select $x_k \in E_k (3 \leq k \leq n)$ and, for $A \in L(E_1, \dots, E_n; F)$, consider the projection $A_{x_3,\dots,x_n} : E_1 \times E_2 \to F$. If (x_{1j}) and (x_{2j}) are unconditionally convergent sequences in E_1, E_2 respectively, then for some absolute constant $K > 0$, we have, thanks to the closed graph theorem,

$$\begin{aligned}
&\sum_{j=1}^{\infty} ||A_{x_3,\dots,x_n}(x_{1j}, x_{2j})|| \\
&\leq ||A_{x_3,\dots,x_n}|| \sum_{j=1}^{\infty} ||x_{1j}|| \, ||x_{2j}|| \\
&\leq ||A|| \, ||x_3|| \dots ||x_n|| \left(\sum_{j=1}^{\infty} ||x_{1j}||^2 \right)^{1/2} \left(\sum_{j=1}^{\infty} ||x_{2j}||^2 \right)^{1/2} \\
&\leq K ||A|| \, ||x_3|| \dots ||x_n|| \, ||(x_{1j})||_{1,w} ||(x_{2j})||_{1,w}.
\end{aligned}$$

In other words, $A_{x_3,\dots,x_n}$ is absolutely summing with

$$||A_{x_3,\dots,x_n}||_{as} \le K||A||\,||x_3||\dots||x_n||.$$

The corollary now follows from theorem 4. Q.E.D.

The last corollary we give is a multilinear extension of Grothendieck's theorem that every bounded linear map from ℓ_1 to ℓ_2 is absolutely summing.

Corollary 6. *For any Banach spaces* $E_2,\dots,E_n$, *we have*

$$L(\ell_1, E_2,\dots,E_n;\ell_2) = L_{as}(\ell_1, E_2,\dots,E_n;\ell_2).$$

Proof. Let $A \in L(\ell_1, E_2,\dots,E_n; F)$ and for $x_k \in E_k (2 \le k \le n)$ consider the projection $A_{x_2,\dots,x_n} : \ell_1 \to \ell_2$. By Grothendieck's theorem, this is absolutely summing, and

$$\begin{aligned}||A_{x_2,\dots,x_n}||_{as} &\le K_G||A_{x_2,\dots,x_n}|| \\ &\le K_G||A||\,||x_2||\dots||x_n||\end{aligned}$$

where K_G is the absolute constant of Grothendieck. By theorem 4, we find that $||A||_{as} \le K_G||A||$. Q.E.D.

Now let $L_s(^nE)$ denote the subspace of $L(^nE)$ consisting of the *symmetric* n-linear forms. As we noted before, for each $P \in P(^nE)$ there is a unique element $A \in L_s(^nE)$ such that $P(x) = A(x^n)$ for every $x \in E$, where $A(x^n) = A(x,\dots,x)$ (n times). Clearly, we have $||P|| \le ||A||$. The following *polarization formula* can be used to recover A from P:

$$A(x_1,\dots,x_n) = \frac{1}{n!}\int_0^1 s_1(t)\dots s_n(t)P(s_1(t)x_1 + \dots + s_n(t)x_n)\,dt,$$

where $s_j(t)$ are the classical Rademacher functions. From this identity we obtain an upper bound for the norm of A:

$$||A|| \le \frac{n^n}{n!}||P||.$$

Since the classical Rademacher functions are real valued, this inequality holds for both the real and complex scalar fields. In the case $E = \ell_1$, the polynomial $P(x) = x_1x_2\dots x_n$ has the property that $||A|| = \frac{n^n}{n!}||P||$, so the inequality given above is the best possible.

Harris [H] proved a finer inequality than the above in the case where some of the variables are repeated, and the scalar field is the complex numbers. This inequality is a consequence of the following polarization formula.

Theorem 7. *Let the scalar field be $\mathbb{C}$. Let P be the continuous n-homogeneous polynomial on E generated by the symmetric n-linear form A. Let $n_1, \ldots, n_k$ be non-negative integers with $n_1 + \ldots + n_k = n$, and let (s_j) be the generalized Rademacher functions corresponding to n. Then*

$$A(x_1^{n_1}, \ldots x_k^{n_k}) = \frac{n_1! \ldots n_k!}{n!} \int_0^1 s_1^{n-n_1}(t) \ldots s_k^{n-n_k}(t) P(s_1(t)x_1 + \ldots + s_k(t)x_k) dt$$

for every $x_1, \ldots, x_k \in E$, where $A(x_1^{n_1}, \ldots x_k^{n_k})$ means that A is evaluated at the point $(x_1, \ldots, x_1 (n_1$ times$), \ldots, x_k, \ldots, x_k(n_k$ times$))$.

