REMARKS ON SOME BASIC PROPERTIES OF TSIRELSON'S SPACE
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Abstract. This note presents a new approach to some known, but difficult to prove, results: it is shown how all the basic properties, and some other less well-known, of Tsirelson space and its dual follow from an inequality stated in Tsirelson's original paper [9].

1. BACKGROUND

We base our approach to the properties of Tsirelson's space on the use of weakly-$p$-summable sequences. Throughout the paper $p^*$ denotes the conjugate number of $p$; if $p = 1$, $l_{\infty}$ plays the role of $c_0$.

Preliminary Definitions. A sequence $(x_n)$ in a Banach space $X$ is said to be weakly-$p$-summable ($p \geq 1$) if there is a $C > 0$ such that

$$\sup_n \left\| \sum_{k=1}^{n} \xi_k x_k \right\| \leq C \cdot \left\| (\xi_n) \right\|_{l^p},$$

for any $(\xi_n) \in l^p$.

It is said to be $p$-Banach-Saks, $1 < p < +\infty$, if

$$\left\| \sum_{k=1}^{n} x_k \right\| \leq C \cdot n^{1/p}$$

for some constant $C > 0$ and all $n \in \mathbb{N}$.

We shall say that the sequence $(x_n)$ is weakly-$p$-convergent (resp. $p$-Banach-Saks convergent) to $x \in X$ if the sequence $(x_n - x)$ is weakly-$p$-summable (resp. $p$-Banach-Saks).

These sequences allow us now to introduce two special classes of reflexive spaces, $W_p$ and $p$-Banach-Saks.

Definition. Let $1 \leq p < +\infty$. A subset $K$ of a Banach space $X$ is said to be relatively weakly-$p$-compact if any sequence in $K$ admits a weakly-$p$-convergent sub sequence. We say that $X \in W_p$ if its closed unit ball is weakly-$p$-compact.

Example of spaces in $W_p$ are: $l_p \in W_{p^*}, 1 < p < +\infty$; and $L_p \in W_r$ where $r = \max\{p^*, 2\}, 1 < p < +\infty$. In general, James' characterization of super-reflexivity [4], combined with the Bessaga-Pelczynski selection principle, implies:

Proposition 1 [1]. Let $X$ be an infinite-dimensional super reflexive Banach space. Then there are numbers $p > q > 1$ such that $X \in W_p$ but $X \not\in W_q$.

Definition [5]. Let $1 < p < +\infty$. A Banach space is said to have the $p$-Banach-Saks property when each bounded sequence $(x_m)$ admits a sub-sequence $(x_n)$ and a point $x$ such that $(x_n - x)$ is a $p$-Banach-Saks sequence.

One of the main results of [1] may now be stated:

Proposition 2. Let $1 < p < +\infty$. Every $X \in W_p$ has the $p^*$-Banach-Saks property. Every Banach space $X$ with the $p$-Banach-Saks property belongs to $W_r$ for all $r > p^*$.

Many arguments in what follows will be simplified by the introduction of another class of Banach spaces, in some sense the dual of $W_p$: we say that a Banach space $X \in C_p$ if weakly-$p$-summable sequences are norm null. Two simple examples are: $l_p \in C_r$ for all $r < p^*, 1 \leq p < +\infty$; and $L_p \in C_r$, for $r < \min\{p^*, 2\}, 1 \leq p < +\infty$. Another example results from the application of Orlicz' theorem: that spaces of cotype $s$ belongs to $C_r$ for all $r < s^*$.

It is clear that an infinite-dimensional Banach space cannot simultaneously belong to $C_p$ and $W_p$. It is also clear that subspaces of spaces in $C_p$ (resp. $W_p$) themselves belong to $C_p$ (resp. $W_p$). Quotients of spaces in $W_p$ belong to $W_p$ (obviously false for $C_p$).

