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BIBDs THAT ARE BOTH QUASI-DERIVED AND QUASI-RESIDUAL

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Abstract. *A BIBD that is both quasi-derived and quasi-residual is embeddable as a residual design if and only if it is embeddable as a derived design.*

We treat BIBDs as incidence structures in the manner of [1]. The set of blocks of a BIBD on a point P is denoted by (P).

We begin with a couple of definitions and some basic facts. A (v, b, r, k, λ)-design is said to be *quasi-derived* if $k = \lambda + 1$. Such a design is an $\left(r, \frac{r(r-1)}{\lambda}, r-1, \lambda, \lambda-1\right)$-design, upon replacing k by λ and v by r, and might be a derived design of an $\left(\frac{r(r-1)}{\lambda}+1, r, \lambda\right)$-design. A (v, b, r, k, λ)-design is said to be *quasi-residual* if $k = r - \lambda$. Such a design is an $\left(\frac{(r-\lambda)(r-1)}{\lambda}, \frac{r(r-1)}{\lambda}, r, r-\lambda, \lambda\right)$-design and might be a residual design of an $\left(\frac{r(r-1)}{\lambda}+1, r, \lambda\right)$-design. It is easy to show that the complement of a quasi-residual design is a quasi-derived design and *vice versa.* A design is both quasi-derived and quasi-residual if and only if it is a $(2n, 4n-2, 2n-1, n, n-1)$-design for some integer $n \geq 2$.

In [2] Hall and Connor established the following result concerning quasi-residual designs.

Result 1. *Let* $D = (\mathcal{P}, \mathcal{B}, \mathcal{I})$ *be an* $\left(\frac{(r-\lambda)(r-1)}{\lambda}, \frac{r(r-1)}{\lambda}, r, r-\lambda, \lambda\right)$*-design.* D *is embeddable as residual design in an* $\left(\frac{r(r-1)}{\lambda}+1, r, \lambda\right)$*-design if and only if there is a system* S *of sets of blocks of* D *such that*

(a) $|(P) \cap X| = \lambda$ *for every* $P \in \mathcal{P}, X \in S$,

(b) $|X \cap Y| = \lambda - 1$ *for every* $X, Y \in S$ *such that* $X \neq Y$, *and*

(c) every block of $\mathcal{B}$ *is contained in exactly* λ *sets* $X \in S$.

An analogous result for quasi-derived designs is

Result 2. *Let* $D = (\mathcal{P}, \mathcal{B}, \mathcal{I})$ *be an* $\left(r, \frac{r(r-1)}{\lambda}, r-1, \lambda, \lambda-1\right)$*-design.* D *is embeddable as a derived design in an* $\left(\frac{r(r-1)}{\lambda}+1, r, \lambda\right)$*-design if and only there is a system* S *of sets of blocks of* D *such that*

(a) $|(P) \cap X| = \lambda$ *for every* $P \in \mathcal{P}, X \in S$,

(b) $|X \cap Y| = \lambda$ *for every* $X, Y \in S$ *such that* $X \neq Y$*, and*
(c) *every block of* $\mathcal{B}$ *is contained in exactly* $r - \lambda$ *sets* $X \in S$.

Results 1 and 2 can be established in a similar manner. Note that we must have $|S| = r$ and $|X| = r - 1$, for all $X \in S$, in Result 1. Also, we must have $|S| = \dfrac{(r-\lambda)(r-1)}{\lambda}$ and $|X| = r$, for all $X \in S$, in Result 2.

Results 1 and 2 can be used to establish the following theorem.

Theorem 3. *Let* D *be a BIBD which is both quasi-derived and quasi-residual, say a* $(2K, 4K-2, 2K-1, K, K-1)$*-design.* D *is embeddable as a residual design in a* $(4K-1, 2K-1, K-1)$*-design if and only if* D *is embeddable as a derived design in a* $(4K-1, 2K, K)$*-design.*

Proof. Suppose D is embeddable as a derived design in a $(4K-1, 2K, K)$-design. By Result 2, there is a system S of sets of blocks of D such that

(a) $|(P) \cap X| = K$ for all $P \in \mathcal{P}, X \in S$,
(b) $|X \cap Y| = K$ for all $X, Y \in S$ such that $X \neq Y$, and
(c) every block of D is contained in exactly K sets of X of S.

We also have that $|S| = 2K - 1$ and $|X| = 2K$ for all $X \in S$.

Now

$$\begin{aligned}|(P) \cup X| &= |(P)| + |X| - |(P) \cap X| \\ &= (2K-1) + 2K - K \\ &= 3K - 1.\end{aligned}$$

So $|((P) \cup X)^c| = 4K - 2 - (3K-1) = K - 1$, hence $|(P)^c \cap X^c| = K - 1$, and so we have

$$\begin{aligned}|(P) \cap X^c| &= |X^c| - |(P)^c \cap X^c| \\ &= (4K-2) - 2K - (K-1) \\ &= K - 1\end{aligned}$$

for all $P \in \mathcal{P}, X \in S$.

A similar, but simpler, argument shows that $|X^c \cap Y^c| = K - 2$ for all $X, Y \in S$ such that $X \neq Y$. Clearly, every block of $\mathcal{B}$ is in $|S| - K = (2K-1) - K = K - 1$ sets X^c. Thus we have that $S' = \{X^c : X \in S\}$ satisfies the conditions of Result 1 (with $\lambda = K - 1$). Applying Result 1 we have that D is embeddable as a residual design in a $(4K-1, 2K-1, K-1)$-design.

The converse can be established in a similar fashion. □

Theorem 3 should be contrasted with the following well known result: A quasi-residual design D is embeddable as a residual design if and only if the complement of D is embeddable as a derived design.

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REFERENCES

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