DUALS OF INDUCTIVE AND PROJECTIVE LIMITS OF MOSCATELLI TYPE

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Abstract. In this note we shall provide duality results between the general inductive and projective limits of Moscatelli type. These extend the corresponding duality results in the case of Fréchet and LB-spaces due to J. Bonet and S. Dierolf.

1. PRELIMINARIES AND DEFINITIONS

From now on, \((L, ||||)\) will denote a normal Banach sequence space i.e. a Banach sequence space satisfying:

\((\alpha)\) \(\varphi \subset L \subset \omega\) algebraically and the inclusion \((L, ||||) \rightarrow \omega\) is continuous (here \(\omega\) and \(\varphi\) stand for \(K^N = \prod_{k \in \mathbb{N}} K\) and \(\bigoplus_{k \in \mathbb{N}} K\) respectively).

\((\beta)\) \(\forall a = (a_k) \in L, \forall b = (b_k) \in \omega\) with \(|b_k| \leq |a_k| (k \in N)\), we have \(b \in L\) and \(||b|| \leq ||a||\).

Clearly every projection onto the first \(n\) coordinates \(p_n : \omega \rightarrow \omega, (a_k)_{k \in \mathbb{N}} \rightarrow ((a_k)_{k \leq n}, (0)_{k > n})\) induces a norm-decreasing endomorphism on \(L\). Other properties on \((L, ||||)\) we may require are the following:

\((\gamma)\) \(||a|| = \lim ||p_n(a)|| (n \rightarrow \infty), \forall a \in L\).

\((\varepsilon)\) \(\lim ||a - p_n(a)|| = 0 (n \rightarrow \infty), \forall a \in L\) (i.e. \(\varphi\) is dense in \((L, ||||)\)).

\((\delta)\) If \(a \in \omega\), and \(\sup_{n \in \mathbb{N}} ||p_n(a)|| < \infty\), then \(a \in L\) and \(||a|| = \lim ||p_n(a)|| (n \rightarrow \infty)\).

Unexplained terminology as in [6, 7, 10].

Let \((L, ||||)\) be a normal Banach sequence space, let \(Y\) and \(X\) be locally convex spaces and \(f : Y \rightarrow X\) a continuous linear mapping. For every \(n \in \mathbb{N}\), we define \(F_n := \prod_{k < n} \times L((X)_{k \geq n})\) provided with the topology of such a finite topological product. Note that this topology is generated by the semi norms: \((x_k)_{k \in \mathbb{N}} \in F_n \rightarrow ||(q(x_k))_{k < n}, (p(x_k))_{k \geq n})||\) with \(q \subset cs(Y), p \subset cs(X)\).

For every \(n \in \mathbb{N}\), we also define the mapping: \(g_n : F_{n-1} \rightarrow F_n, (x_k)_{k \in \mathbb{N}} \rightarrow ((x_k)_{k < n}, f(x_n), (x_k)_{k > n})\). Clearly \(g_n (n \in \mathbb{N})\) is a continuous linear mapping and \(F_n (n \in \mathbb{N})\) is Hausdorff (resp. complete, metrizable, normable) if and only if \(X\) and \(Y\) have the same property. Moreover the spaces \(X\) and \(Y\) are complemented subspaces of \(F_n\) for \(n \geq 1\) and \(n \geq 2\) respectively.

We define the projective limit \(F^*\) of Moscatelli type w.r.t. (with respect to) \((L, ||||), Y, X\) and \(f : Y \rightarrow X\) by \(F = \text{proj}_{n \in \mathbb{N}} (F_n, g_n)\).
Observe that $F$ is Hausdorff, (resp. complete, metrizable) if and only if $Y$ and $X$ have the same property.

We shall first present another description of the space $F$ similar to the one given in [3] for the case of Banach spaces $X$ and $Y$.