Proof. Since $P(s_1(t)x_1 + \ldots + s_k(t)x_k) = A((s_1(t)x_1 + \ldots + s_k(t)x_k)^n)$, we may expand by the multinomial theorem, and integrate term by term. The result then follows from an application of Lemma 1, part (3).

Corollary 8 [Harris]. *If $x_1, \ldots, x_k$ are unit vectors, then*

$$|A(x_1^{n_1}, \ldots, x_k^{n_k})| \leq \frac{n_1! \ldots n_k! \, n^n}{n_1^{n_1} \ldots n_k^{n_k} \, n!} ||P||$$

Proof. Applying the theorem to the vectors $\frac{n_1 x_1}{n}, \ldots, \frac{n_k x_k}{n}$ gives

$$\left| \frac{n_1^{n_1} \ldots n_k^{n_k}}{n^n} A(x_1^{n_1}, \ldots, n_x^{n_k}) \right| = \left| A\left(\left(\frac{n_1 x_1}{n}\right)^{n_1}, \ldots, \left(\frac{n_k x_k}{n}\right)^{n_k} \right) \right|$$
$$= \left| \frac{n_1! \ldots n_k!}{n!} \int_0^1 s_1^{n-n_1}(t) \ldots s_k^{n-n_k}(t) P\left(\frac{n_1 s_1(t) x_1}{n} + \ldots + \frac{n_k s_k(t) x_k}{n} \right) dt \right|$$
$$\leq \frac{n_1! \ldots n_k!}{n!} ||P||,$$

since $\left\| \frac{n_1 s_1(t) x_1}{n} + \ldots + \frac{n_k s_k(t) x_k}{n} \right\| \leq 1$.

Once again, our proof depends on the fact that the scalar field is $\mathbb{C}$. This inequality does not hold for the case $K = \mathbb{R}$ [S, p. 26].

Now let us recall a couple of definitions from [P]. Let E denote a Banach space and let s be a real number with $0 \leq s < 1$. A sequence (x_m) is E is said to be *τ_s-convergent to* 0 (or *τ_s-null*) if there is a constant $C > 0$ such that for any positive integer k, for arbitrary indices $m_1, \ldots, m_k$, and for every sequence (ε_i) of scalars with $|\varepsilon_i| = 1$, the following inequality is satisfied:

$$||\varepsilon_1 x_{m_1} + \ldots + \varepsilon_k x_{m_k}|| \leq C k^s.$$

The sequence (x_m) is said to be τ_s*-convergent to* x if the sequence $(x_m - x)$ is τ_s-null. It is plain that every τ_s-convergent sequence is weakly convergent so that if the τ_s-limit exists, then it is unique. A Banach space is said to *have rank* r (*to have loose rank* r) if the norm is τ_s-continuous for every $0 \le s \le r (0 \le s < r)$.

It is easy to see that in c_0, every weakly null sequence admits a τ_0-null subsequence. It was also observed in [P] that it follows from a theorem of Banach that for $1 \le p < \infty$, ℓ_p has loose rank $1/p$.

Pelczynski's result can now be stated as follows.

Theorem 9 (Pelczynski). *Let n be an integer ≥ 1, let E and F be two complex Banach spaces, and suppose that F has (loose) rank r. If $ns \le r (ns < r)$, then any bounded n-linear map $A : E \times \ldots \times E \to F$ is τ_s-to-norm continous.*

The beauty of this theorem, from our personal point of view, lies in its powerful consequences. Before proceeding with the proof, let us mention some of the corollaries that can be drawn from it (see [P]).

Corollary 10 (Pelczynski). *Every multilinear form on c_0 is weakly continuous. Consequently, every polynomial on c_0 is weakly continuous.*

Corollary 11 (Pelczynski). *Let $1 < p, q < \infty$ and assume $p > nq$. Then any bounded n linear map $A : \ell_p \times \ldots \times \ell_p \to \ell_q$ is weak-to-norm continuous.*

Notice that the classical Pitt-Rosenthal Theorem is a particular case of the above corollary for $n = 1$. Also, it follows that for $n < p$, every polynomial of degree $\le n$ on ℓ_p is weakly continuous. An alternative proof of the last remark can be found in [B & F].

Corollary 12 (Pelczynski). *If there is a τ_0-null sequence in E that is not norm-null (e.g., if $E = C(K), K$ an infinite compact Hausdorff) then the norm is not a uniform limit of a sequence of polynomials on the unit ball of E.*

Proof of Theorem 9. We just show continuity at the origin; continuity at other points can be easily derived from here by induction on n, in the same fashion as in [P].