2. PROPERTIES OF $T$ AND $T^*$

We shall now apply the preceding ideas to obtain most of the basic properties of Tsirelson's space $T^*$. This is a reflexive Banach sequence space which does not contain a copy of $l_p$ for $1 \leq p < +\infty$, or $c_0$. The following is Tsirelson's original definition:

Let $K$ be a weakly compact set of $c_0$ such that

1) $K$ is contained in the unit ball of $c_0$;

2) for any sequence $x \in K$, if $y$ is another sequence such that $|y(n)| \leq |x(n)|$ for all $n \in \mathbb{N}$, then $y \in K$;

3) given $x_1, \ldots, x_N \in K$ such that if $x_n(i) \neq 0$ then $x_m(j) = 0$ for all $m > n$ and $j \leq i$, then the element $y$, defined as $y(n) = (x_1(n) + \ldots + x_N(n))/2$ for $n \geq N$ and 0 otherwise, belongs to $K$;

4) given $x \in K$, then there is a $k \in \mathbb{N}$ such that the element $y$, defined as $y(n) = 2x(n)$ for $n \geq k$ and 0 otherwise, belongs to $K$.

Then the space $T^*$ is defined as the span of the absolutely convex closed hull of $K$ in $c_0$ with the norm having $K$ as the unit ball.

The first result we want to show is the following:
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Proposition 3. \( T^* \in W_p \) for all \( p > 1 \).

Current method of proof: Reference ([3], p.52) provides a sketch of a proof of this result that is quite difficult. In essence is as follows.

First of all, one needs a profound theorem of Krivine [6] asserting that if \( X \) satisfies a lower-\( p \)-estimate then, for all \( n \), \( X \) contains \( n \) disjointly supported vectors equivalent to the canonical basis of \( l_p^n \). Then one passes to \( T \), and sees that disjointly supported vectors in \( T \) must be equivalent to the unit vector basis of \( l_p^n \) (see[3], Prop.V.8). One next shows that in \( T \) (and therefore in \( T^* \)) the canonical basis dominates, and is dominated by, its blocks (see[3], Chapter II).

The proof itself now follows: If, for some \( p \), \( T \) does not admit a lower-\( p \)-estimate then the number \( \inf \{ q : T \text{ satisfies lower-}q\text{-estimates} \} \) is greater than 1, and thus \( T \) admits disjointly supported vectors equivalent to the unit vector basis of \( l_p^n \) for all \( n \) uniformly, which is a contradiction. So we have that \( T \) admits lower-\( p \)-estimates for all \( p \). It is a standard duality argument that in that case \( T^* \) admits upper-\( p \)-estimates for all \( p \). Finally, given a weakly null sequence in \( T^* \), if it is norm null then there is nothing to prove; if not, we apply the Bessaga-Pelczynski selection principle to obtain a basic sub-sequence equivalent to certain blocks of the unit vector basis of \( T^* \). These blocks satisfy an upper-\( p \)-estimate for all \( p \) since they are dominated by the basis, and thus a sub-sequence of our original sequence is weakly-\( p \)-summable.

Simpler proof: Now our approach: It is easy to see that \( T^* \) has, for all \( p \), the \( p \)-Banach-Saks property. This follows from the inequality given by Tsirelson ([9], p.140):

\[
||\lambda_1 x_{N+1}, \ldots, \lambda_N x_{2N}|| \leq 2 \cdot \max_{1 \leq i \leq N} |\lambda_i|
\]

which implies that

\[
||x_1 + \ldots + x_N|| \leq K \cdot \log N
\]

is valid for normalized blocks \( \{ x_i \} \) of the canonical basis. This and the use of the Bessaga-Pelczynski selection principle in the form indicated above prove our assertion.

Proposition 2 then implies that \( T^* \) is of the class \( W_p \) for all \( p > 1 \).

The next result was the motive for the construction of Tsirelson's space:

Proposition 4. No subspace or quotient of \( T \) or \( T^* \) is isomorphic to \( l_p \) for \( 1 \leq p < +\infty \), or \( c_0 \).