1.1 Proposition. Let $(L, ||||)$ be a normal Banach sequence space, let $Y$ and $X$ be locally convex spaces and $f : Y \to X$ a continuous linear mapping. The corresponding projective limit of Moscatelli type $F$ coincides algebraically with $\{ y = (y_k)_{k \in \mathbb{N}} \in \prod_{k \in \mathbb{N}} Y : (f(y_k))_{k \in \mathbb{N}} \in L(X) \}$ and $F$ has the initial topology w.r.t. the inclusion $j : F \to \prod_{k \in \mathbb{N}} Y$ and the linear mapping $f : F \to L(X), (y_k)_{k \in \mathbb{N}} \to (f(y_k))_{k \in \mathbb{N}}$.

Proof. Denote by $H$ the space $\{ (y_k)_{k \in \mathbb{N}} \in \prod_{k \in \mathbb{N}} Y : (f(y_k))_{k \in \mathbb{N}} \in L(X) \}$ carrying the initial topology w.r.t. $j$ and $f$ and define $\gamma : F \to H, (z^k)_{k \in \mathbb{N}} \to (z^{k+1})_{k \in \mathbb{N}}$. It is easy to show that $\gamma$ is linear, bijective, continuous and open.

Note that we obtain the same space $F$ if we take the closure of $f(Y)$ in $X$ instead of $X$. Therefore we may assume without loss of generality that $f$ has dense range.

2. BOUNDED SETS

Before dealing with the duality between inductive and projective limits of Moscatelli type we need some preparation. Our first two results are essentially well-known (see[10] for $L = l_1$ or [12])

If $Z$ is a locally convex space, $b(Z)$ will denote the family of all the bounded sets in $Z$ and we shall always consider them closed and absolutely convex. Besides $Z'_b := [Z, \beta(Z', Z)]$ will stand for the strong dual of $Z$.

2.1 Lemma. Let $(L, ||||)$ be a normal Banach sequence space and let $Z$ be a metrizable space. Then for every $B \in b(L(Z))$, there exist $B \in b(Z)$ such that $B \in b(L(Z_B))$, i.e., $(p_B(x_k))_{k \in \mathbb{N}} \in L$ for every $x = (x_k)_{k \in \mathbb{N}} \in B$ and $\sup_{x \in B} ||(p_B(x_k))_{k \in \mathbb{N}}|| < +\infty$.

We refer to [6] for df-spaces and properties of this class of locally convex spaces. Every DF-space of Grothendieck (in particular strong duals of Fréchet spaces) and LB-spaces belong to this class.

2.2 Lemma. Let $(L, ||||)$ satisfy $(\delta)$ and let $Z$ be a df-space. Then for every $B \in b(L(Z))$, there is $B \in b(Z)$ such that $B \in b(L(Z_B))$
The condition that \((L, ||\cdot||)\) satisfies \((\delta)\) is needed in lemma 2.2. In fact if \(Z\) is any locally convex space and \((L, ||\cdot||) = (c_0, ||\cdot||_\infty)\) the fact that for every \(B \in b(c_0(Z))\), there is \(B \in b(Z)\) such that \(B \in b(c_0(Z_B))\) implies that \(Z\) has the Mackey convergence condition (see e.g. [10]). There are locally convex spaces \(Z\) which do not satisfy the Mackey convergence condition.

We would like to recall-as it was done in [3]-that whenever \((L, ||\cdot||)\) is a normal Banach sequence space satisfying property \((\varepsilon)\), its dual space \((L', ||\cdot||')\) coincides with the \(\alpha\)-dual \(L^*\) and \((L'||\cdot||')\) has properties \((\beta)\) and \((\delta)\).

2.3 Proposition. Let \((L, ||\cdot||)\) be a normal Banach sequence space which fulfills property \((\varepsilon)\) and let \(Z\) be a locally convex space such that

i) For every \(B \in b(L(Z))\), there is \(B \in b(Z)\) with \(B \in b(L(Z_B))\).

ii) For every \(u \in L'(Z'_B)\), there is an absolutely convex equicontinuous set \(M \subset Z'\) with \(u \in L'(Z'_M)\).

