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INTEGRATION OF FRENET EQUATIONS IN THE ISOTROPIC SPACE $I_3^{(1)} = P^3_{12|000}$ BY MEANS OF QUADRATURES

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The natural (=Frenet) equations for a curve in isotropic space with given isotropic curvature and torsion are explicitly solved.

If the curvature k and torsion τ of a three times continuously differentiable space curve c in Euclidean space $E^3 = P^3_{1|00}$ are given as functions of the arc length of c, then (according to the fundamental theorem of curve theory of Euclidean differential geometry) c is uniquely determined up to (proper) motions. But until now it has not been possible, to integrate the Frenet equations, which are used in the proof of this fundamental theorem, in order to determine c (or a curve congruent to c). For a twice continuously differentiable curve d in the Euclidean plane $E^2 = P^2_{1|00}$, whose curvature k is given as a function of the arc length of d, this integration can be accomplished. In this case the Frenet equations for the moving frame $\mathfrak{e}_1, \mathfrak{e}_2$ of d are

$$(1)\qquad \begin{aligned} \mathfrak{e}_1' &= k \cdot \mathfrak{e}_2, \\ \mathfrak{e}_2' &= -k \cdot \mathfrak{e}_1, \end{aligned}$$

where $'$ means differentiation with respect to the arc length of d. For any set of initial values, the solution of this system of linear differential equations for the coordinates of $\mathfrak{e}_1$ and $\mathfrak{e}_2$ with four equations and four unknown functions is uniquely determined. By suitably choosing the initial values we effect that $\mathfrak{e}_1, \mathfrak{e}_2$ do not only solve the system of differential equations but are also the vectors of a moving frame of a curve d in $E^2 = P^2_{1|00}$, which has the arc length of d as its parameter. Then d is the plane curve (which is uniquely determined up to motions) with curvature k (as function of the arc length of d) and we have (up to motions):

$$(2)\qquad \mathfrak{e}_1(s) = \begin{pmatrix} c_1 \cdot \cos \int_{s_0}^{s} k(\sigma)\,d\sigma - c_2 \cdot \sin \int_{s_0}^{s} k(\sigma)\,d\sigma \\ c_2 \cdot \sin \int_{s_0}^{s} k(\sigma)\,d\sigma + c_1 \cdot \cos \int_{s_0}^{s} k(\sigma)\,d\sigma \end{pmatrix}$$

where $c_1, c_2 \in \mathbb{R}, c_1^2 + c_2^2 = 1$. For the coordinate vector $\mathfrak{r}$ of the running point of the curve as a function of the arc length of d we have in some cartesian coordinate system of E^2:

$$(3)\qquad \mathfrak{r}(s) = \mathfrak{a} + \int_{s_0}^{s} \mathfrak{e}_1(\sigma)\,d(\sigma)$$

where $\mathfrak{a} \in \mathbf{R}^2$ is an integration constant.

A proof for this well known fact is found (with somewhat different notations) for example in Strubecker [5], pp. 44-45.

A theorem analogous to the fundamental theorem of curve theory in the Euclidean differential geometry of E^3 or E^2 is known for many geometries. But an explicit integration of the Frenet equations seems to be unknown even for such «simple» cases as the hyperbolic or the elliptic plane geometry. Therefore it seems remarkable that in the oftenly studied isotropic space $I_3^{(1)} = P^3_{12|000}$ the explicit integration of the Frenet equations can be accomplished up to quadratures. (For the two notations $I_3^{(1)}$ and $P^3_{12|000}$ compare for example Sachs [3], who studies the isotropic space $I_3^{(1)}$ in detail and gives extensive literature references, and Giering [1], who studies the isotropic space $P^3_{12|000}$ in the context of the more general theory of Cayley/Klein-spaces and uses a unified notation for all Cayley/Klein-spaces. According to Sachs the points of the absolute plane are not points of the isotropic space $I_3^{(1)}$, according to Giering they are points of the isotropic space $P^3_{12|000}$, but they do not belong to the «Schauplatz» of the different isotropic geometries, for example the isotropic motion geometry).

The curve theory in isotropic space $I_3^{(1)}$ has been developed by Strubecker in [4]. The Frenet equations for this isotropic space have been known since then ([4], p. 21) and are given here in the notation of Sachs ([3], p. 108):

$$\vec{t}' = k \cdot \vec{n}, \tag{4}$$

$$\vec{n}' = -k \cdot \vec{t} + \tau \cdot \vec{b}, \tag{5}$$

$$\vec{b}' = \vec{o}. \tag{6}$$

There $\vec{t}, \vec{n}$ and $\vec{b}$ are functions of the isotropic arc length s of an isotropic space curve c with curvature k and torsion τ, and $'$ means differentiation with respect to s. The coordinates of $\vec{b}$ are constantly equal to (0,0,1). An attempt to integrate these Frenet equations, leads to the following differential equation for the z-coordinate of the coordinate vector $\vec{x}$ of the space curve in a suitable xyz-coordinate system

$$k \cdot z''' - k' \cdot z'' + k^3 \cdot z' = k^2 \cdot \tau \tag{7}$$

(see [4], p. 21). This differential equation seems not to have been explicitly solved until now. (In spite of its most promising title in the context of this problem, Pavković's paper [2] and the preceding papers deal with a completely different question).

