Note di Matematica Vol. XIII - n. 1, 41-73 (1993)

THEORY AND PRACTICE OF DIRICHLET SERIES WITH FUNCTIONAL EQUATION

ANDREAS GUTHMANN

Let $a : \mathbf{N} \to \mathbf{C}$ be an arithmetical function and

$$(1) \qquad \varphi(s) = \sum_{n=1}^{\infty} a(n) n^{-s}, s = \sigma + it, a \neq 0 .$$

be its associated Dirichlet series.

We consider the class $\mathbf{D}$ of Dirichlet series satisfying the following conditions:

I) The series (1) converges absolutely for $\sigma > \sigma_a > 0$ (σ_a not necessarily minimal).

II) a)There exist constants $A > 0, 0 < r < \sigma_a$ and integers $r_1 \geq 0, r_2 \geq 0$ with $r_1 + r_2 > 0$, such that the function R defined by

$$R(s) := A^{-s} \Gamma\left(\frac{s}{2}\right)^{r_1} \Gamma(s)^{r_2} \varphi(s), \sigma > \sigma_a$$

can be analytically continued to the entire complex plane to a meromorphic function.

b) R has its only singularities at the points $s = r$ and $s = 0$ which are poles of order $m \geq 0$ and $l \geq 0$ respectively.

c) If $\Xi(s) := s^l (s-r)^m R(s)$, then Ξ is an entire function of order 1.

III) R satisfies the functional equation

$$R(s) = \gamma \overline{R(r - \overline{s})}$$

where $\gamma \in \mathbf{C}, |\gamma| = 1$.

The complex zeros of Dirichlet series are of particular interest, since their distribution is strongly related to the asymptotic behaviour of the aritmetical function a. In this paper our goal is to give a method for the computation of φ in the critical strip $0 < \sigma < r$. This enables us to compute the complex zeros of φ on the critical line $\sigma = \frac{r}{2}$. See §3 A for more details concerning these matters.

The functional equation (III) implies that non real zeros of φ are symmetric with respect to $\sigma = \frac{r}{2}$ and the *Riemann hypothesis* for φ states that $\Re(\varrho) = \frac{r}{2}$ for these zeros ϱ. We shall give a method which allows in principle to test the Riemann hypothesis for any given interval $0 \leq t \leq t_0$.

The prototype of all Dirichlet series is the Riemann zeta function $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$. In this case extensive numerical investigations have been performed. The computations of van de Lune et al. [19, 20] showed that the first $1.5 \cdot 10^9$ complex zeros lie on the critical line $\sigma = \frac{1}{2}$ and are simple. On the other hand there are only scattered results for other types of Dirichlet series. Dirichlet L-functions have been investigated by Davies[5], Davies and Haselgrove[6] and Spira[27], Hecke L-functions by Davies[4], Artin L-functions by Lagarias and Odlyzko[12]. Spira[26] and Ferguson et al.[7] computed zeros of Dirichlet series associated to modular forms. But it seems that these methods have some disadvantage, relating to numerical instabilities. In a sequel to this paper we shall give an effective version with our method[9].

Even a proof of the Riemann hypothesis does not make our method superfluos. For instance, Odlyzko[21] studied the statistical distribution of the complex zeros of the zeta function which necessitated extensive computations. Finally, it is possible to get some information on the asymptotic behaviour of the arithmetical function a from explicit numerical values of the zeros . Some examples have been given by Pintz[24], te Riele[25] and Odlyzko and te Riele who gave a disproof of Mertens' conjecture[22].

In the present investigation we extend the methods for the verification of the Riemann hypoyhesis in a given interval $t_1 \leq t \leq t_2$ to the Dirichlet series from the class $\mathbf{D}$. Here we compile some of the necessary tools. But for each concrete example of a given Dirichlet series further estimations are needed. How to do this for a particular case has been demonstrated in [9] for the simplest case $(n = r_1 + 2r_2 = 1)$. If $r_1 + 2r_2 > 1$ then we need extensive numerical estimates which are not yet finished. The case $r_1 + 2r_2 = 2$ includes among others the zeta function of quadratic number fields. But is possible to compute these functions effectively with the methods presented here and in [9] and [10].

This paper is organized as follows. After compiling some technical tools in §0 we transform the series (1) into a series which converges absolutely in the critical strip. This transformation is due to Lavrik and it gives a method for computing $\varphi(s)$. The summation involves certain inverse Mellin transforms which generalize the incomplete gamma function. Analytical properties and numerical methods for these function have been given in [10]. The zeros of φ can be found by sign changes of a function Z which is considered in §2 .

In §3 we make some general remarks concerning the zeros and their distribution in the critical strip. In the third part of §3 we generalize Gram's Law and a method of Turing[29] for the verification of the Riemann hypothesis an any given t interval.

Throughout we apply the following notation. $\int_{(c)} = \int_{c-i\infty}^{c+i\infty}$ denotes an integration along the line with real part equal to c and imaginary part increasing. $f(x) = O(g(x))$ or $f(x) \ll g(x)$ for $x \to x_0$ are standard abbrevations for the fact that $|f(x)/g(x)| \leq C$ for $x \to x_0$ and suitable constant $C > 0$. In most cases $x_0 = \infty$.

0. SOME TECHNICAL TOOLS

In §1 we modify Lavrik's derivation [13, 14, 15] of the approximate functional equation for φ. For this, we need some results, which we insert here. The proofs are to be found in [10].

Let λ and ν be non negative integers with $\lambda + \nu \geq 1$. For $x > 0$ we define

$$E_{\lambda,\nu} = \frac{1}{2\pi i}\int_{(c)} \Gamma\left(\frac{w}{2}\right)^{\lambda} \Gamma(w)^{\nu} x^{-w} dw, \tag{2}$$

and the *incomplete gamma function of order* (λ, ν) by

$$\Gamma_{\lambda,\nu}(a, x) = \int_x^{\infty} E_{\lambda,\nu}(t) t^{a-1} dt. \tag{3}$$

For numerical purposes and aspecially in our investigations it is advantageous to consider the corresponding normalized incomplete gamma function $Q(a, x)$, i.e.

$$Q(a, x) = \frac{\Gamma_{\lambda,\nu}(a, x)}{\Gamma(a/2)^{\lambda}\Gamma(a)^{\nu}}.$$

We need [10].

Lemma 1. *$E_{\lambda,\nu}$ can be continued to a function analytic in the complex plane slit along the negative real axis. For $|\arg(x)| \leq \delta < \frac{\pi}{2}$ we have*

$$E_{\lambda,\nu}(x^{n/2}) = O\left(e^{-bx} x^{-\frac{\lambda+\nu-1}{2}}\right), x \to \infty,$$

where $n = \lambda + 2\nu$ and $b = 2^{-2\nu/n}$.

Further properties of the incomplete gamma function, for example an asymptotic expansion for $a \to \infty$ can be found in [10].

For the use in Turing's method in §4 we need the following facts.

Lemma 2. *(Lehman's Lemma[16]) Let $f(s)$ be regular in $|s - s_0| \leq R'$ with $f(s_0) \neq 0$ and $|f(s)/f(s_0)| \leq M$ for $|s - s_0| = R'$. Let $s_1, \ldots, s_n$ be the zeros of f in $|s - s_0| \leq R$, where $0 < R < R'$ and all zeros are counted according to their multiplicities. Then*

$$\left|\log|f(s)| - \left\{\log|f(s_0)| + \sum_{k=1}^{n} \log\left|\frac{s - s_k}{s_0 - s_k}\right|\right\}\right| \leq \frac{2r}{R - r}\left\{\log M + n \log\frac{R}{R' - R}\right\}$$

for $|s - s_0| \leq r < R$.

The next result is due to Turing[29].

Lemma 3. *Let $a \in \mathbf{C}, a \neq 0, \Re(a) \geq 0$. Then*

$$\int_{a-1}^{a} \log \left| \frac{z}{z+1} \right| dz \geq -1.48 \Re \frac{1}{a}.$$

We need the following generalization.

Lemma 4. *Let $b \in \mathbf{C}^*, \Re(b) \geq 0, \delta > 0$. Then we have*

$$\int_{b-\delta}^{b} \log \left| \frac{z}{z+\delta} \right| dz \geq -1.48 \delta^2 \Re \frac{1}{b}.$$

Proof. The substitution $z = \delta w$ gives

$$\int_{b-\delta}^{b} \log \left| \frac{z}{z+\delta} \right| dz = \delta \int_{b/\delta-1}^{b/\delta} \log \left| \frac{w}{w+1} \right| dw,$$

and our claim follows from Turing's Lemma.

1. LAVRIK'S APPROXIMATE FUNCTIONAL EQUATION

Here we give another proof of Lavrik's representation [13, 14, 15] of a Dirichlet series with functional equation.

For short, we set $E = E_{r_1,r_2}$. Substituting $t = Anx, n \in \mathbf{N}$,

$$\Gamma \left(\frac{s}{2}\right)^{r_1} \Gamma(s)^{r_2} = \int_0^\infty E(t) t^{s-1} dt, \Re(s) > 0,$$

(follows from (2) by inversion) we get

$$A^{-1} \Gamma \left(\frac{s}{2}\right)^{r_1} \Gamma(s)^{r_2} n^{-s} = \int_0^\infty E(Anx) x^{s-1} dx.$$

With (1) and (II) we obtain from this for $\sigma > \sigma_a$

$$R(s) = \sum_{n=1}^\infty a(n) \int_0^\infty E(Anx) x^{s-1} dx = \int_0^\infty \sum_{n=1}^\infty a(n) E(Anx) x^{s-1} dx,$$

where the interchange of summation and integration is justified by absolute convergence of the left hand side. Hence we have

$$R(s) = \int_0^\infty \psi(x) x^{s-1} dx, \sigma > \sigma_a. \tag{4}$$

where

$$\psi(x) = \sum_{n=1}^{\infty} a(n) E(Anx), x > 0.$$

Inversion of the Mellin integrals in (4) gives for $x > 0$

$$\psi(x) = \frac{1}{2\pi i} \int_{(c)} R(w) x^{-w} dw, c > \sigma_a.$$

Shifting the integral to the left to the line $\Re(w) = \frac{r}{2}(< c$ by (II)), gives

$$\psi(x) = res_1 + \frac{1}{2\pi i} \int_{(\frac{r}{2})} R(w) x^{-w} dw, \tag{5}$$

where res_1 denotes the residue of the integrand in $w = r$. If $m = 0$ then $res_1 = 0$ and if $m > 0$ then for $|\delta|$ sufficently small we have an expansion $R(r + \delta) = \sum_{\nu=-m}^{\infty} b_\nu \delta^\nu$. Therefore

$$res_1 = x^{-r} \sum_{\nu=0}^{m-1} B_\nu (\log x)^\nu, B_\nu = b_{-\nu-1} \frac{(-1)^\nu}{\nu!}. \tag{6}$$

We apply the functional equation (III) to the integral in (5), where we define

$$\varphi^* = \overline{\varphi(\overline{s})} = \sum_{n=1}^{\infty} \overline{a(n)}\, n^{-s}, R^*(s) = \overline{R(\overline{s})} = A^{-s}\Gamma\left(\frac{s}{2}\right)^{r_1} \Gamma(s)^{r_2} \varphi^*(s).$$