Then \(L(Z)_b\) is canonically algebraically and topologically isomorphic to \(L'(Z'_b)\).

Remark. In particular i) and ii) are satisfied if either \(L\) satisfies \((\delta)\) and \(Z\) is a quasi-barrelled DF-space or \(L\) satisfies \((\varepsilon)\) and \(Z\) is metrizable.

Proof. For every \(k \in \mathbb{N}\), we denote \(j_k : Z \to L(Z), x \to (\delta_k x)_{j \in \mathbb{N}}\). Now we define \(\varphi : L(Z)_b \to L'(Z'_b), v \to (v \circ j_k)_{k \in \mathbb{N}}\)

1. \(\varphi\) is well defined. \((\varphi)\) is clearly linear. Fix \(u \in L(Z)_b\). There must be \(U \in H_\sigma(Z)\) such that \(||(u, x)|| \leq ||(P_U(x_k))_{k \in \mathbb{N}}||, \forall x \in L(Z)\). Take any \(B \in b(Z)\). We denote \(p_{B^*}(u) := \sup_{z \in B} ||(u, z)||, (u \in Z')\). We must show that \((p_{B^*}(u \circ j_k))_{k \in \mathbb{N}} \in L^*\), that is, \(\sum_{k \in \mathbb{N}} p_{B^*}(u \circ j_k) < \alpha_k < +\infty\), for all \(\alpha \in L\) with \(\alpha_k > 0(k \in \mathbb{N})\). Fix \(n \in \mathbb{N}\). For every \(z_1, \ldots, z_n\) belonging to \(B\) we may write:

\[
\sum_{k=1}^{n} | u \circ j_k, z_k | | \alpha_k | \leq \sum_{k=1}^{n} | u \circ j_k, \alpha_k z_k |
\]

\(= (\text{for suitable } \beta_k \in K, | \beta_k | = 1(k \leq n)) =
\]

\[= \sum_{k=1}^{n} | u \circ j_k, \beta_k \alpha_k z_k | = u \left( \sum_{k=1}^{n} j_k (\beta_k \alpha_k z_k) \right) \leq ||(p_u(\beta_k \alpha_k z_k))_{k \leq n},
\]

\((0)_{k>n}|| \leq \mu ||\alpha||.

where \(\mu > 0\) is such that \(p_u(y) \leq \mu\) for all \(y \in B\). Consequently \(\sum_{k=1}^{n} p_{B^*}(u \circ j_k) \alpha_k \leq ||\alpha|| \mu\). Since \(n\) is arbitrary, the proof of 1. is complete.
2. \( \varphi \) is continuous. Fix \( C \in b(Z) \). We define the following bounded set in \( L(Z) : C := \{ \sum_{k=1}^{n} j_k(\beta_k \alpha_k z_k) : \beta_k \in \mathbb{K}, |\beta_k| = 1 (k \leq n), z_1, \ldots, z_n \in C, n \in \mathbb{N} \}, \) and \( \alpha = (\alpha_k)_{k \in \mathbb{N}} \in L \) with \( \|\alpha_k\| = 1 \).

Take \( v \in C^\circ \). Let us check that \( \| (p_\varphi(\psi \circ j_k))_{k \in \mathbb{N}} \| \leq 1 \) or equivalently that for every \( n \in \mathbb{N} \), \( \| (p_\varphi(\psi \circ j_k))_{k \leq n}, (0)_{k > n} \| \leq 1 \). Fix \( n \in \mathbb{N} \). On one hand

\[
\| (p_\varphi(\psi \circ j_k))_{k \leq n}, (0)_{k > n} \| = \sup_{\|\alpha\| = 1} \left| \sum_{k=1}^{n} p_\varphi(\psi \circ j_k) \alpha_k \right|.
\]