Proof: Pitt's theorem states that all operators of \( \mathcal{L}(l_p, l_q) \) are compact for \( q > p \). Therefore \( l_p \in C_r \) for \( r < p^* \) (in fact the two statements are equivalent), and thus \( l_p \) cannot be a
subspace or quotient of a space in $W_p$ for all $p > 1$. In conclusion, our assertion is true for $T^*$. A duality argument gives the same result for $T$.

A different approach to proving the assertion for $T$ is to recognize that $X \in W_p$ implies $X^* \in C_r$ for $r < p^*$ (see [2] for details). Since $T^* \in W_p$ for all $p > 1, T \in C_r$ for all $r$, and thus $l_p$ cannot be a subspace or quotient of such a space.

This last paragraph can also be expressed in the following proposition.

**Proposition 5.** For any $p > 1$, $\mathcal{L}(L_p, T) = \mathcal{K}(L_p, T)$. For any $p$ there is an operator $l_p \to T^*$ which is not compact.

**Proof.** $L_p$ spaces belong to some $W_r$, and $T \in C_p$ for all $p$. On the other hand, $T^*$ does not belong to $C_r$ for any $r > 1$. One can easily see that a Banach space $X \in C_p$ if and only if all operators of $\mathcal{L}(l_p, X)$ are compact.

**Remark.** Straueuli [8] has proved that any Banach-Saks operator with values in $T$ is compact. This proves the first assertion of Proposition 5.

**Proposition 6.** $T$ and $T^*$ admit no non-trivial type infinite-dimensional subspaces or quotients, and in particular, no super-reflexive subspaces or quotients.

**Proof.** Consider first $T^*$. Recall that spaces of cotype $s$ belong to $C_r$ for all $r < s^*$. This shows that no finite cotype subspaces or quotients of $T^*$ are allowed. The assertion for non-trivial type and super-reflexive spaces follows from this (but is also a consequence of the fact that super-reflexive spaces do not belong to some of the classes $W_q$).

The assertions for $T$ are established by duality.

**Remark.** The special feature of $T^*$, that it belongs to $W_p$ for all $p > 1$, determines all of its properties (and those of its dual), in particular that of not containing $l_p$. The latter not determine the former, however. Let us recall here the existence of a 2-convexified Tsirelson space $T_2$ (see [3]): this space is uniformly convex (and thus super-reflexive) and does not contain any copy of $l_p$ for $1 \leq p < +\infty$, or $c_0$. Since $T_2$ and $T_2^*$ are super-reflexive, neither of them can belong to every $W_p$.

Nevertheless, it is possible to clarify somewhat the structure of $T_2$, in terms of $C_p$ and $W_p$. Since $T_2$ is of type 2 and cotype $q > 2$, it must belong to $W_2$ and to $C_r$ for all $r < 2$. This is the best that can be achieved for $r$. Since $T_2^*$ is of type $p < 2$ for all $p$ and of cotype 2, it must belong to $C_r$ for all $r < 2$ and to $W_r$ for all $r > 2$. The question is whether $T_2^*$ belongs to $W_2$ or to $C_2$.

Assume that it does not belong to $C_2$. Then the criterion, developed in [2], for detecting copies of $l_p$ comes into play:
If $X \not\in C_p$, and $X^* \in W_p$ then $X$ contains a copy of $l_{p^*}$.

Thus $T_2^*$ should contain $l_2$, a contradiction.

If we recall that its dual space $T_2$ has, with respect to $C_p$ and $W_p$, exactly the same behaviour as $l_2$, that is, $T_2 \in C_p$ for $p < 2$ and $T_2 \in W_2$, then the behaviour of $T_2^*$ may be somewhat surprising: $T_2^* \in C_2$ and $T_2^* \in W_p$ for $p > 2$.

The reason however is precisely that $T_2$ cannot contain $l_2$.

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