Then we have

$$\frac{1}{2\pi i} \int_{(\frac{r}{2})} R(w) x^{-w} dw = \gamma \frac{1}{2\pi i} \int_{(\frac{r}{2})} R^*(r - w) x^{-w} dw = \gamma x^{-r} \frac{1}{2\pi i} \int_{(\frac{r}{2})} R^*(z) x^z dz.$$

Shift the last integral back to the line $\Re(z) = c > \sigma_a$:

$$\begin{aligned} \frac{1}{2\pi i} \int_{(\frac{r}{2})} R(w) x^{-w} dw &= -\gamma x^{-r} res_2 + \gamma x^{-r} \frac{1}{2\pi i} \int_{(c)} R^*(z) x^z dz \\ &= -\gamma x^{-r} res_2 + \gamma x^{-r} \psi^* \left(\frac{1}{x}\right), \end{aligned} \tag{7}$$

where res_2 denotes the residue of the integrand at $z = r$ and $\psi^*(x) = \sum \overline{a(n)}\, E(Anx)$. Again we have

$$res_2 = x^r \sum_{\nu=0}^{m-1} (-1)^\nu \overline{B}_\nu (\log x)^\nu \tag{8}$$

and from (5), (6), (7), (8) we see that

$$\psi(x) = \gamma x^{-r} \psi^* \left(\frac{1}{x}\right) + x^{-r} \sum_{\nu=0}^{m-1} B_\nu (\log x)^\nu - \gamma \sum_{\nu=0}^{m-1} (-1)^\nu \overline{B}_\nu (\log x)^\nu . \tag{9}$$

This type of functional equation for ψ has been obtained by Berndt[2]. For the reader's convenience we gave its proof here in a somewhat modified form. Set $\xi > 0$. If we let $x = y\xi$ in (4), we get

$$R(s) = \xi^s \int_0^\infty \psi(y\xi) y^{s-1} dy = \xi^s \left(\int_0^1 + \int_1^\infty\right) \psi(y\xi) y^{s-1} dy.$$

Apply the functional equation (9) to the first integral. Thus,

$$\begin{aligned}
\xi^s \int_0^1 \psi(y\xi) y^{s-1} dy &= \gamma \xi^s \int_0^1 (y\xi)^{-r} \psi^* \left(\frac{1}{y\xi}\right) y^{s-1} dy \\
&\quad + \xi^s \sum_{\nu=0}^{m-1} B_\nu \int_0^1 (y\xi)^{-r} (\log y\xi)^\nu y^{s-1} dy \\
&\quad - \gamma \xi^s \sum_{\nu=0}^{m-1} (-1)^\nu \overline{B}_\nu \int_0^1 (\log y\xi)^\nu y^{s-1} dy \\
&= \gamma \xi^{s-r} \int_0^1 \psi^* \left(\frac{1}{y\xi}\right) y^{s-r-1} dy \\
&\quad + \xi \sum_{\nu=0}^{m-1} B_\nu \int_0^1 (\log y\xi)^\nu (y\xi)^{s-r-1} dy \\
&\quad - \gamma \xi \sum_{\nu=0}^{m-1} (-1)^\nu \overline{B}_\nu \int_0^1 (\log y\xi)^\nu (y\xi)^{s-1} dy.
\end{aligned}$$

Define

$$I(\nu, \xi, a) = \int_0^1 (\log y\xi)^\nu (y\xi)^{a-1} dy. \tag{10}$$

If we further let $\frac{1}{y} = x$, we get

$$\xi^s \int_0^1 \psi(y\xi) y^{s-1} dy = \gamma \xi^{s-r} \int_1^\infty \psi^* \left(\frac{x}{\xi}\right) x^{r-s-1} dx$$
$$+ \xi \sum_{\nu=0}^{m-1} B_\nu I(\nu, \xi, s-r) - \gamma \xi \sum_{\nu=0}^{m-1} (-1)^\nu \overline{B}_\nu I(\nu, \xi, s).$$

Here

$$\gamma \xi^{s-r} \int_1^\infty \psi^* \left(\frac{x}{\xi}\right) x^{r-s-1} dx = \gamma \xi^{s-r} \int_1^\infty \sum_{n=1}^\infty \overline{a(n)} E(Anx/\xi) x^{r-s-1} dx$$
$$= \gamma \xi^{s-r} \sum_{n=1}^\infty \overline{a(n)} \int_1^\infty E(Anx/\xi) x^{r-s-1} dx$$
$$= \gamma A^{s-r} \sum_{n=1}^\infty \overline{a(n)} n^{s-r} \Gamma(r-s, An\xi^{-1})$$

by (3) and Lemma 1 with $\Gamma(a, x) = \Gamma_{r_1, r_2}(a, x)$. Similarly

$$\xi^s \int_1^\infty \psi(y\xi) y^{s-1} dy = \xi^s \sum_{n=1}^\infty a(n) \int_1^\infty E(Any\xi) y^{s-1} dy = A^{-s} \sum_{n=1}^\infty a(n) n^{-s} \Gamma(s, An\xi),$$

and finally

$$R(s) = A^{-s} \sum_{n=1}^\infty a(n) n^{-s} \Gamma(s, An\xi) + \gamma A^{s-r} \sum_{n=1}^\infty \overline{a(n)} n^{s-r} \Gamma(r-s, An\xi^{-1})$$
$$+ \xi \sum_{\nu=0}^{m-1} B_\nu I(\nu, \xi, s-r) - \gamma \xi \sum_{\nu=0}^{m-1} (-1)^\nu \overline{B}_\nu I(\nu, \xi, s). \tag{11}$$

This equation has been proved for $\sigma > \sigma_a, \xi > 0$. But by Lemma 1 (with $\Gamma = \Gamma_{r_1, r_2}$)

$$\Gamma(s, An\xi) = (An\xi)^s \int_1^\infty E(An\xi x) x^{s-1} dx \ll n^\alpha \int_1^\infty \exp\left\{-b\Re(An\xi)^{2/N}\right\} x^{\sigma-1} dx$$
$$\ll \exp\left\{-b(An)^{2/N}\delta\right\} n^\beta,$$

for $N = r_1 + 2r_2, \delta = \Re(\xi^{2/N})$ and suitable constants $\alpha, \beta > 0$. Hence, the series in (11) are absolutely convergent for all s with $0 < \Re(s) < r$, provided $\delta > 0$, i.e. $|\arg \xi| < \frac{\pi N}{4}$. So (11) holds for these values of s and we have proved

Theorem 1. *Let* $N = r_1 + 2r_2 > 0, r > 0, |\arg\xi| < \frac{\pi N}{4}$. *Then, for* $0 < \Re(s) < r$

$$\varphi(s) = \sum_{n=1}^{\infty} a(n)n^{-s}Q_{r_1,r_2}(s, An\xi) + \gamma A^{2s-r}X(s)\sum_{n=1}^{\infty}\overline{a(n)}n^{s-r}Q_{r_1,r_2}(r-s, An\xi^{-1})$$

$$+ \xi A^s \Gamma\left(\frac{s}{2}\right)^{-r_1}\Gamma(s)^{-r_2}\sum_{\nu=0}^{m-1}[B_\nu I(\nu,\xi,s-r) - \gamma(-1)^\nu \overline{B}_\nu I(\nu,\xi,s)],$$

where

$$X(s) = \frac{\Gamma\left(\frac{r-s}{2}\right)^{r_1}\Gamma(r-s)^{r_2}}{\Gamma\left(\frac{s}{2}\right)^{r_1}\Gamma(s)^{r_2}}.$$

This is our version of Lavrik's approximate functional equation. Our proof as given here seems to be more natural, since it uses on the one hand Riemann's own starting point for the zeta function. On the other hand it shows clearly how the absolute convergent series arise which are used later for the computation of φ in the critical strip $0 < \Re(s) < r$.

Theorem 1 allows in principle to compute $\varphi(s)$ for all $s \neq 0, r$, since $I(\nu,\xi,s)$ can be determined explicity (vgl. (19), (20)). Here we can choose the parameter ξ in a suitable manner. Later we shall define $|\xi| = 1, \arg\xi = (\arg s)^{N/2}$. Then $An\xi s^{-N/2}$ is real and the result from [10] apply, giving an asymptotic expansion of $\Gamma(a,x)$ for $a \to \infty$, which can also be used for numerical computations.

2. THE FUNCTION Z

We now rewrite the functional equation (III) for our purposes. First

$$A^{-s}\Gamma\left(\frac{s}{2}\right)^{r_1}\Gamma(s)^{r_2}\varphi(s) = R(s) = \gamma\overline{R(r-\overline{s})}$$
$$= \gamma A^{s-r}\Gamma\left(\frac{r-s}{2}\right)^{r_1}\Gamma(r-s)^{r_2}\overline{\varphi(r-\overline{s})},$$

so that

$$\varphi(s) = \gamma A^{2s-r}X(s)\overline{\varphi(r-\overline{s})} \tag{12}$$

with $X(s)$ from Theorem 1. Now let $s = \frac{r}{2} + it, t \in \mathbf{R}$. Then

$$X\left(\frac{r}{2}+it\right) = \left[\Gamma\left(\frac{r}{4}-i\frac{t}{2}\right)\Gamma\left(\frac{r}{4}+i\frac{t}{2}\right)^{-1}\right]^{r_1}\left[\Gamma\left(\frac{r}{2}-it\right)\Gamma\left(\frac{r}{2}+it\right)^{-1}\right]^{r_2}.$$

If we for $\Re(a) > 0$ write

$$\Gamma(a) = |\Gamma(a)|e^{i\Delta(a)}, \Delta(a) = \arg\Gamma(a) = \Im\log\Gamma(a), \tag{13}$$

taking the principal value of the argument (i.e. $\Delta(a) = 0$ for $a > 0$), we see that (note $\Gamma(\overline{a}) = \overline{\Gamma(a)}$)

$$X\left(\frac{r}{2}+it\right) = \exp\left\{-2ir_1\Delta\left(\frac{r}{4}+i\frac{t}{2}\right) - 2ir_2\Delta\left(\frac{r}{2}+it\right)\right\},$$

and with (12)

$$\varphi\left(\frac{r}{2}+it\right) = \gamma A^{2it}\exp\left\{-2i\left[r_1\Delta\left(\frac{r}{4}+i\frac{t}{2}\right) + r_2\Delta\left(\frac{r}{2}+it\right)\right]\right\}\overline{\varphi\left(\frac{r}{2}+it\right)}. \tag{14}$$

Since $|\gamma| = 1$ we may let $\gamma = e^{i\delta}, -\pi \leq \delta < \pi$, and (14) reads

$$\varphi\left(\frac{r}{2}+it\right) = e^{-2i\vartheta}\overline{\varphi\left(\frac{r}{2}+it\right)}, \tag{15}$$

where

$$\vartheta = \vartheta(t) = -\frac{\delta}{2} - t\log A + r_1\Im\log\Gamma\left(\frac{r}{4}+i\frac{t}{2}\right) + r_2\Im\log\Gamma\left(\frac{r}{2}+it\right). \tag{16}$$