On the other hand for every \( \alpha = (\alpha_k)_{k \in \mathbb{N}} \in L \) with \( \|\alpha\| = 1 \) and every \( z_1, \ldots, z_n \in C \),

\[
\sum_{k=1}^{n} \langle v \circ j_k, \alpha_k z_k \rangle = (\text{for suitable } \beta_k \in \mathbb{K}, |\beta_k| = 1 (k \leq n)) = \sum_{k=1}^{n} \langle v \circ j_k, \beta_k \alpha_k z_k \rangle = v \left( \sum_{k=1}^{n} j_k(\beta_k \alpha_k z_k) \right).
\]

Since \( v \in C^\circ \), the definition of \( C \) proves 2.

3. The mapping

\[
\psi : L'(Z'_o) \rightarrow L(Z)'_o
\]

\[
u = (u_k)_{k \in \mathbb{N}} \rightarrow \psi(u) : L(Z) \rightarrow \mathbb{K}
\]

\[
x = (x_k)_{k \in \mathbb{N}} \rightarrow \sum_{k \in \mathbb{N}} \langle u_k, x_k \rangle
\]

is well defined. Fix \( x = (x_k)_{k \in \mathbb{N}} \in L(Z) \). By i), there must be \( C \in b(Z) \) such that \( x \in L(Z_c) \). Since \( (p_\varphi(u_k))_{k \in \mathbb{N}} \in L' \),

\[
\sum_{k=1}^{\infty} \langle u_k, x_k \rangle \leq \sum_{k=1}^{\infty} p_\varphi(u_k) p_c(x_k) \leq \| (p_\varphi(u_k))_{k \in \mathbb{N}} \| \| (p_c(x_k))_{k \in \mathbb{N}} \|
\]

that is, \( \psi(u) \) is well defined (\( \psi(u) \) is clearly linear). By ii), there is \( U \in H_0(Z) \) such that \( (p_{U^\circ}(u_k))_{k \in \mathbb{N}} \in L' \). Thus for every \( x \in L(Z) \), we obtain

\[
|\psi(u)(x)| = \sum_{k=1}^{\infty} \langle u_k, x_k \rangle \leq \sum_{k=1}^{\infty} p_{U^\circ}(u_k) p_{U'}(x_k) \leq \| (p_{U^\circ}(u_k))_{k \in \mathbb{N}} \| \| (p_{U'}(x_k))_{k \in \mathbb{N}} \|
\]

that is, \( \psi(u) \) is continuous and 3. is established.
Clearly $\psi$ is linear, $\Psi \circ \phi = 1_{L(X)'_p}$ and $\phi \circ \Psi = 1_{L(X)'_p}$.

4. $\psi$ is continuous. Take $B \in b(L(Z))$. By i), there is $B \in b(L(Z_B))$. In particular there is $\mu > 0$ with $\|(p_B(x_k))_{k \in \mathbb{N}}\| \leq \mu, (x = (x_k)_{k \in \mathbb{N}} \in \mathfrak{B})$. Thus $\psi(u) \in \mathfrak{B}^0$ for all $u \in L'(Z'_p)$ with $\|(p_B(u_k))_{k \in \mathbb{N}}\| \leq \mu^{-1}$. Indeed for every $x = (x_k)_{k \in \mathbb{N}} \in \mathfrak{B}, 1 - <\psi(u), x> \leq \sum_{k=1}^{\infty} | < u_k, x_k > | \leq \sum_{k=1}^{\infty} p_B(u_k) p_B(x_k) \leq \mu \mu^{-1} = 1$.

3. DUALITY

Let us first recall the definition of a general inductive limit of Moscatelli type.

Let $(L, |||)$ be a normal Banach sequence space, let $Y$ and $X$ be locally convex space, $Y$ continuously included in $X$. For every $n \in \mathbb{N}$, the space $E_n := \prod_{k<n} X \times L((Y)_{k \geq n})$ has the obvious meaning and should be provided with the canonical product topology. Now we define the inductive limit of Moscatelli type w.r.t. $(L, |||), X, Y$ (and the continuous canonical inclusion $j : Y \to X$) as $E = \operatorname{ind}_{n \in \mathbb{N}} E_n$ (we refer to [8] for details).