The parameter transformation

$$u(s) := \int_{s_0}^{s} k(\sigma) d(\sigma) \tag{8}$$

together with the notation

$$(9) \qquad T(u) := \frac{\tau(s(u))}{k(s(u))}, \quad \text{or} \quad T := \frac{\tau \circ u^{-1}}{k \circ u^{-1}}$$

leads to the simpler system of differential equations

$$(10) \qquad \dot{\vec{t}} = \vec{n},$$

$$(11) \qquad \dot{\vec{n}} = -\vec{t} + T \cdot \vec{b},$$

$$(12) \qquad \dot{\vec{b}} = \vec{o}.$$

Here means differentiation with respect to u, a notation which is well known from curve theory and is suitable in this context, and u^{-1} is the inverse function of u. If we know a solution of the system (10) - (12), we are able to substitute $u(s)$ from (8) to get a solution of the system (4) - (6).

Now we use the general notation of Giering [1] for all Cayley/Klein-spaces with partitioned coordinate vectors for the running point of the curve as well as for the vectors of the moving frame of the curve and get

$$\vec{x}_{00} := (1), \qquad \vec{x}_{10} := (0), \qquad \vec{x}_{20} := (0),$$

$$\vec{x}_{01} := \begin{pmatrix} x \\ y \end{pmatrix}, \qquad \vec{x}_{11} := \begin{pmatrix} t_1 \\ t_2 \end{pmatrix}, \qquad \vec{x}_{21} := \begin{pmatrix} n_1 \\ n_2 \end{pmatrix},$$

$$\vec{x}_{02} := (z), \qquad \vec{x}_{12} := (t_3), \qquad \vec{x}_{22} := (n_3),$$

$$\vec{x}_{30} := (0),$$

$$\vec{x}_{31} := \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

$$\vec{x}_{32} := (1),$$

where x, y, z are the coordinates of the running point of the curve, t_1, t_2, t_3 the coordinates of $\vec{t}$, n_1, n_2, n_3 the coordinates of $\vec{n}$ and $b_1 = 0, b_2 = 0, b_3 = 1$ the coordinates of $\vec{b}$ in an appropriate constant affine xyz – coordinate system in $I_3^{(1)}$. The system (10) - (12) reduces to

$$(13) \qquad \dot{\vec{x}}_{11} = \vec{x}_{21},$$

$$(14)\qquad \dot{\vec{x}}_{21} = -\vec{x}_{11},$$

$$(15)\qquad \dot{\vec{x}}_{12} = \vec{x}_{22},$$

$$(16)\qquad \dot{\vec{x}}_{22} = -\vec{x}_{12} + (T).$$

For the differential equations (13), (14) we get as general solution

$$(17)\qquad \vec{x}_{11} = c_1 \cdot \begin{pmatrix} \cos u \\ \sin u \end{pmatrix} + c_2 \cdot \begin{pmatrix} -\sin u \\ \cos u \end{pmatrix},$$

$$(18)\qquad \vec{x}_{21} = c_1 \cdot \begin{pmatrix} -\sin u \\ \cos u \end{pmatrix} + c_2 \cdot \begin{pmatrix} -\cos u \\ \sin u \end{pmatrix}.$$

Necessary for such a solution of the system (13), (14) to lead to a moving frame of a curve is $c_1^2 + c_2^2 = 1$. But the solution (17), (18) is not very innovative, for (17) is by virtue of (8) essentially identical with (2), and the ground plan

$$(19)\qquad \vec{x}_{01}(s) := \vec{x}_{01}(s_0) + \int_{s_0}^{s} \vec{x}_{11}(u(\sigma))d\sigma$$

of a space curve with curvature $k(s)$ in isotropic space has already been explicitly known. For as $k(s)$ is in the same time the Euclidean curvature of the ground plan of the space curve, the finding out of the ground plan is the same as the determination of a curve in the Euclidean plane from its given curvature which has been described in the beginning of this paper in (2) and (3) (see for example [4], p. 18).

As $\vec{x}_{12}$ consist only of one coordinate, namely t_3, the differential equations (15), (16) lead to one inhomogeneous linear differential equation of the second order with constant coefficients:

$$(20)\qquad \ddot{t}_3 + t_3 = T$$

Variation of constants yields

$$(21)\qquad t_{3p} = \sin u \cdot \int_{u_0}^{u} T(v) \cdot \cos v dv - \cos u \cdot \int_{u_0}^{u} T(v) \cdot \sin v dv$$

as a particular solution of (20). Therefore the general solution of the differential equation is

$$(22)\qquad t_3 = \sin u \cdot \left[d_1 + \int_{u_0}^{u} T(v) \cdot \cos v dv \right] - \cos u \cdot \left[d_2 + \int_{u_0}^{u} T(v) \cdot \sin v dv \right]$$

with arbitrary real integration constants d_1, d_2.