Hence

$$Z(t) := e^{i\vartheta}\varphi\left(\frac{r}{2}+it\right) = e^{-i\vartheta}\overline{\varphi\left(\frac{r}{2}+it\right)} = \overline{Z(t)}, \tag{17}$$

i.e. $Z(t)$ is real provided t is and the zeros of φ on the line $\sigma = \dfrac{r}{2}$ are exactly the real zeros of Z. Now we want to express Z with the help of Theorem 1. For this we need an explicit representation of $I(\nu,\xi,a)$. From (10) we get $I(0,\xi,a) = \xi^{a-1}a^{-1}$ and

$$\frac{d^\nu}{da^\nu}I(0,\xi,a) = \frac{d^\nu}{da^\nu}\int_0^1 (y\xi)^{a-1}dy = \int_0^1 (\log y\xi)^\nu (y\xi)^{a-1}dy = I(\nu,\xi,a),$$

so that

$$\begin{aligned} I(\nu,\xi,a) &= \frac{d^\nu}{da^\nu}[\xi^{a-1}a^{-1}] = \xi^{-1}\sum_{j=0}^{\nu}\binom{\nu}{j}\frac{d}{da^j}(a^{-1})\frac{d}{da^{\nu-j}}(e^{a\log\xi}) \\ &= \xi^{a-1}a^{-1}\sum_{j=0}^{\nu}(-1)^j\binom{\nu}{j}j!a^{-j}(\log\xi)^{\nu-j}. \end{aligned} \tag{18}$$

Let $\xi = e^{i\psi}, \psi \in \mathbf{R}, a = \frac{r}{2} + it$. Then

(19)

$$I\left(\nu, e^{i\psi}, \frac{r}{2} + it\right) = e^{i\psi\left(\frac{r}{2}+it\right)-i\psi}\left(\frac{r}{2} + it\right)^{-1}\sum_{j=0}^{\nu}(-1)^j\binom{\nu}{j}j!\left(\frac{r}{2} + it\right)^{-j}(i\psi)^{\nu-j}$$

$$= \xi^{-1}i^{\nu}e^{i\frac{r\psi}{2}-\psi t}\left(\frac{r}{2} + it\right)^{-1}\sum_{j=0}^{\nu}i^j\binom{\nu}{j}j!\left(\frac{r}{2} + it\right)^{-j}\psi^{\nu-j},$$

and similarly with $a = s - r = -\frac{r}{2} + it$

(20)

$$I\left(\nu, e^{i\psi}, -\frac{r}{2} + it\right) = -\xi^{-1}i^{\nu}e^{-i\frac{r\psi}{2}-\psi t}\left(\frac{r}{2} - it\right)^{-1}\sum_{j=0}^{\nu}(-i)^j\binom{\nu}{j}j!\left(\frac{r}{2} - it\right)^{-j}\psi^{\nu-j}.$$

If we let

$$S(\nu, \psi, t) = e^{i\frac{r\psi}{2}}\left(\frac{r}{2} + it\right)^{-1}\sum_{j=0}^{\nu}i^j\binom{\nu}{j}j!\left(\frac{r}{2} + it\right)^{-j}\psi^{\nu-j}, \tag{21}$$

we get from (19) and (20)

$$I\left(\nu, e^{i\psi}, \frac{r}{2} + it\right) = \xi^{-1}i^{\nu}e^{-\psi t}S(\nu, \psi, t),$$

$$I\left(\nu, e^{i\psi}, -\frac{r}{2} + it\right) = -\xi^{-1}i^{\nu}e^{-\psi t}\overline{S(\nu, \psi, t)}.$$

With $s = \frac{r}{2} + it, R = |\Gamma\left(\frac{s}{2}\right)^{-r_1}\Gamma(s)^{-r_2}|$ we get in Theorem 1

(22)

$$A^s\xi\Gamma\left(\frac{s}{2}\right)^{-r_1}\Gamma(s)^{-r_2}\sum_{\nu=0}^{m-1}[B_\nu I(\nu, \xi, s - r) - \gamma(-1)^\nu\overline{B}_\nu I(\nu, \xi, s)]$$

$$= A^{\frac{r}{2}+it}Re^{-\psi t - ir_1\Delta\left(\frac{r}{4}+i\frac{t}{2}\right)-ir_2\Delta\left(\frac{r}{2}+it\right)}\sum_{\nu=0}^{m-1}i^{\nu}[B_\nu\overline{S(\nu, \psi, t)}$$

$$+ (-1)^\nu e^{i\delta}\overline{B}_\nu S(\nu, \psi, t)]$$

$$= A^{\frac{r}{2}}Re^{-\psi t - i\vartheta}\sum_{\nu=0}^{m-1}i^{\nu}[e^{-i\frac{\delta}{2}}B_\nu\overline{S(\nu, \psi, t)} + (-1)^\nu e^{i\frac{\delta}{2}}\overline{B}_\nu S(\nu, \psi, t)]$$

$$= A^{\frac{r}{2}}Re^{-\psi t - i\vartheta}\sum_{\nu=0}^{m-1}P_\nu(t),$$

where the functions

$$P_\nu(t) = i^\nu \left[e^{-i\frac{\delta}{2}} B_\nu \overline{S(\nu,\psi,t)} + (-1)^\nu e^{i\frac{\delta}{2}} \overline{B}_\nu S(\nu,\psi,t) \right]$$

are real valued. From Theorem 1, (17) and (22) and using $Q_{r_1,r_2}(\overline{a},\overline{x}) = \overline{Q_{r_1,r_2}(a,x)}$ we obtain

$$(23)\quad Z(t) = 2\Re\left\{ e^{i\vartheta} \sum_{n=1}^{\infty} a(n) n^{-\frac{r}{2}-it} Q_{r_1,r_2}\left(\frac{r}{2}+it, Ane^{i\psi}\right)\right\} + A^{\frac{r}{2}} Re^{-\psi t} \sum_{\nu=0}^{m-1} P_\nu(t).$$

This formula can be used to compute $Z(t)$ for $t \to \infty$ and it has proved to be useful in certain applications (see [9], [11]). In order to compute $Q_{r_1,r_2}(a,x)$ with $a = \frac{r}{2} + it, x = Ane^{i\psi} = x(n)$ the asymptotic expansion from [10] can be used. For this it is necessary that $l = l(n) = xa^{-N/2}, N = r_1 + 2r_2$, is real. Hence we choose $\psi = \frac{N}{2}\arg\left(\frac{r}{2}+it\right) \sim \frac{\pi N}{4}$ for $t \to \infty$, so that

$$l(n) = An\left|\frac{r}{2}+it\right|^{-\frac{N}{2}}.$$

Since $|\Gamma(s)| \sim (2\pi)^{1/2} e^{-\pi t/2} t^{\sigma-1/2}, t \to \infty$, we have in (23)

$$R = \left|\Gamma\left(\frac{r}{4}+i\frac{t}{2}\right)^{-r_1} \Gamma\left(\frac{r}{2}+it\right)^{-r_2}\right| \sim Ce^{\frac{\pi N t}{4}} t^{\frac{r_1+r_2}{2}-\frac{Nr}{4}}, C > 0, t \to \infty,$$

i.e. the second term in (23) satisfies for $\psi = N/2 \arg\left(\frac{r}{2}+it\right)$ and since $P_\nu(t) = O(t^{-1})$ the estimate

$$\ll t^{\frac{r_1+r_2}{2}-\frac{Nr}{4}-1}, t \to \infty,$$

This tends to zero, provided

$$\frac{r_1+r_2}{2} < 1 + \frac{Nr}{4}$$

which we will assume tacitly in our applications. We further remark that $\psi = \frac{N}{2}\arg\left(\frac{r}{2}+it\right)$ coincides exactly which Lavrik's choice of this parameter. He gives in this case an estimation for $Q_{r_1,r_2}\left(\frac{r}{2}+it, Ane^{i\psi}\right)$, while we gave an asymptotic expansion which is useful if $l(n)$ is not too small.

The reader should be aware of the fact that for an effective use of Theorem 1 it is necessary to supply explicit estimations of the error in computing the incomplete gamma functions. How to do this has been demonstrated in [9]. The number of terms needed from the infinite series in Theorem 1 is approximately $O(t^{N/2+\varepsilon})$ for each $\varepsilon > 0$.

3. THE ZEROS

A. Generalities. Since $\Gamma\left(\frac{s}{2}\right)^{r_1}\Gamma(s)^{r_2}$ has a pole of order r_2 at $s = -(2\nu+1)$ and pole of order $r_1 + r_2$ at $s = -2\nu, \nu \in \mathbf{N}$, it follows that φ has a zeros at $s = -k, k \in \mathbf{N}_0$, because $s^l(r - s)^m R(s)$ is an entire function by (II). If n_k denotes the mutiplicity of the zero, we see that $n_0 = r_1 + r_2 - l$, if $l > 0$ and $n_0 \geq r_1 + r_2$, if $l = 0$. Moreover, $n_{2k} \geq r_1 + r_2$ if $k \geq 1$ and $n_{2k+1} \geq r_2$ if $k \geq 0$. These zeros at the non positive integers will be called *trivial.*

By our assumptons $a \neq 0$. Hence, there is a smallest n with $a(n) \neq 0$. Then, for $\sigma \to \infty$

$$|\varphi(s)| = |a(n)|n^{-\sigma} + O((n+1)^{-\sigma}) > 0,$$

provided σ is large enough. So there exists $\sigma_k > 0$ with

$$\varphi(s) \neq 0 \quad \text{if } \sigma \geq \sigma_k. \tag{24}$$

But then also $\gamma\overline{R(r - \overline{s})} = R(s) \neq 0$ for $\sigma \geq \sigma_k$, i.e. $R(s) \neq 0$ for $\sigma \leq r - \sigma_k$. In particular, $n_{2k} = r_1 + r_2$, if $-2k \leq r - \sigma_k$ and $n_{2k+1} = r_2$, if $-(2k+1) \leq r - \sigma_k$. Hence, all zeros of R lie in the «critical strip»

$$r - \sigma_k < \sigma < \sigma_k.$$

Since $s^{l-n_0}(s - r)^m R(s)$ is an entire function of order 1, Hadamard's Theorem[3] implies

$$s^{l-n_0}(s - r)^n R(s) = e^{c+bs}\prod_{\varrho}\left(1 - \frac{s}{\varrho}\right)e^{\frac{s}{\varrho}}, \tag{25}$$

with suitable constants b and c and the product runs over all zeros ϱ of R and converges absolutely. Since $\log\Gamma(s) \sim s\log s$ for $s \to +\infty$ we have $R(s) = O(e^{|s|^{1+\varepsilon}})$ for all $\varepsilon > 0$, but not $R(s) = O(e^{|s|})$ for $s \to \infty$ and so we get

$$\sum_{\varrho}\frac{1}{|\varrho|^{1+\varepsilon}} \text{convergent}, \quad \sum_{\varrho}\frac{1}{|\varrho|} \text{divergent},$$

which implies in particular that R has infinitely many zeros in the critical strip.

Of course, one would like to choose σ_k as small as possible. But if still $\sigma_k > r$, it is possible that the critical strip contains trivial zeros. If $-k$ is of this type and $R(-k) = 0$, the functional equation (III) implies also $0 = R(-k) = \gamma\overline{R(r+k)}$, i.e. $\varphi(r+k) = 0$. Such zeros of φ will be called *induced.* They can only occur if $\sigma_k > r$ and $n_{2k} > r_1 + r_2$. All other zeros of R (hence of φ) are called *non trivial.* If $\varrho = \beta + i\tau$ is a non trivial zero of φ, then so is $\varphi(r - \overline{\varrho}) = \varphi(r - \beta + i\tau) = 0$, i.e. the non trivial zeros are symmetric with respect to the line $\sigma = \frac{r}{2}$ and the *Riemann hypothesis* for φ states that all non trivial zeros lie on the *critical line* $\sigma = \frac{r}{2}$. It is our goal here to find, for any given φ, the zeros on the critical line and also possible counterexamples to the Riemann hypothesis. The introduction on the real valued function Z in the last section gives a method to find a lower bound for the number of zeros on $\sigma = \frac{r}{2}$ by detecting sign changes of Z. Further information is given in §4.