Let $(L, |||)$ be a normal Banach sequence space, let $Y$ and $X$ be locally convex space, $f : Y \to X$ a continuous linear mapping and $F$ the corresponding projective limit of Moscatelli type. For every sequence of subsets $(B_k)_{k \in \mathbb{N}}$ in $b(Y)$ and every subset $B \in b(X)$, which we shall always choose closed and absolutely convex, we define the space $F_{B(B_k)} := L(\{(B_k \cap f^{-1}(B)) \cdot p_B \cap f^{-1}(B)\}_{k \in \mathbb{N}})$ (compare with the definition in [3]). Here $[A]$ means the linear span of $A$ and $p_A$ is the Minkowski functional of $A$.

The space $F_{B(B_k)}$ is continuously embedded in $F((B_k)_{k \in \mathbb{N}} \in b(Y), B \in b(X))$.

3.1 Proposition. Let $(L, |||)$ be a normal Banach sequence space, let $Y$ and $X$ be locally convex spaces and $f : Y \to X$ a continuous linear mapping. Let $F$ be the corresponding projective limit of Moscatelli type and let $F_{B(B_k)}$ be as above. If either $X$ is a metrizable space or the space $(L, |||)$ satisfies $(\delta)$ and $X$ is a df-space, then for every $B \in b(L(X))$, there are $B \in b(Y) \text{ and } (B_k)_{k \in \mathbb{N}} \text{ in } b(Y)$ such that $B \in b(F_{B(B_k)})$. In particular, $\operatorname{ind}(F_{B(B_k)}) : B \in b(X), (B_k)_{k \in \mathbb{N}} \text{ in } b(Y))$ is the bornological space associated to $F$.

Proof. Given $B \in b(F)$, we have $\tilde{f}(B) \in b(L(X))$ (cf.1.1). By lemma (2.1) or (2.2) we can find $B \in b(X)$ such that $\tilde{f}(B) \subset \{(x_k)_{k \in \mathbb{N}} \in L(X) : (p_B(x_k))_{k \in \mathbb{N}} \in L$ and $\|(p_B(x_k))_{k \in \mathbb{N}}\| \leq 1 \}$. Define $B_k := \eta_k^{-1} p_{r_k}(B)(k \in \mathbb{N})$ where $(\eta_k)_{k \in \mathbb{N}} \in L$ with $\|(\eta_k)_{k \in \mathbb{N}}\| = 1$ and $\eta_k > 0, (k \in \mathbb{N})$. Thus $B \in b(L(B_k \cap f^{-1}(B)), p_{B_k \cap f^{-1}(B)})$ since for every $x = (x_k)_{k \in \mathbb{N}} \in \mathfrak{B}, \text{ we may write:}$

$$p_{B_k \cap f^{-1}(B)}(x_k) = \max(p_{B_k}(x_k), p_f(x_k)) \leq \max(\eta_k, p_B(f(x_k)))(k \in \mathbb{N}).$$
3.2 Proposition. Let \((L, \|\|)\) be a normal Banach sequence space with property \((\varepsilon)\), let \(Y\) and \(X\) be locally convex spaces and \(f : Y \to X\) a continuous linear mapping with dense range. Let \(F\) be the corresponding projective limit of Moscataelli type. Let \(E\) be the inductive limit of Moscataelli type w.r.t. the duals \((L', \|\|')\), \(X'_b, Y'_b\) and \(f^i : X'_b \to Y'_b\) (that we shall always consider as an inclusion). If the following two conditions are satisfied:

i) For every \(B \in \mathfrak{b}(L(X))\) there is \(B \in \mathfrak{b}(L(X_B))\).

ii) For every \(v \in L'(X'_b)\) there is an absolutely convex \(X\)-equicontinuous set \(M \subset X\) such that \(u \in L'(X'_M)\).