Substituting $u = u(s)$ by virtue of $dv = k(\varrho)d\varrho$ yields

$$
\begin{aligned}
(23)\qquad t_3 = {} & \sin\left(\int_{s_0}^{s} k(\sigma)d\sigma\right)\cdot\left[d_1 + \int_{s_0}^{s}\tau(\sigma)\cdot\cos\left(\int_{s_0}^{\sigma} k(\varrho)d\varrho\right)d\sigma\right] \\
& - \cos\left(\int_{s_0}^{s} k(\sigma)d\sigma\right)\cdot\left[d_2 + \int_{s_0}^{s}\tau(\sigma)\cdot\sin\left(\int_{s_0}^{\sigma} k(\sigma)d\varrho\right)d\sigma\right].
\end{aligned}
$$

If we write down (19) and the integral over (23) explicity, we get the following

Theorem. *Let I be an open intervall in $\mathbb{R}$, $k : I \to \mathbf{R}$ a C^1- and $\tau : I \to \mathbf{R}$ a C^0-function with $k(s) \neq 0$ for all $s \in I$. Then the coordinates of the C^3-curve of the isotropic space $I_3^{(1)} = P^3_{12|000}$ with the isotropic curvature k and the isotropic torsion τ as functions of the arc length, which according to the fundamental theorem of curve theory is uniquely determined up to isotropic motions, are given in an appropriate affine xyz-coordinate system as functions of the arc length by*

$$
\begin{aligned}
(24)\qquad x(s) = {} & x(s_0) + c_1\cdot\int_{s_0}^{s}\cos\left(\int_{s_0}^{\sigma} k(\varrho)d\varrho\right)d\sigma - \\
& - c_2\cdot\int_{s_0}^{s}\sin\left(\int_{s_0}^{\sigma} k(\varrho)d\varrho\right)d\sigma,
\end{aligned}
$$

$$
\begin{aligned}
(25)\qquad y(s) = {} & y(s_0) + c_1\cdot\int_{s_0}^{s}\sin\left(\int_{s_0}^{\sigma} k(\varrho)d\varrho\right)d\sigma + \\
& + c_2\cdot\int_{s_0}^{s}\cos\left(\int_{s_0}^{\sigma} k(\varrho)d\varrho\right)d\sigma,
\end{aligned}
$$

$$
\begin{aligned}
(26)\qquad z(s) = {} & z(s_0) + d_1\cdot\int_{s_0}^{s}\sin\left(\int_{s_0}^{\sigma} k(\varrho)d\varrho\right)d\sigma - \\
& - d_2\cdot\int_{s_0}^{s}\cos\left(\int_{s_0}^{\sigma} k(\varrho)d\varrho\right)d\sigma + \\
& + \int_{s_0}^{s}\sin\left(\int_{s_0}^{\sigma} k(\varrho)d\varrho\right)d\sigma\cdot\int_{s_0}^{\sigma}\tau(\varrho)\cdot\cos\left(\int_{s_0}^{\varrho} k(\lambda)d\lambda\right)d\varrho d\sigma - \\
& - \int_{s_0}^{s}\cos\left(\int_{s_0}^{\sigma} k(\varrho)d\varrho\right)d\sigma\cdot\int_{s_0}^{\sigma}\tau(\varrho)\cdot\sin\left(\int_{s_0}^{\varrho} k(\lambda)d\lambda\right)d\varrho d\sigma,
\end{aligned}
$$

where $x(s_0), y(s_0), z(s_0), d_1, d_2$ *are arbitrary real constants and for* $c_1, c_2 \in \mathbb{R}$ *there is only the condition* $c_1^2 + c_2^2 = 1$.

Remark 1: Looking at the above differential equation (7) of Strubecker ([4], p.23) for z, by the substitutions

$$w := z', u := \int_{s_0}^{s} k(\sigma)\,d\sigma \tag{27}$$

we get the differential equation

$$\ddot{w} + w = \frac{\tau}{k} \tag{28}$$

for w, and using the ground plan given by Strubecker in [4] we only have to solve the differential equation (28).

Remark 2: There are two differences between the Euclidean and the isotropic space which are essential in this context and which effect that there is no similar method for the integration of the Frenet equations in Euclidean space. In the first place the system (4) - (6) of the Frenet equations in isotropic space is essentially simplified by omitting the third coordinate so that the general solution of this simplified system is even already known from plane Euclidean differential geometry. In the second place in isotropic space the parameter transformation (8) leads to a linear differential equation (certainly inhomogeneous) of the second order with constant coefficients and not (as in the Euclidean case) to a homogeneous linear differential equation of the third order with variable coefficients, because the third vector of the moving frame in isotropic space is constant and therefore does not contain an unknow function which has to be determined from the Frenet equations.

Remark 3: There are some obvious applications of the above theorem. One example may be pointed out: Ruled surfaces of type I) in isotropic space $I_3^{(1)}$ (compare [3], p. 193 f.) may explicitly be determined from their curvature, torsion and striction as functions of the arc length of their curve of striction.

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