By logarithmic differentiation in (25) we obtain

$$\frac{R'}{R}(s) = -\frac{l - n_0}{s} - \frac{m}{s - r} + b + \sum_{\varrho} \left(\frac{1}{\varrho} + \frac{1}{s - \varrho} \right). \tag{26}$$

Moreover, from (II) with $\psi(z) = \frac{\Gamma'}{\Gamma}(z)$

$$\frac{R'}{R}(s) = -\log A + \frac{r_1}{2}\psi\left(\frac{s}{2}\right) + r_2\psi(s) + \frac{\varphi'}{\varphi}(s),$$

so

$$\frac{\varphi'}{\varphi}(s) = -\frac{l - n + 0}{s} - \frac{m}{s - r} + \log A - \frac{r_1}{2}\psi\left(\frac{s}{2}\right) - r_2\psi(s) + b + \sum_{\varrho} \left(\frac{1}{\varrho} + \frac{1}{s - \varrho} \right). \tag{27}$$

We now consider more closely the important special case $l = m = n_0 = 0, \sigma_k \leq r$. It includes for instance Dirichlet series attached to modular form. A famous example here is Ramanujan's zeta function[11]. In our case R is an entire function and $R(0) \neq 0$. By (26) and (II) we get

$$b = \frac{R'}{R}(0) = -\overline{\frac{R'}{R}(r)} = \log A - \frac{r_1}{2}\psi\left(\frac{r}{2}\right) - r_2\psi(r) - \overline{\frac{\varphi'}{\varphi}(r)}.$$

Moreover

$$\begin{aligned} b + \sum_{\varrho} \left(\frac{1}{\varrho} + \frac{1}{s - \varrho} \right) &= \frac{R'}{R}(s) = -\overline{\frac{R'}{R}(r - \overline{s})} = -\overline{b} - \sum_{\varrho} \left(\frac{1}{\varrho} + \frac{1}{r - s - \overline{\varrho}} \right), \\ 2\Re(b) &= -\sum_{\varrho} \left(\frac{1}{\varrho} + \frac{1}{s - \varrho} \right) - \sum_{\varrho} \left(\frac{1}{\varrho} + \frac{1}{r - s - \overline{\varrho}} \right). \end{aligned} \tag{28}$$

Let $T > 0$. We split the sums in (28) as

(29)

$$2\Re(b) = -\left(\sum_{|\Im\varrho|\le T} + \sum_{|\Im\varrho|>T}\right)\left(\frac{1}{\varrho} + \frac{1}{s-\varrho}\right) - \left(\sum_{|\Im\varrho|\le T} + \sum_{|\Im\varrho|>T}\right)\left(\frac{1}{\varrho} + \frac{1}{r-s-\overline{\varrho}}\right)$$

$$= -\sum_{|\Im|\le T}\left(\frac{1}{\varrho} + \frac{1}{\overline{\varrho}}\right) - \sum_{|\Im|\le T}\left(\frac{1}{s-\varrho} - \frac{1}{s-(r-\overline{\varrho})}\right) + o(1)$$

for $T \to \infty$ by absolute convergence of the series in (28). If ϱ runs through the zeros , so does $r - \overline{\varrho}$ and hence the second sum in (29) vanishes. If $\varrho = \beta + i\tau, \beta, \tau \in \mathbf{R}$, then $\frac{1}{\varrho} + \frac{1}{\varrho} = \frac{2\beta}{\beta^2 + \tau^2}$ and $T \to \infty$ in (29) gives

$$\Re(b) = -\sum_{\tau} \frac{\beta}{\beta^2 + \tau^2} = -2\sum_{\varrho} \Re\frac{1}{\varrho}$$

and we have

Lemma 5. *Let $n_0 = l = m = 0$ and $\sigma_k \le r$. Then*

$$-\beta\sum_{\tau} \frac{1}{\beta^2 + \tau^2} = \log A - \frac{r_1}{2}\psi\left(\frac{r}{2}\right) - r_2\psi(r) - \Re\left\{\frac{\varphi'}{\varphi}(r)\right\}.$$

B. The Distribution of the Zeros in the Critical Strip

If $T > 0$ then let $N(T)$ denote the number of zeros of R in the critical strip $r - \sigma_k < \sigma < \sigma_k, 0 \le t \le T$, each counted according to its multiplicity, those with ordinates equal to $t = 0$ or $t = T$ counted with weight $\frac{1}{2}$. We may assume $T > 0$, since otherwise one can look at $R(\overline{s})$ instead of $R(s)$.

After Littlewood [18] we define a function S in the following manner. For $\sigma \ge \sigma_k$ we have $\varphi(s) \ne 0$ and then set

$$\log\varphi(s) := \log|\varphi(s)| + i\arg\varphi(s),\ -\pi \le \arg\varphi(s) < \pi.$$

If $\sigma < \sigma_k$ and t is not the ordinate of a zero of φ, then let $\log\varphi(s)$ be the value obtained by continuous continuation along the line with imaginary part equal to t starting from $s_0 = \sigma_0 + it$ with $\sigma_0 > \sigma_a$. If t is the ordinate of a zero of φ, then we define $\log\varphi(\sigma + it) := \frac{1}{2}\lim_{\varepsilon\to 0}[\log\varphi(\sigma + it + i\varepsilon) + \log\varphi(\sigma + it - i\varepsilon)]$. Now let

$$S(t) := \frac{1}{\pi}\arg\varphi\left(\frac{r}{2} + it\right) = \frac{1}{\pi}\Im\log\varphi\left(\frac{r}{2} + it\right). \tag{30}$$

Theorem 2. *If $\varphi \in \mathbf{D}$, then for $T > 0$*

$$N(T) = \frac{1}{\pi}\vartheta(T) + \frac{\delta}{2\pi} + \frac{1}{2}(l+m) + S(T) - S(0).$$

Proof. Since the zeros of R form a discrete set in the complex plane, there exists $\varepsilon > 0$, such that there are no zeros in the horizontal strips $-\varepsilon \le t < 0, 0 < t \le \varepsilon, T-\varepsilon \le t < T, T < t \le T+\varepsilon$. Let $a_0 > 0$, such that $r - \sigma_k < a_0$ and denote by C_1 the rectangle with corners $r-\sigma_k - a_0 - i\varepsilon, \sigma_k + a_0 - i\varepsilon, \sigma_k + a_0 + i(T+\varepsilon), r - \sigma_k - a_0 + i(T+\varepsilon)$ and by C_2 the rectangle with corners $r-\sigma_k-a_0+i\varepsilon, \sigma_k+a_0+i\varepsilon, \sigma_k+a_0+i(T-\varepsilon), r-\sigma_k-a_0+i(T-\varepsilon)$.

If N_i, P_i denotes the number of zeros and poles of R in C_i, then

$$N_i - P_i = \frac{1}{2\pi i}\int_{C_i}\frac{R'}{R}(s)\,ds = \frac{1}{2\pi}\Im\left\{\int_{C_i}\frac{R'}{R}(s)\,ds\right\}.$$

In C_2 there are no real zeros of R and obviously $P_2 = 0$. Let N_{tr} denote the number of trivial zeros of φ, which are also zeros of R (note that $\varphi(s) = 0$ does not necessarily imply that $R(s) = 0$) and N_s the number of non trivial real zeros of φ. Then $N_1 = N_2 + N_{tr} + N_s$ and $N(T) = N_2 + \frac{1}{2}N_{tr} + \frac{1}{2}N_s$. Moreover, $P_1 = l + m$ and thus

(31)

$$N(T) = \frac{1}{2}\lim_{\varepsilon\to 0}(N_1 + N_2) = \frac{1}{4\pi}\lim_{\varepsilon\to 0}\left[\Im\int_{C_1}\frac{R'}{R}(s)\,ds + \Im\int_{C_2}\frac{R'}{R}(s)\,ds\right] + \frac{1}{2}(l+m).$$

Let C_{11} be part of C_1 bounded by the points $\frac{r}{2} - i\varepsilon, \frac{1}{2} + i(T+\varepsilon)$ and C_{12} be the remainder. Now we have with our convention for $\log\varphi$

$$\int_{C_{11}}\frac{R'}{R}(s)\,ds = \int_{\frac{r}{2}-i\varepsilon}^{\frac{r}{2}+i(T+\varepsilon)}\left[-\log A + \frac{r_1}{2}\frac{\Gamma'}{\Gamma}\left(\frac{s}{2}\right) + r_2\frac{\Gamma'}{\Gamma}(s) + \frac{\varphi'}{\varphi}(s)\right]ds$$

$$= -\log A \cdot s\Big|_{\frac{r}{2}-i\varepsilon}^{\frac{r}{2}+i(T+\varepsilon)} + r_1\log\Gamma\left(\frac{s}{2}\right)\Big|_{\frac{r}{2}-i\varepsilon}^{\frac{r}{2}+i(T+\varepsilon)}$$

$$+ r_2\log\Gamma(s)\Big|_{\frac{r}{2}-i\varepsilon}^{\frac{r}{2}+i(T+\varepsilon)} + \log\varphi(s)\Big|_{\frac{r}{2}-i\varepsilon}^{\frac{r}{2}+i(T+\varepsilon)},$$

hence,

$$\Im \int_{C_{11}} \frac{R'}{R}(s)ds = -(T+2\varepsilon)\log A$$

(32)
$$+ r_1 \left[\Im \log \Gamma\left(\frac{r}{4} + i\frac{T}{2} + i\frac{\varepsilon}{2}\right) - \Im \log \Gamma\left(\frac{r}{4} - i\frac{\varepsilon}{2}\right)\right]$$
$$+ r_2 \left[\Im \log \Gamma\left(\frac{r}{2} + iT + i\varepsilon\right) - \Im \log \Gamma\left(\frac{r}{2} - i\varepsilon\right)\right]$$
$$+ \pi S(T+\varepsilon) - \pi S(-\varepsilon).$$

Apply the functional of R to C_{12} :

$$\int_{C_{12}} \frac{R'}{R}(s)ds = -\int_{\frac{r}{2}+i(T+\varepsilon)}^{\frac{r}{2}-i\varepsilon} \frac{R^{*\prime}}{R^*}(r-s)ds = \int_{\frac{r}{2}-i(T+\varepsilon)}^{\frac{r}{2}+i\varepsilon} \frac{R^{*\prime}}{R^*}(s)ds.$$

Since $\varphi^*(s) = \overline{\varphi(\overline{s})}$ we find $\Im \log \varphi^*(\overline{s}) = -\Im \log \varphi(s)$ and similarly to (32)

$$\Im \int_{C_{12}} \frac{R'}{R}(s)ds = -(T+2\varepsilon)\log A$$
$$+ r_1 \left[\Im \log \Gamma\left(\frac{r}{4} + i\frac{T}{2} + i\frac{\varepsilon}{2}\right) - \Im \log \Gamma\left(\frac{r}{4} - i\frac{\varepsilon}{2}\right)\right]$$
$$+ r_2 \left[\Im \log \Gamma\left(\frac{r}{2} + iT + i\varepsilon\right) - \Im \log \Gamma\left(\frac{r}{2} - i\varepsilon\right)\right]$$
$$+ \pi S(T+\varepsilon) - \pi S(-\varepsilon).$$