Then \(F' = E\) algebraically and \(E\) is continuously embedded in \(F'_b\).

Proof. By proposition (2.3), we have \(F'_{n,b} = \prod_{k < n} Y'_b \times L'((X'_b)_{k \geq n}) = E_n(n \in \mathbb{N})\) algebraically and topologically. Besides, the continuous limiting mapping:

\[ g_n^i : F'_{n,b} = \prod_{k < n} Y'_b \times L'((X'_b)_{k \geq n}) \to F'_{n+1,b} = \prod_{k < n+1} Y'_b \times L'((X'_b)_{k \geq n+1}) \]

\[ v \to v \circ g_n \]

that is, \(g_n^i(v) = v \circ g_n = ((v_{k})_{k < n}, f^i(v_n), (v_{k})_{k \geq n+1})(v \in F'_{n,b})\), coincides with the canonical inclusion \(E_n \to E_{n+1}(n \in \mathbb{N})\).

But \(F\) is reduced (which is easily derived from the fact that \(f\) has dense range) and therefore by [7, 22.6(6)], we obtain \(F' = \text{ind}_{n \in \mathbb{N}} (F'_n, g_n^i)\) = \(E\) algebraically. Besides, the inclusion \(E \subset F'\) is continuous because of the definition of the inductive limit topology.

The topological identity in the theorem above is rather delicate. We refer to [3] and [5] for the case of Banach spaces \(Y\) and \(X\).

3.3 Proposition. Let \((L, \|\|)\) be a normal Banach sequence space with property \((\varepsilon)\), let \(Y\) and \(X\) be locally convex spaces, \(Y\) continuously included in \(X\). Let \(E\) be the inductive limit of Moscataelli type w.r.t. \((L, \|\|), Y, X\) (and the continuous canonical inclusion \(j : Y \to X\)). Let \(F\) be the corresponding projective limit of Moscataelli type w.r.t. the duals \((L', \|\|')\), \(X'_b, Y'_b\) and \(j^i\). If the following two conditions are satisfied:

i) For every \(B \in \mathfrak{b}(L(Y))\) there is \(B \in \mathfrak{b}(L(Y_B))\).

ii) For every \(u \in L'(Y'_b)\), there is an absolutely convex \(Y\)-equicontinuous set \(M \subset Y\) such that \(u \in L'(Y'_M)\).

Then \(E' = F\) algebraically: the inclusion \(E'_b \subset F\) is continuous and \(E'_b = F\) algebraically and topologically whenever \(E\) is regular.

Proof. By proposition (2.3), we have \(E'_{n,b} = \prod_{k < n} X'_b \times L'((Y'_b)_{k \geq n}) = E_n(n \in \mathbb{N})\) algebraically and topologically. If we denote \(j_n : E_n \to E_{n-1} = ((x_{k})_{k \in \mathbb{N}}, (j(x_{k})_{k \in \mathbb{N}})\).
(x_k)_{k \geq n} \quad (n \in \mathbb{N}) \text{ we obtain that}

\[ j_n^t : E'_n \times \prod_{k < n+1} X'_b \times L'((Y'_b)_{k \geq n+1}) \to E'_{n,b} = \prod_{k < n} X'_b \times L'((Y'_b)_{k \geq n}) \]

\[ \nu \to \nu \circ j_n \]

coincides with the canonical mapping from \( F'_{n+1} \) to \( F'_n \) \( (n \in \mathbb{N}) \).

Thus, by [7, 22.6.(4)], we have \( E' = F \) (algebraically) and the inclusion \( E'_b \subset F \) is continuous because of the definition of the projective limit topology. Besides, if \( E \) is regular, we have \( E'_b = \text{proj } (E'_{n,b}, J_n) = \text{proj } F_n = F \); algebraically and topologically.

\[ \blacksquare \]

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