Let $(x_m^{(1)})_m, \ldots, (x_m^{(n)})_m$ be τ_s-null sequences in E, i.e., there is a constant $C > 0$ such that, for $j = 1, \ldots, n$, the inequality

$$\|\varepsilon_1 x_{m_1}^{(j)} + \ldots + \varepsilon_k x_{m_k}^{(j)}\| \le Ck^s$$

holds for arbitrary indices $m_1, \ldots, m_k$, and for every sequence (ε_i) of complex numbers with $|\varepsilon_i| = 1$.

Fix a positive integer k, indices $m_1, \ldots, m_k$ and signs $|\varepsilon_1| = \ldots = |\varepsilon_k| = 1$. For every $i = 1, \ldots, k$, let η_i be an n^{th} root of ε_i.

Consider for $1 \leq j \leq n$ the vector-valued function $f_j : [0,1] \to X$ defined by

$$f_j(t) = \sum_{i=1}^{k} \eta_i s_i(t) x_{m_i}^{(j)}.$$

Also, consider the function $g : [0,1] \to Y$ defined by $g(t) = A(f_1(t), \dots, f_n(t))$. Since g is piecewise constant, it is Bochner integrable. We have by assumption, for every $j = 1, \dots, n$,

$$\sup_{0 \leq t \leq 1} \|f_j(t)\| \leq Ck^s,$$

so that

$$\sup_{0 \leq t \leq 1} \|g(t)\| \leq \|A\| C^n k^{ns}.$$

Finally, Lemma 1 in combination with A's multilinearity gives

$$\begin{aligned}
\int_0^1 g(t)\,dt &= \int_0^1 A(f_1(t), \dots, f_n(t))\,dt \\
&= \int_0^1 A\left(\sum_{i_1=1}^{k} \eta_{i_1} s_{i_1}(t) x_{m_{i_1}}^{(1)}, \dots, \sum_{i_n=1}^{k} \eta_{i_n} s_{i_n}(t) x_{m_{i_n}}^{(n)}\right) dt \\
&= \sum_{i_1, \dots, i_n = 1}^{k} \eta_{i_1} \cdots \eta_{i_n} \left(\int_0^1 s_{i_1}(t) \dots s_{i_n}(t)\,dt\right) A(x_{m_{i_1}}^{(1)}, \dots, x_{m_{i_n}}^{(n)}) \\
&= \sum_{i=1}^{k} \varepsilon_i A(x_{m_i}^{(1)}, \dots, x_{m_i}^{(n)}).
\end{aligned}$$

This identity leads to

$$\begin{aligned}
\left\|\sum_{i=1}^{k} \varepsilon_i A(x_{m_i}^{(1)}, \dots, x_{m_i}^{(n)})\right\| &= \left\|\int_0^1 g(t)\,dt\right\| \\
&\leq \int_0^1 \|g(t)\|dt \\
&\leq \sup_{0 \leq t \leq 1} \|g(t)\| \leq \|A\| C^n k^{ns}.
\end{aligned}$$

Since Y has (loose) rank r and $(ns < r)\, ns \leq r$ we conclude that

$$\lim_{m \to \infty} \|A(x_m^{(1)}, \dots, x_m^{(n)})\| = 0.$$

REFERENCES

[A & G] R.M. Aron, J. Globevnik, *Analytic functions on* c_0, Revista Matemática (Madrid), **2** (1989), 27-34.

[A & M] R. Alencar, M. Matos, *Some classes of multilinear mappings between Banach spaces*, Preprint, Univ. Complutense de Madrid, 12 (1989).

[B & F] R. Bonic, J. Frampton, *Smooth functions on Banach manifolds*, Journal of Mathematics and Mechanics, **15-5** (1966).

[H] L.A. Harris, *Bounds on the derivatives of holomorphic functions of vectors*, Colloque d'Analyse, Rio de Janeiro (1972), ed. L. Nachbin, 145-163.

[L & T] J. Lindenstrauss, L. Tzafriri, *Classical Banach spaces I*, Springer-Verlag (1979).

[P] A. Pelczynski, *A property of multilinear operations*, Studia Mathematica **16**, (1957-58).

[S] I.C. Sarantopoulos, *Polynomials and multilinear mappings in Banach spaces*, Ph. D. Thesis, Brunel University, (1986).

[Z] I. Zalduendo, *An estimate for multilinear forms on* ℓ_p*-spaces*, to appear in Proc. Royal Irish Academy.

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