Together with (32) this gives

$$\Im \int_{C_1} \frac{R'}{R}(s)ds = -2(T+2\varepsilon)\log A$$

(33)
$$+ 2r_1 \left[\Im \log \Gamma\left(\frac{r}{4} + i\frac{T}{2} + i\frac{\varepsilon}{2}\right) - \Im \log \Gamma\left(\frac{r}{4} - i\frac{\varepsilon}{2}\right)\right]$$
$$+ 2r_2 \left[\Im \log \Gamma\left(\frac{r}{2} + iT + i\varepsilon\right) - \Im \log \Gamma\left(\frac{r}{2} - i\varepsilon\right)\right]$$
$$2\pi S(T+\varepsilon) - 2\pi S(-\varepsilon).$$

By the same reasoning we obtain

$$\Im \int_{C_2} \frac{R'}{R}(s)\,ds = -2(T-2\varepsilon)\log A$$
$$+2r_1\left[\Im \log\Gamma\left(\frac{r}{4}+i\frac{T}{2}-i\frac{\varepsilon}{2}\right)-\Im\log\Gamma\left(\frac{r}{4}+i\frac{\varepsilon}{2}\right)\right] \tag{34}$$
$$+2r_2\left[\Im\log\Gamma\left(\frac{r}{2}+iT-i\varepsilon\right)-\Im\log\Gamma\left(\frac{r}{2}+i\varepsilon\right)\right]$$
$$+2\pi S(T-\varepsilon)-2\pi S(\varepsilon).$$

By (31), (33), (34) and (16) we get the conclusion of our Theorem.

Finally, we give an explicit estimation for the number of zeros in an interval of the form $T \le t \le T+h, T>0, h>0$.

Lemma 6. *Let* $s_0 = \sigma_0 + it_0, \sigma_0 > \sigma_k, 0 < \alpha < 1, R > 0, 0 < \sigma_0 - \frac{r}{2} < \alpha R < t_0, \delta := \left[(\alpha R)^2 - \left(\frac{r}{2}-\sigma_0\right)^2\right]^{1/2}$. *For* $t > 0$ *and* $\sigma_0 - R \le \sigma \le \sigma_0 + R$ *assume that* $|\varphi(s)| \le c_1 t^{c_2}, c_1 > 0, c_2 > 0$. *Then*

$$N(t_0+\delta) - N(t_0-\delta) \le \frac{2}{\log\frac{1}{\alpha}}\{\log c_1 + c_2\log t - \log|\varphi(s_0)|\}.$$

Proof. Let $f(z) = \varphi(s_0+z)$. Then $f(0) \neq 0$ and Jensen's formula [3] gives

$$\sum_{j=1}^{n}\log\frac{R}{|\varrho_j|} = \frac{1}{2\pi}\int_0^{2\pi}\log|f(Re^{i\psi}|d\psi - \log|f(0)|$$
$$= \frac{1}{2\pi}\int_0^{2\pi}\log|\varphi(s_0+Re^{i\psi})|d\psi - \log|\varphi(s_0)|,$$

where $\vartheta_1, \ldots, \vartheta_n$ are the zeros of f in $|z| < R$, i.e. the zeros of φ in $|z-s_0| < R$. On the other hand, we have for $0 < \alpha < 1$

$$\sum_j \log\frac{R}{|\varrho_j|} \ge \sum_{|\varrho_j|\le\alpha R}\log\frac{R}{|\varrho_j|} \ge \sum_{|\varrho_j|\le\alpha R}\log\frac{R}{\alpha R} = M\log\frac{1}{\alpha},$$

where now M denotes the number of zeros of φ in $|z-s_0| \le \alpha R$. The circle $|z-s_0| = \alpha R$ intersects the line $\sigma = \frac{r}{2}$ in the two points $s_{1,2} = \frac{r}{2} + i(t_0 \pm \delta)$, where δ is defined above. Therefore,

$$\frac{1}{2}[N(t_0+\delta) - N(t_0-\delta)] \le M \le \frac{1}{\log\frac{1}{\alpha}}\sum_j\log\frac{R}{|\varrho_j|}$$

and

$$N(t_0 + \delta) - N(t_0 - \delta) \leq \frac{2}{\log \frac{1}{\alpha}} \left\{ \frac{1}{2\pi} \int_0^{2\pi} \log |\varphi(s_0 + Re^{i\psi})| - \log |\varphi(s_0)| \right\},$$

which gives the Lemma.

An estimation for $|\varphi(s)|$ as it is used in the Lemma can be obtained from Theorem 1 and an estimation for the incomplete gamma function. More details on this can be found in [9] for the special case $r_1 + 2r_2 = 1$.

4. GRAM'S LAW AND TURING'S METHOD

In this section we shall assume that the function a is normalized, i.e. $a(1) = 1$. By our assumptions this will be the case provided a is mutiplicative.

In order to compute Z and the zeros of φ we use the representation (23), where we assume t to be large enough. If this is not the case, we may use the functional equation for the incomplete gamma function. This and many other properties are to be found in [10]. If φ is chosen properly, namely

$$\varphi = M \arg \left(\frac{r}{2} + it \right) \sim \frac{\pi M}{2}, t \to \infty, M := r_1 + 2r_2,$$

the second term in (23) will be of order $O(t^{-c})$ for some $c > 0$, if $\frac{1}{2}(r_1 + r_2) < 1 + \frac{Mr}{4}$, which we may assume. The first summand in the infinite series is given by (with $Q = Q_{r_1,r_2}$)

$$Q\left(\frac{r}{2} + it, Ae^{i\psi}\right) = Q\left(\frac{r}{2} + it, \lambda \left(\frac{r}{2} + it\right)^M\right),$$

where λ ir real and

$$\lambda = Ae^{i\psi} \left(\frac{r}{2} + it\right)^{-M} \sim At^{-M}, t \to \infty.$$

For such a small λ we know that $Q(a, \lambda a)$ is close to 1 and the summands are

$$\approx 1 + \text{ oscillating terms},$$

where the oscillating terms come from $n^{-\frac{r}{2}-it}$ for $n \geq 2$. Hence we can expect some cancellation, and we expect

$$Z(t) \approx 2\Re\{e^{i\vartheta} + \text{oscillating terms}\} \approx 2\cos\vartheta + \text{oscillating terms}, \vartheta = \vartheta(t).$$

At the extreme values of $\cos\vartheta$, i.e. at the points $t \in \{t \in \mathbf{R} \,|\, \vartheta(t) = \pi\nu, \nu \in \mathbf{Z}\}$ we therefore expect that the signs of Z and $\cos\vartheta$ often will agree.

Let $T_G \geq 0$ be chosen such that $\vartheta'(t) \neq 0$ for $t \geq T_G$ (for many Dirichlet series $T_G = 0$). The points t_ν where

$$t_\nu \geq T_G, \vartheta(t_\nu) = \pi\nu, \nu \in \mathbf{Z},$$

will be called *Gram points* and *Gram's Law*[8]

$$sgn Z(t_\nu) = sgn \cos\vartheta(t_\nu) = (-1)^\nu$$

states that there is a zero of Z of odd order between two Gram points.

Now the Riemann hypothesis for φ in some interval $I = [T, T_1]$ with $0 < T < T_1$ is checked in two stages: First, we try to find in I as many zeros of Z as there are Gram intervals in I (i.e. intervals of the form $[t_\nu, t_{\nu+1}]$), for example by computing Z at the Gram points $t_\nu \in I$. This gives a lower bound for the number of zeros of φ on the critical line $s = \frac{r}{2} + it, t \in I$. In the second stage, we try to show that φ has no more zeros in $r - \sigma_k \leq \sigma \leq \sigma_k, t \in I$. This verifies the Riemann hypothesis for φ in the interval I of the critical line.

There are two possibilities for the second stage, which are due to Backlund[1] and Turing[29]. Backlund's methos involves much work in practice since it is necessary to compute values of $\varphi(s)$ for $\sigma > \frac{r}{2}$. Turing's method is more complicated to derive but much easier to apply in practice since only values of φ on the critical line, i.e. values of Z have to be computed and this has already been done in the first stage. Turing's idea is very elegant and rests on an estimation of Littlewood [18] for

$$S_1(T) := \int_0^T S(t)\,dt, T > 0, \tag{35}$$

where S was defined in (30). It is well known [28, p. 220 ff.] that for $0 < T < T_1$

$$\begin{aligned} \pi[S_1(T_1) - S_1(T)] &= \Re\left\{\int_{\frac{r}{2}+iT_1}^{\infty+iT_1} \log\varphi(s)\,ds - \int_{\frac{r}{2}+iT}^{\infty+iT} \log\varphi(s)\,ds\right\} \\ &= \int_{\frac{r}{2}+iT_1}^{\infty+iT_1} \log|\varphi(s)|\,ds - \int_{\frac{r}{2}+iT}^{\infty+iT} \log|\varphi(s)|\,ds, \end{aligned} \tag{36}$$

with $\log\varphi(s)$ as in §3 B. We now give two estimations for S_1, of which the first is generally applicable but the second gives sharper bounds in special cases.

First, we note that integrals of the form

$$(37)\qquad J(\sigma,H,t) := \int_{\sigma+it}^{\sigma+H+it} \log|\varphi(s)|ds, \sigma > \max\{\sigma_a,\sigma_k\}, H > 0, t > 0,$$

are relatively easy to estimate, since the Dirichlet series (1) is absolutely convergent for the admissible values of the parameters. Note, that in this section we assume $a(1) = 1$. If φ has an Euler product the calculations will be simplified. We refer to the examples in [9].

Theorem 3. *Let* $\varphi \in \mathbf{D}, a(1) = 1, s_0 = \sigma_0 + it_0, \sigma_0 > \max\{\sigma_a,\sigma_k\}, 0 < R < R' < t_0, \left(\sigma_0 - \frac{r}{2}\right) \le r' > R$. *Assume that*

$$(38)\qquad \left|\frac{\varphi(s)}{\varphi(s_0)}\right| \le M, \text{ if } |s-s_0| = R'.$$

Define

$$a_0 = \frac{1}{2}\left(\sigma_0 - \frac{r}{2}\right),$$

$$c_2 = 2a_0 \log\left(1 + \frac{2a_0}{\sigma_0 - \sigma_k}\right),$$

$$c_3 = 2\int_0^{a_0} \log u du,$$

$$c_4 = \max\{c_2, |c_3 - 2a_0 \log R|\}.$$

Finally, assume that

$$(39)\qquad \frac{4a_0 r'}{R - r'} \log\frac{R}{R' - R} + c_4 \le 0.$$

Then

$$\left|\int_{\frac{r}{2}+it_0}^{\infty+it_0} \log|\varphi(s)|ds\right| \le 2a_0\left(\frac{2r'}{R-r'}\log M + |\log|\varphi(s_0)||\right) + |J(\sigma_0,\infty,t_0)|.$$

Proof. (Compare Lehman's method[16]) We have

$$(40)\qquad \int_{\frac{r}{2}+it_0}^{\infty+it_0} \log|\varphi(s)|ds = U(t_0) + J(\sigma_0,\infty,t_0)$$

with

$$U(t_0) := \int_{\frac{r}{2}+it_0}^{\sigma_0+it_0} \log|\varphi(s)|ds.$$

By assumption $\varphi(s)$ is regular in $|s-s_0| \le R'$, since this circle entirely lies in the upper half plane. Let $\varrho_1, \dots, \varrho_n$ be the zeros of φ in $|s-s_0| \le R < R'$, counted according to their multiplicity. Lehman's Lemma gives

(41)
$$\begin{aligned} U(t_0) &= \int_{\frac{r}{2}+it_0}^{\sigma_0+it_0} \left\{ \log|\varphi(s)| - \log|\varphi(s_0)| - \sum_{k=1}^{n} \log\left|\frac{s-\varrho_k}{s_0-\varrho_k}\right| \right\} ds \\ &+ \left(\sigma_0 + \frac{r}{2}\right) \log|\varphi(s_0)| + \sum_{k=1}^{n} \int_{\frac{r}{2}+it_0}^{\sigma_0+it_0} \log\left|\frac{s-\varrho_k}{s_0-\varrho_k}\right| ds \\ &\le 2a_0 \frac{2r'}{R-r'} \left(\log M + n \log \frac{R}{R'-R}\right) + 2a_0 |\log|\varphi(s_0)|| \\ &+ \sum_{k=1}^{n} \left| \int_{\frac{r}{2}+it_0}^{\sigma_0+it_0} \log\left|\frac{s-\varrho_k}{s_0-\varrho_k}\right| ds \right|. \end{aligned}$$

We need an estimation for

$$I(s_0, \sigma_k) := \int_{\frac{r}{2}+it_0}^{\sigma_0+it_0} \log\left|\frac{s-\varrho_k}{s_0-\varrho_k}\right| ds, 1 \le k \le n.$$

Let $\varrho_k = \beta_k + i\gamma_k$. Then $\sigma_0 - R \le \beta_k \le \sigma_k$, because $\varphi(s) \ne 0$ for $\sigma \ge \sigma_k$. If $s = s_0 + \delta$, we have on the line segment $s = \sigma + it_0, \frac{r}{2} \le \sigma \le \sigma_0$

$$-2a_0 \le \delta \le 0.$$

Since $\log x$ increases for $x > 0$

(42)
$$I(s_0, \varrho_k) = \int_{-2a_0}^{0} \log\left|1 + \frac{\delta}{s_0-\varrho_k}\right| d\delta \le 2a_0 \log\left(1 + \frac{2a_0}{\sigma_0 - \sigma_k}\right) = c_2, 1 \le k \le n.$$

On the other hand $|s_0 - \varrho_k| \le R$, and so

(43)
$$\begin{aligned} I(s_0, \varrho_k) &\ge \int_{\frac{r}{2}}^{\sigma_0} \log \frac{|\sigma + it_0 - \varrho_k|}{R} d\sigma \ge \int_{\frac{r}{2}}^{\sigma_0} \log \frac{|\sigma - \beta_k|}{R} d\sigma \\ &= \int_{\frac{r}{2}}^{\sigma_0} \log|\sigma - \beta_k| d\sigma - 2a_0 \log R. \end{aligned}$$

Now set

$$f(\beta) := \int_{\frac{r}{2}}^{\sigma_0} \log|\sigma - \beta_k| d\sigma = \int_{\frac{r}{2}-\beta_k}^{\sigma_0-\beta_k} \log|u| du.$$

As remarked already, $\sigma - R \leq \beta_k \leq \sigma_k$. If $\beta_k \leq \frac{r}{2}$, then

$$f(\beta_k) = \int_{\frac{r}{2}-\beta_k}^{\sigma_0-\beta_k} \log u du = \int_0^{2a_0} \log\left(y + \frac{r}{2} - \beta_k\right) dy \geq \int_0^{2a_0} \log y dy \geq c_3.$$

Assume $\beta_k > \frac{r}{2}$, so that $\beta_k = \frac{1}{2}\left(\sigma_0 + \frac{r}{2}\right) + \delta$, and $-a_0 < \delta < a_0$. We then have for $\delta > 0$

$$\begin{aligned} f(\beta_k) &= \int_{\frac{r}{2}-\beta_k}^{0} \log|u| du + \int_0^{\sigma_0-\beta_k} \log|u| du = \int_0^{a_0+\delta} \log u du + \int_0^{a_0-\delta} \log u du \\ &= \int_0^{a_0} \log u du + \int_{a_0}^{a_0+\delta} \log u du + \int_0^{a_0} \log u du - \int_{a_0-\delta}^{a_0} \log u du \geq c_3, \end{aligned}$$

and in case $\delta \leq 0$ with $\delta' = -\delta \geq 0$

$$f(\beta_k) = \int_0^{a_0-\delta'} \log u du + \int_0^{a_0+\delta'} \log u du \geq c_3$$

as above. Hence $f(\beta_k) \geq c_3$, $1 \leq k \leq n$, and therefore with (43) and (42)

$$|I(s_0, \varrho_k)| \leq c_4, 1 \leq k \leq n. \tag{44}$$

The conclusion of the Theorem now follows immediately from (41), (44) and the assumption (39).

We remark that the condition (39) can be satisfied always. Since a_0, r', R and c_4 do not depend on R', (39) is always satisfied provided R' is large enough. On the other hand M in (39) is growing with R'. How to choose the parameters optimally depends on φ and its parameters σ_a, σ_k, r etc.

We now give a further estimation for S_1 with a method due to Turing[17, 29], which often gives sharper results. First some preliminary tools are needed.

Lemma 7. *Let* $\varphi \in \mathbf{D}$ *and*

$$b := \lim_{s\to 0}\left[\frac{R'}{R}(s) + \frac{l-n_0}{s} + \frac{m}{s-r}\right]$$

as in (25). Then

$$-\Re(b) = \sum_{\varrho} \Re\frac{1}{\varrho},$$

where ϱ runs over all zeros of R.

The proof is similary to that of Lemma 5.

Lemma 8. *Let $\varphi \in \mathbf{D}, \delta > 0, s \neq \varrho, s+\delta \neq \varrho$ for all zeros ϱ of R. Then we have*

$$\left|\frac{\varphi(s)}{\varphi(s+\delta)}\right| = (n_0 - l)\log\left|\frac{s}{s+\delta}\right| - m\log\left|\frac{s-r}{s-r+\delta}\right| - \delta\log A$$
$$- r_1 \log\left|\frac{\Gamma\left(\frac{s}{2}\right)}{\Gamma\left(\frac{s+\delta}{2}\right)}\right| - r_2\log\left|\frac{\Gamma(s)}{\Gamma(s+\delta)}\right| + \sum_{\varrho}\log\left|\frac{s-\varrho}{s+\delta-\varrho}\right|$$

and the sum over ϱ converges absolutely.

Proof. As usual, we take the principal branch of the logarithms occurring. By (25) we have

$$\begin{aligned} \log|\varphi(s)| &= \Re\log\varphi(s) \\ &= (n_0 - l)\log|s| - m\log|s-r| + \sigma\log A \\ &\quad - r_1\log\left|\Gamma\left(\frac{s}{2}\right)\right| - r_2\log|\Gamma(s)| + \Re(c) + \sigma\Re(b) \\ &\quad + \sum_{\varrho}\left(\Re\frac{s}{\varrho} + \log\left|1 - \frac{s}{\varrho}\right|\right). \end{aligned} \tag{45}$$

Let $T > 0$. Since the sum in (45) converges absolutely, we may write

$$\sum_{\varrho}\left(\Re\frac{s}{\varrho} + \log\left|1-\frac{s}{\varrho}\right|\right) = \sum_{|\varrho|\le T}\left(\Re\frac{s}{\varrho} + \log\left|1-\frac{s}{\varrho}\right|\right) + o(1),\ T \to \infty.$$

Insert this in (45) and subtract the expression obtained by substituting s by $s+\delta$. This leads to

$$\log\left|\frac{\varphi(s)}{\varphi(s+\delta)}\right| = (n_0 - l)\log\left|\frac{s}{s+\delta}\right| - m\log\left|\frac{s-r}{s-r+\delta}\right| - \delta\log A$$
$$- r_1\log\left|\frac{\Gamma\left(\frac{s}{2}\right)}{\Gamma\left(\frac{s+\delta}{2}\right)}\right| - r_2\log\left|\frac{\Gamma(s)}{\Gamma(s+\delta)}\right| - \delta\Re(b)$$
$$- \delta\sum_{|\varrho|\le T}\Re\frac{1}{\varrho} + \sum_{|\varrho|\le T}\log\left|\frac{s-\varrho}{s+\delta-\varrho}\right| + o(1).$$

Letting $T \to \infty$ this gives with Lemma 7 the first claim, and the second follows from

$$\log \left| \frac{s-\varrho}{s+\delta-\varrho} \right| = O(\gamma^{-2}),$$

where $\varrho = \beta + i\gamma$ and $|\gamma| \to \infty$.

Lemma 9. *Let $s \neq -\nu, \nu \in \mathbf{N}_0, s \neq r, s \neq \varrho$ for all ϱ. Then*

$$\sum_{\varrho} \Re \frac{1}{s-\varrho} = \Re \frac{\varphi'}{\varphi}(s) + (l-n_0)\Re \frac{1}{s} + m\Re \frac{1}{s-r} - \log A + \frac{r_1}{2}\Re\psi\left(\frac{s}{2}\right) + r_2 \Re\psi(s).$$

Proof. Follows from (27) and Lemma 7 as in the proof of Lemma 8. Observe, that $\sum_{\varrho}$ is again absolutely convergent.

Theorem 4. *Let $\varphi \in \mathbf{D}$ with $a(1) = 1$. Assume that $\sigma_0 > \max\{\sigma_a, \sigma_k\}, t > 0$ and $\sigma_0 - \frac{r}{2} < \left|\frac{r}{2} + it\right|$. Moreover, assume that there exists c_1, c_2, c_3 with $c_i \geq 0$, such that*

$$|\varphi(s)| \leq c_1 t^{c_2 - \sigma c_3} \tag{46}$$

for all $\sigma \in \left[\frac{r}{2}, \sigma_0\right]$. For $u > 0, v \in \mathbf{C}$ and $|u/v| < 1$ let $B(u,v) := \frac{1}{3}\left(1 - \left|\frac{u}{v}\right|^{-1}\right)$ $\left(\frac{u^2}{2} + u\left|u - \frac{1}{2}\right|\right)$. Define $\delta := \sigma_0 - \frac{r}{2}, z := \frac{r}{2} + it, \varepsilon := \delta|z|^{-1}$ (hence $\varepsilon < 1$) as well as

$$c_1' = \max\{\delta^2|z|^{-1}, -\delta\log(1-\varepsilon)\},$$
$$c_2' = \delta\log(1 - \delta|z-r+\delta|^{-1}),$$
$$c_3' = |l-n_0|\frac{\frac{r}{2}+\delta}{|z+\delta|^2} + m\frac{\left|-\frac{r}{2}+\delta\right|}{|z-r-\delta|^2} - \log A,$$

with $c_2' < 0$ *and* $\delta < |z - r + \delta|$. *Then*

$$a) \int_{\frac{r}{2}+it}^{\infty, t} \log|\varphi(s)|ds \leq \delta \log c_1 + \delta c_2 \log t - \frac{1}{2}\left(\sigma_0^2 - \frac{r^2}{4}\right) c_3 \log t + J(\sigma_0, \infty, t).$$

$$\begin{aligned} b) \int_{\frac{r}{2}+it}^{\infty, t} \log|\varphi(s)|ds \geq & - |n_0 - l|c_1' + mc_2' - \delta^2 \log A \\ & - 1.48\delta^2 c_3' - 1.48\delta^2 \Re\frac{\varphi'}{\varphi}(z+\delta) \\ & + J\left(\frac{r}{2} + \delta, \delta, t\right) + J\left(\frac{r}{2} + \delta, \infty, t\right) \\ & - |z|^{-1}\left[2r_1 AB\left(\frac{\delta}{2}, \frac{z}{2}\right) + r_2 \delta B(\delta, z)\right] \\ & - 1.48\delta^2 |z+\delta|^{-1}(r_1 + r_2) \\ & - \delta^2 r_1 \left[0.74 \log\left|\frac{z+\delta}{2}\right| - \frac{1}{2}\log\left|\frac{z}{2}\right|\right] \\ & - \delta^2 r_2 [1.48 \log|z+\delta| - \log|z|]. \end{aligned}$$

Proof. a)Because log is monotonic, we immediately obtain

$$\begin{aligned} \int_{\frac{r}{2}+it}^{\infty+it} \log|\varphi(s)|ds &= \int_{\frac{r}{2}+it}^{\sigma_0+it} \log|\varphi(s)|ds + J(\sigma_0, \infty, t) \\ &\leq \delta \log c_1 + \log t \int_{\frac{r}{2}}^{\sigma_0} (c_2 - \sigma c_3) d\sigma + J(\sigma_0, \infty, t). \end{aligned}$$

b)First, observe that $a(1) = 1$, hence $\log|\varphi(s)| = O(2^{-\sigma})$ for $\sigma \to \infty$ and therefore the integrals are absolutely convergent. We have

(47)

$$\int_{\frac{r}{2}-it}^{\infty-it} \log|\varphi(s)|ds = \int_{\frac{r}{2}+it}^{\frac{r}{2}-\delta-it} \log\left|\frac{\varphi(s)}{\varphi(s+\delta)}\right| ds + J\left(\frac{r}{2} + \delta, \delta, t\right) + J\left(\frac{r}{2} + \delta, \infty, t\right),$$

and Lemma 8 gives

$$
\begin{aligned}
\int_{\frac{T}{2}+it}^{\frac{T}{2}+\delta+it} \log\left|\frac{\varphi(s)}{\varphi(s+\delta)}\right| ds = (n_0 - l) &\int_{\frac{T}{2}+it}^{\frac{T}{2}+\delta+it} \log\left|\frac{s}{s+\delta}\right| ds \\
-m &\int_{\frac{T}{2}+it}^{\frac{T}{2}+\delta+it} \log\left|\frac{s-r}{s+\delta-r}\right| ds - \delta^2 \log A \\
-r_1 &\int_{\frac{T}{2}+it}^{\frac{T}{2}+\delta+it} \log\left|\frac{\Gamma\left(\frac{s}{2}\right)}{\Gamma\left(\frac{s+\delta}{2}\right)}\right| ds \\
-r_2 &\int_{\frac{T}{2}+it}^{\frac{T}{2}+\delta+it} \log\left|\frac{\Gamma(s)}{\Gamma(s+\delta)}\right| ds \\
+\sum_{\varrho} &\int_{\frac{T}{2}+it}^{\frac{T}{2}+\delta+it} \log\left|\frac{s-\varrho}{s+\delta-\varrho}\right| ds,
\end{aligned}
\tag{48}
$$

where the interchange of summation and integration is permitted by absolute convergence (Lemma 8). We consider the summands in (48) separately. Clearly,

$$
-\int_z^{z+\delta} \log\left|\frac{s}{s+\delta}\right| ds = \int_z^{z+\delta} \log\left|1+\frac{\delta}{s}\right| ds \le \int_z^{z+\delta} \log\left(1+\frac{\delta}{|s|}\right) ds \le \delta^2 |z|^{-1},
$$

which implies

$$
\int_z^{z+\delta} \log\left|\frac{s}{s+\delta}\right| ds \ge -\delta^2 |z|^{-1}.
$$

Moreover we see that

$$
\int_z^{z+\delta} \log\left|1+\frac{\delta}{s}\right| ds \ge \int_z^{z+\delta} \log\left(1-\frac{\delta}{|s|}\right) ds \ge \delta \log(1-\delta|z|^{-1}),
$$

and we conclude

$$
\left|\int_z^{z+\delta} \log\left|\frac{s}{s+\delta}\right| ds\right| \le c_1'. \tag{49}
$$

In a similar fashion we show that

$$
\int_z^{z+\delta} \log\left|\frac{s-r}{s+\delta-r}\right| ds = \int_{z-r}^{z-r+\delta} \log\left|\frac{s}{s+\delta}\right| ds \le -\delta c_2'. \tag{50}
$$

We apply Turing's Lemma to the integrals

$$I_\varrho := \int_z^{z+\delta} \log \left| \frac{s-\varrho}{s+\delta-\varrho} \right| ds = \int_{z-\varrho}^{z+\delta-\varrho} \log \left| \frac{s}{s+\delta} \right| ds.$$

Set $b := z + \delta - \varrho$ in Turing's Lemma. Then $\Re(b) > 0$ and we get

$$I_\varrho \geq -1.48\delta^2 \Re \frac{1}{z+\delta-\varrho},$$

and Lemma 9 together with this estimation gives

$$\begin{aligned}
\sum_\varrho I_\varrho \geq & -1.48\delta^2 \left\{ \Re\frac{\varphi'}{\varphi}(z+\delta) + (l-n_0)\Re\frac{1}{z+\delta} + m\Re\frac{1}{z-r+\delta} - \log A \right.\\
& \left. + \frac{r_1}{2}\Re\psi\left(\frac{z+\delta}{2}\right) + r_2\Re\psi(z+\delta)\right\} \\
\geq & -1.48\delta^2 c_3' - 1.48\Re\frac{\varphi'}{\varphi}(z+\delta) \\
& -1.48\delta^2\left[\frac{r_1}{2}\Re\psi\left(\frac{z+\delta}{2}\right) + r_2\Re\psi(z+\delta)\right].
\end{aligned} \tag{51}$$

Here we have uses the well known estimate ([23], p. 295)

$$\Re\psi(u) = \log|u| + R_u,\ |R_u| \leq |u|^{-1},\ \Re(u) > 0,$$

such that by (51)
(52)

$$\begin{aligned}
\sum_\varrho I_\varrho \geq & -1.48\delta^2 c_3' - 1.48\Re\frac{\varphi'}{\varphi}(z+\delta) - 0.74\delta^2 r_1 \log\left|\frac{z+\delta}{2}\right| - 0.74\delta^2 r_2 \left|\frac{z+\delta}{2}\right|^{-1} \\
& -1.48\delta^2 r_2 \log|z+\delta| - 1.48\delta^2 r_2 |z+\delta|^{-1}
\end{aligned}$$

holds. The gamma function integrals in (48) are simply estimated by the Mean Value Theorem:

$$\begin{aligned}
-r_1\int_z^{z+\delta} \log\left|\frac{\Gamma\left(\frac{s}{2}\right)}{\Gamma\left(\frac{s+\delta}{2}\right)}\right| ds &= r_1 \int_{\frac{r}{2}}^{\frac{r}{2}+\delta} \log\left|\frac{\Gamma\left(\frac{\sigma+\delta+it}{2}\right)}{\Gamma\left(\frac{\sigma+it}{2}\right)}\right| ds \\
&= r_1 \log\left|\frac{\Gamma\left(\frac{\sigma^*+\delta+it}{2}\right)}{\Gamma\left(\frac{\sigma^*+it}{2}\right)}\right|, \frac{r}{2} \leq \sigma^* \leq \frac{r}{2}+\delta, \\
&\geq \delta^2\frac{r_1}{2}\log\left|\frac{z}{2}\right| - r_1\delta B\left(\frac{\delta}{2},\frac{z}{2}\right)\left|\frac{z}{2}\right|^{-1},
\end{aligned} \tag{53}$$

by appling Stirling's formula ([23], p.297). In the same manner

$$(54)\qquad -r_2 \int_z^{z+\delta} \log \left| \frac{\Gamma(s)}{\Gamma(s+\delta)} \right| ds \geq \delta^2 r_2 \log |z| - \delta r_2 B(\delta, z) |z|^{-1}.$$

(53), (54), (52) and (49), (50) now are inserted in (48) and this proves part b).

It is clear that Theorems 3 and 4 imply that

$$(55)\qquad S_1(T) = O(\log T),\ T \to \infty,$$

provided, $\varphi(s)$ grows only polynomial as $t \to \infty$. How to satisfy the conditions (38) and (46) has been demonstrated in [9]. As already mentioned, an explicit version of (55) is fundamental to Turing's method, which we now apply to the zeros of φ. We shall give two variants, to allow some flexibility in the applications.

Assume again $T > 0, Z((0) \neq 0 \neq Z(T)$, so that $N(T)$ is a natural number (observe that zeros not lying on the critical line occues in pairs). Assume moreover that in the interval $0 < t < T$ the zeros $0 < \gamma_1 < \ldots < \gamma_k < T$ have been found, each with multiplicity≥ 1. If $N(T) = k$, we have found all zeros and Riemann's hypothesis is true for φ in the interval $[0,T]$. Now assume $N(T) = k + \Delta$ for some $\Delta \in \mathbf{N}_0$. Choose $T_1 > T$ and find zeros $\gamma_{k+1} < \ldots < \gamma_{k+p}$ of Z in the interval $[T, T_1]$. Then, obviously,

$$(56)\qquad N(\gamma_{k+j} + 0) \geq N(T) + j = k + j + \Delta,\ 1 \leq j \leq p.$$

By Theorem 2 we have (with $\gamma = e^{i\delta}$ as in (16))

$$S(t) = N(t) - \frac{1}{\pi}\vartheta(t) - \frac{\delta}{2\pi} - \frac{1}{2}(l+m) + S(0),$$

hence

$$(57)\qquad \int_T^{T_1} S(t)\,dt = \int_T^{T_1} N(t)\,dt - \frac{1}{\pi}\int_T^{T_1} \vartheta(t)\,dt - \left[\frac{\delta}{2\pi} + \frac{1}{2}(l+m) - S(0)\right](T_1 - T).$$

Here we have

$$(58)\qquad \begin{aligned} \int_T^{T_1} N(t)\,dt &= \int_T^{\gamma_{k+1}} N(t)\,dt + \sum_{j=1}^{p-1} \int_{\gamma_{k+1}}^{\gamma_{k+j+1}} N(t)\,dt + \int_{\gamma_{k+p}}^{T_1} N(t)\,dt \\ &\geq (\gamma_{k+1} - T)(k+\Delta) + \sum_{j=1}^{p-1} (\gamma_{k+j+1} - \gamma_{k+j})(k+j+\Delta) \\ &\quad + (T_1 - \gamma_{k-p})(k+p+\Delta) \end{aligned}$$

by (56). The sum over j equals

$$\sum_{j=1}^{p-1}(\gamma_{k+j+1}-\gamma_{k+j})(k+j+\Delta) = (\gamma_{k+p}-\gamma_{k+1})(k+\Delta)+p\gamma_{k+p}-\sum_{j=1}^{p}\gamma_{k+j}$$

and inserted in (58) gives

$$\int_T^{T_1} N(t)dt \geq (k+\Delta)(T_1-T)+pT_1-\sum_{j=1}^{p}\gamma_{k+j}.$$

Next, we have in (57) by using (16)

$$\frac{1}{\pi}\int_T^{T_1}\vartheta(t)\,dt = -\frac{\delta}{2\pi}(T_1-T)-\frac{\log A}{2\pi}(T_1^2-T^2)+\frac{r_1}{\pi}\int_T^{T_1}\Im\log\Gamma\left(\frac{r}{4}+i\frac{t}{2}\right)dt$$
$$+\frac{r_2}{\pi}\int_T^{T_1}\Im\log\Gamma\left(\frac{r}{2}+it\right)dt$$

such that

$$\int_T^{T_1}S(t)\,dt = \int_T^{T_1}N(t)\,dt+\frac{\log A}{2\pi}(T_1^2-T^2)-\frac{r_1}{\pi}\int_T^{T_1}\Im\log\Gamma\left(\frac{r}{4}+i\frac{t}{2}\right)dt$$
$$+\frac{r_2}{\pi}\int_T^{T_1}\Im\log\Gamma\left(\frac{r}{2}+it\right)dt-\left[\frac{1}{2}(l+m)-S(0)\right](T_1-T)$$
$$=\int_T^{T_1}N(t)\,dt+\frac{\log A}{2\pi}(T_1^2-T^2)-\frac{2r_1}{\pi}H\left(\frac{T}{2},\frac{T_1}{2},\frac{r}{4}\right)$$
$$-\frac{r_2}{\pi}H\left(T,T_1,\frac{r}{2}\right)-\left[\frac{l+m}{2}-S(0)\right](T_1-T)$$

where

$$H(a.b,\sigma)=\int_a^b\Im\log\Gamma(\sigma+it)\,dt,\ 0<a\leq b,\sigma>0. \tag{59}$$

In summary, we have the first version of Turing's method:

Theorem 5. *Let* $T > 0, Z(0) \neq 0 \neq Z(T), T_1 > T$. *Assume that in* $[0,T]$ *we found* k *zeros of* Z *and in* $[T,T_1]$ *the zeros* $\gamma_{k+1} < \ldots < \gamma_{k+p}, p \geq 1$. *If* $N(T) = k + \Delta$ *with* $\Delta \in \mathbf{N}_0$, *then*

$$\int_T^{T_1} S(t)dt \geq k(T_1 - T) + pT_1 - \sum_{j=1}^{p} \gamma_{k+j} + \frac{\log A}{2\pi}(T_1^2 - T^2) - \left[\frac{1}{2}(l+m) - S(0)\right]$$
$$- \frac{2r_1}{\pi} H\left(\frac{T}{2}, \frac{T_1}{2}, \frac{r}{4}\right) - \frac{r_2}{\pi} H\left(T, T_1, \frac{r}{2}\right) + \Delta(T_1 - T),$$

where H *is defined in (59).*

If the Riemann hypothesis for φ in $[0,T]$ is correct, then the assumption $\Delta \geq 1$ in Theorem 5 will quickly lead to a contradiction with Theorem 3 or Theorem 4. See [9] for a numerical example. If the assumption $\Delta \geq 1$ does not lead to a contradiction, then we can say nothing: The Riemann hypoyhesis could either be false or the mutiplicity of a zero was greater than 1. Since there is a considerable freedom in the choice of the parameters (e.g. T_1) the method will in most cases achieve its goal.

The computation of H in (59) is of course easy, using, for example, Stirling's formula. In general, the zeros of γ_{k+j} in $[T,T_1]$ can not be calculated exactly but only some approximations $\tilde{\gamma}_{k+j}$ with

$$|\tilde{\gamma}_{k+j} - \gamma_{k+j}| \leq \varepsilon,\ 1 \leq j \leq p,\ \varepsilon \geq 0.$$

In this case we simply replace

$$\sum_{j=1}^{p} \gamma_{k+j} \text{ by } \sum_{j=1}^{p} \tilde{\gamma}_{k+j} + \varepsilon p$$

in Theorem 5. We now give the second version of Turing's method.

Theorem 6. *Let* $k, n \in \mathbf{N}, I_j := (t_{n+j}, t_{n+j+1}), 0 \leq j < k$, *be the* $(n+j)th$ *Gram interval,* $Z(0) \neq 0 \neq Z(T_n)$. *Assume that in* $[0, t_n]$ *we have found* N' *and in* I_j *we have found* n_j *zeros of* Z. *If we have* $N(t_n) = N' + \Delta$ *for some* $\Delta \in \mathbf{N}_0$, *then*

$$\int_{t_n}^{t_{n+k}} S(t)dt \geq (N' + \Delta - n - c - 1)(t_{n+k} - t_n) + \sum_{j=1}^{k-1}(t_{n+j-1} - t_{n+j})(s_j - j)$$

with

$$c := \frac{\delta}{2\pi} + \frac{1}{2}(l+m) - S(0),\ s_j := \sum_{\nu=0}^{j-1} n_\nu.$$

Proof. We have

$$N(t_{n+j}) \geq N(t_n) + \sum_{\nu=0}^{j-1} n_\nu = N' + \Delta + \sum_{\nu-0}^{j-1} n_\nu, 0 \leq j < k.$$

Inserted in

$$S(t) = N(t) - \frac{1}{\pi}\vartheta(t) - c \tag{60}$$

this leads to

$$S(t_{n+j}) = N(t_{n+j}) - n - j - c \geq N' + \Delta + \sum_{\nu=0}^{j-1} n_\nu - n - j - c, 0 \leq j < k. \tag{61}$$

From (60) it follows moreover that $S(t)$ can decrease at most one unit in I_j, i.e. for some $t \in I_j$ we have because of (61)

$$S(t) \geq N' + \Delta + \sum_{\nu=0}^{j-1} n_\nu - n - j - c - 1$$

and so

$$\int_{t_n}^{t_{n+k}} S(t)dt = \sum_{j=0}^{k-1} \int_{I_j} S(t)dt \geq \sum_{j=0}^{k-1}(t_{n+j+1} - t_{n+j})(N' + \Delta + \sum_{\nu=0}^{j-1} n_\nu - j - n - c - 1)$$

which is our Theorem.

Finally we remarks that we may even assume that $\Delta \equiv 0(2)$ because zeros off the critical line always occur in pairs. The same holds for zeros between two sign changes of Z.

REFERENCES

[1] R. BACKLUND, *Sur les Zéros de la Fonction $\zeta(s)$ de Riemann*, C.R. Acad. Sci. Paris **158**, (1914), 1979-1982.

[2] B.C. BERNDT, *Identities Involving a Class of Dirichlet Series*, Trans. AMS **137**, (1969), 345-359.

[3] H. DAVENPORT, *Multiplicative Number Theory*, Springer, Berlin 1980.

[4] D. DAVIES, *The Computation of the Zeros of Hecke Zeta-Functions in the Gaussian Field*, Proc. Roy. Soc. London(A) **264**, (1961), 496-502.

[5] D. DAVIES, *An Approximate Functional Equation for Dirichlet L-Functions*, Proc. Roy. Soc. London(A) **284**, (1965), 224-236.

[6] D. DAVIES, C.B. HASELGROVE, *The Evaluation of Dirichlet L-Functions*, Proc. Roy. Soc. London(A) **264**, (1961), 122-132.

[7] H.R.P. FERGUSON, R.D. MAJOR, K.E. POWELL, H.G. THROOLIN, *On Zeros of Mellin Transforms of $SL_2(\mathbf{Z})$ Cusp Forms*, Math. Comp. **42**, (1984), 241-255.

[8] J.P. GRAM, *Note sur les Zéros de la Fonction $\zeta(s)$ de Riemann*, Acta Math. **27**, (1903), 289-304.

[9] A. GUTHMANN, *Die Berechnung Dirichletscher Reihen, die Funktionalgleichungen mit einfachem Gammafaktor erfüllen im kritischen Streifen*, Preprint, Univ. Kaiserslautern 1990.

[10] A. GUTHMANN, *Asymptotische Entwicklungen für unvollständige Gammafunktionen*, Forum Math. **2**, (1990), 1-37.

[11] A. GUTHMANN, *Ramanujans Tau-Funktion*, Preprint, Univ. Kaiserslautern 1988.

[12] J.C. LAGARIAS, A.M. ODLYZKO, *On computing Artin L-Functions in the Critical Strip*, Math. Comp. **33**, (1975), 1081-1095.

[13] A.F. LAVRIK, *An Approximate Functional Equation for the Dirichlet L-Function*, Trans. Moscow Math. Soc. **18**, (1968), 101-115.

[14] A.F. LAVRIK, *An Approximate Functional Equation for the Dirichlet Functions*, Math. USSR-Izvestija **2**, (1968), 129-179.

[15] A.F. LAVRIK, *A Functional Equation for Dirichlet L-Series and the Problem of Divisors in arithmetic Progressions*, AMS Transl.(2) **82**, (1969), 47-65.

[16] R.S. LEHMAN, *Separation of Zeros of the Riemann Zeta-Function*, Math. Comp. **20**, (1966), 523-541.

[17] R.S. LEHMAN, *On the Distribution of the Zeros of the Riemann Zeta-Function*, Proc. London Math. Soc.(3) **20**, (1970), 303-320.

[18] J.E. LITTLEWOOD, *On the Zeros of the Riemann Zeta Function*, Proc. Camb. Philos. Soc. **22**, (1924), 295-318.

[19] J. VAN DE LUNE, H.J.J. TE RIELE, *On the Zeros of the Riemann Zeta function in the Critical Strip III*, Math. Comp. **41**, (1983), 759-767.

[20] J. VAN DE LUNE, H.J.J. TE RIELE, *On the Zeros of the Riemann Zeta Function in the Critical Strip IV*, Math. Comp. **46**, (1986), 667-681.

[21] A.M. ODLYZKO, *On the Distribution of Spacings Between Zeros of the Zeta Function*, Math. Comp. **48**, (1987), 273-308.

[22] A.M. ODLYZKO, H.J.J. TE RIELE, *Disproof of Mertens' Conjecture*, Crelles Journal **357**, (1985), 138-160.

[23] F.W.J. OLVER, *Asymptotics and Special Functions*, Academic Press, New York 1974.

[24] J. PINTZ, *On the Sign Changes of $M(x) = \sum_{n\leq x}\mu(n)$*, Analysis **1**, (1981), 191-195.

[25] H.J.J. TE RIELE, *On the Sign of the Difference $\pi(x) - li(x)$*, Math. Comp. **48**, (1987), 323-328.

[26] R. SPIRA, *Calculation of the Ramanujan τDirichlet Series*, Math. Comp. **27**, (1973), 379-385.

[27] R. SPIRA, *Calculation of Dirichlet L-Functions*, Math. Comp. **23**, (1969), 489-497.

[28] E.C. TITCHMARSH, *The Theory of the Riemann Zeta-function*, Oxford 1951, Sec. Ed. revised by D.R. Heath-Brown, Oxford 1986. Pages refer to the second edition.

[29] A.M. TURING, *Some Calculations of the Riemann Zeta-Function*, Proc. London Math. Soc.(3) **3** (1953), 99-117.

Received March 6, 1991
Andreas Guthmann
Fachbereich Mathematik
Universität Kaiserslautern
Pfaffenbergstr. 95
D-6750 Kaiserslautern
Germany